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TWIN POSITIVE SOLUTIONS FOR RESONANT SINGULAR (p,q)-EQUATIONS

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Abstract. We consider a Dirichlet (p,q)-equation with a reaction having the combined effects of a singular term and of a resonant perturbation. Using an auxiliary problem to bypass the singularity and variational tools from critical point theory, with truncation and comparison techniques, we show that the problem has two positive smooth solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the existence of positive solutions for the Dirichlet (p,q)-equation with singular reaction

$$\begin{gathered} -\Delta_p u(z) - \Delta_q u(z) = a(z)u(z)^{-\eta} + f(z,u(z)) \text{ in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad 1 < q < p,\ 0 < \eta < 1,\ u > 0. \end{gathered} \tag{1.1}$$

For every $r \in (1,\infty)$, we denote by Δ_r the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|Du|^{r-2}Du) \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

In the reaction there are both a singular term $a(z)x^{-\eta}$ and a Carathéodory perturbation $f(z,x)$ (that is, for all $x \in \mathbb{R}$ the mapping $z \mapsto f(z,x)$ is measurable and for a.a. $z \in \Omega$ the function $x \mapsto f(z,x)$ is continuous). We assume that the Carathéodory perturbation $f(z,\cdot)$ exhibits $(p-1)$ linear growth as $x \to +\infty$ and, in fact, asymptotically we can have a resonance with respect to the principal eigenvalue of the Dirichlet p-Laplacian. Actually the resonance occurs from the right of the principal eigenvalue, making the energy functional of the problem indefinite, that is, noncoercive. So, in problem (1.1), the reaction exhibits the combined effects of a singular term and of a resonant perturbation.

We point out that our problem is nonparametric. Usually singular problems involve a parameter, see for example the works of Sun, Wu and Long [25], Giacomoni, Schindler and Takač [8], Lü and Xie [13], Papageorgiou, Rădulescu and Repovš [17], Papageorgiou, Vetro and Vetro [21], Papageorgiou and Winkert [22]. When the equation is parametric, by varying the parameter, we can achieve certain

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geometric configurations, which permit the use of minimax theorems from critical point theory. This is also the case of nodal solutions of nonsingular $(p,2)$-equations with competing nonlinearities in the reaction (see Papageorgiou and Scapellato [19] and Papageorgiou and Zhang [23]).

Nonlinear nonparametric singular equations were investigated by Bai, Gasiński and Papageorgiou [2], Papageorgiou, Rădulescu and Repovš [16], Papageorgiou, Vetro and Vetro [20]. The first two papers consider equations driven by the p-Laplacian, while the third one deals with $(p,2)$-equations and has a perturbation term $f(z,\cdot)$ which is $(p-1)$-superlinear.

We mention that the exponent of the singular term satisfies $\eta \in (0,1)$. The more difficult case $\eta \geq 1$ (strong singularity), was examined by Lazer and McKenna [11] in the context of semilinear equations driven by the Dirichlet Laplacian. An overview of singular problems with a rich bibliography, can be found in the book of Ghergu and Rădulescu [7].

2. Mathematical background and hypotheses

The main spaces in the study of problem (1.1) are the Sobolev space $W_0^{1,p}(\Omega)$ and the ordered Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\}$. By $\|\cdot\|$ we denote the norm of $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we can have

$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The positive (order) cone for the space $C_0^1(\overline{\Omega})$ is

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0\}$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

Given $r \in (1,\infty)$, we denote by $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^*$, $\big(\frac{1}{r} + \frac{1}{r'} = 1\big)$ the nonlinear operator defined by

$$\langle A_r(u), h\rangle = \int_\Omega |Du|^{r-2}(Du, Dh)_{\mathbb{R}^N}\,dz \quad \text{for all } u, h \in W_0^{1,r}(\Omega).$$

The next proposition summarizes the well-known properties of this map. We refer to Problem 2.192 of Gasiński and Papageorgiou [5, p.279] for a more general result.

Proposition 2.1. *The operator $A_r(\cdot)$ is bounded (that is, maps bounded sets to bounded sets) continuous, strictly monotone (hence maximal monotone, too) and of type $(S)_+$, that is,*

if $u_n \xrightarrow{w} u$ in $W_0^{1,r}(\Omega)$ and $\limsup_{n\to\infty}\langle A_r(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ in $W_0^{1,r}(\Omega)$.

If $x \in \mathbb{R}$, then we set $x^\pm = \max\{\pm x, 0\}$. For $u \in W_0^{1,p}(\Omega)$ we define $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We know that

$$u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Given $u, v : \Omega \to \mathbb{R}$ two measurable functions, with $u(z) \le v(z)$ for a.a. $z \in \Omega$, we define

$$[u,v] = \{h \in W_0^{1,p}(\Omega) : u(z) \le h(z) \le v(z) \text{ for a.a. } z \in \Omega\},$$
$$[u) = \{h \in W_0^{1,p}(\Omega) : u(z) \le h(z) \text{ for a.a. } z \in \Omega\},$$
$$\mathrm{int}_{C_0^1(\overline{\Omega})}[u,v] \text{ is the interior in } C_0^1(\overline{\Omega}) \text{ of } [u,v] \cap C_0^1(\overline{\Omega}).$$

Also we write $u \preceq u$ if and only if for every compact $K \subseteq \Omega$ we have $0 < c_K \le v(z) - u(z)$ for a.a. $z \in K$. Evidently, if $u, v \in C(\Omega)$ and $u(z) < v(z)$ for all $z \in \Omega$, then $u \preceq v$.

By $\hat{\lambda}_1(p)$ we denote the first eigenvalue of $\big(-\Delta_p, W_0^{1,p}(\Omega)\big)$. We know that $\hat{\lambda}_1(p) > 0$, it is simple, isolated and admits the following variational characterization

$$\hat{\lambda}_1(p) = \inf\Big\{\frac{\|Du\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega), u \ne 0\Big\}. \tag{2.1}$$

The infimum in (2.1) is realized on the corresponding one-dimensional eigenspace, the elements of which have fixed sign. By $\hat{u}_1(p)$ we denote the positive, L^p-normalized (that is, $\|\hat{u}_1(p)\|_p = 1$) eigenfunction corresponding to $\hat{\lambda}_1(p)$. We know that $\hat{u}_1(p) \in \mathrm{int}\, C_+$ (by the nonlinear maximum principle, see Pucci and Serrin [24]). We will also consider a weighted version of the eigenvalue problem. So, let $m \in L^\infty(\Omega)$, $m(z) \ge 0$ for a.a. $z \in \Omega$, $m \ne 0$ and consider the nonlinear eigenvalue problem

$$-\Delta_p u(z) = \tilde{\lambda} m(z)|u(z)|^{p(z)-2}u(z) \text{ in } \Omega, \quad u\Big|_{\partial\Omega} = 0.$$

This problem too has a smallest eigenvalue $\tilde{\lambda}_1(m,p) > 0$ which has the same properties as $\hat{\lambda}(p)$. Note that if $m \equiv 1$, then $\tilde{\lambda}_1(m,p) = \hat{\lambda}_1(p)$. Moreover, the map $m \mapsto \hat{\lambda}_1(m,p)$ has the following monotonicity property. We refer to Proposition 9.47(d) of Motreanu, Motreanu and Papageorgiou [14, p.250] for details and a complete proof.

Proposition 2.2. *If $m, m' \in L^\infty(\Omega)$, $0 \le m(z) \le m'(z)$ for a.a. $z \in \Omega$, $m \not\equiv 0$, $m \not\equiv m'$, then $\tilde{\lambda}(m',p) < \tilde{\lambda}_1(m,p)$.*

Finally, we mention that $\tilde{\lambda}_1(m,p) > 0$ is the only eigenvalue with eigenfunctions of constant sign. All the other eigenvalues have eigenfunctions which are nodal (that is, sign changing).

Let X be a Banach space and $\varphi \in C^1(X)$. We say that $\varphi(\cdot)$ satisfies the "C-condition", if it has the following property:

> Every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded, and $(1+\|u_n\|)\varphi'(u_n) \to 0$ in X^*, admits a strongly convergent subsequence.

By K_φ we denote the critical set of φ, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

Now we introduce the hypotheses on the data of problem (1.1).

(H0) $a \in C_0^1(\overline{\Omega})$, $a(z) > 0$ for all $z \in \Omega$.

(H1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

- (i) $|f(z,x)| \le a(z)(1 + x^{p-1})$ for a.a. $z \in \Omega$, all $x \ge 0$, with $a \in L^\infty(\Omega)$;

(ii) $\hat{\lambda}_1(p) \le \liminf_{x\to+\infty} \frac{f(z,x)}{x^{p-1}}$ uniformly for a.a. $z \in \Omega$;
(iii) if $F(z,x) = \int_0^x f(z,x)ds$ then there exists $\tau \in (q,p)$ such that
$$0 < \beta_0 \le \liminf_{x\to+\infty} \frac{pF(z,x) - f(z,x)x}{x^\tau}$$
uniformly for a.a. $z \in \Omega$;
(iv) there exist $\mu \in (1,q)$ and $\delta, \vartheta > 0$ such that $C_0 x^\mu \le f(z,x)x \le \mu F(z,x)$ for a.a. $z \in \Omega$, all $0 \le x \le \delta$, some $C_0 > 0$, and
$$a(z)\vartheta^{-\eta} + f(z,\vartheta) \le -\hat{C} < 0 \quad \text{for a.a. } z \in \Omega;$$
(v) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function
$$x \mapsto f(z,x) + \hat{\xi}_\rho x^{p-1}$$
is nondecreasing on $[0,\rho]$.

We point out that we assume $a \in C_0^1(\overline{\Omega})$ since we use the Hardy inequality in the proof of Proposition 3.1 and because we need $a(\cdot)u_0(\cdot)^{-\eta} \in L^\infty(\Omega)$ and $0 \preceq a(\cdot)u_0(\cdot)^{-\eta}$ in the proof of Proposition 4.1 in order to apply the strong comparison principle (see Gasiński and Papageorgiou [6]). We also mention that in hypothesis (H1)(iv) we simply say that there exists $\hat{C} > 0$ such that
$$a(z)\vartheta^{-\eta} + f(z,\vartheta) \le -\hat{C} < 0 \text{ for a.a. } z \in \Omega.$$
In other words, the mapping $a(\cdot)\vartheta^{-\eta} + f(\cdot,\vartheta)$ is bounded away from zero uniformly for a.a. $z \in \Omega$.

Remark 2.3. Since we look for positive solutions and the hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$ only, without any loss of generality we may assume that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \le 0$. Hypothesis $H_1(ii)$ implies that as $x \to +\infty$ we can have resonance with respect to the principal eigenvalue. Hypothesis (H1)(iv) implies the presence of a "concave" nonlinearity near 0^+. Indeed integrating the first inequality in (H1)(iv), we obtain $C_1 x^\mu \le F(z,x)$ for a.a. $z \in \Omega$, all $0 \le x \le \delta$, some $C_1 > 0$. The second inequality in (H1)(iv) implies an oscillatory behavior for the reaction near 0^+.

By a solution of problem (1.1), we mean a function $u \in W_0^{1,p}(\Omega)$ such that $u^{-\eta}h \in L^1(\Omega)$ for all $h \in W_0^{1,p}(\Omega)$ and
$$\langle A_p(u), h\rangle + \langle A_q(u), h\rangle = \int_\Omega [a(z)u^{-\eta} + f(z,u)]h \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The presence of the singular term implies that $\varphi(\cdot)$ is not C^1 and so we cannot use the results of critical point theory directly on this functional. We need to find ways to bypass the singularity and deal with a C^1-functional. For this reason, in the next section we consider an auxiliary problem, the solution of which will be used to bypass the singularity as indicated above.

3. An auxiliary problem

Let $r \in (p, p^*)$. On account of hypotheses (H1)(i) and (H1)(iv), we have
$$f(z,x) \ge C_0 x^{\mu-1} - C_2 x^{r-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0, \text{ some } C_2 > 0. \tag{3.1}$$

Then we introduce the Carathéodory function

$$k(z,x)=\begin{cases} C_0(x^+)^{\mu-1}-C_2(x^+)^{r-1}, & \text{if } x\le\vartheta\\ C_0\vartheta^{\mu-1}-C_2\vartheta^{r-1}, & \text{if } \vartheta<x. \end{cases} \tag{3.2}$$

We consider the Dirichlet (p,q)-equation

$$\begin{gathered} -\Delta_p u(z)-\Delta_q u(z)=k(z,u(z)) \quad \text{in } \Omega,\\ u\big|_{\partial\Omega}=0, \quad 1<q<p, u>0. \end{gathered} \tag{3.3}$$

Proposition 3.1. *Problem* (3.3) *admits a unique positive solution* $\underline{u}\in \operatorname{int} C_+$, $\underline{u}(z)\le\vartheta$ *for all* $z\in\overline{\Omega}$ *and* $a(\cdot)\underline{u}^{-\eta}\in L^\infty(\Omega)$.

Proof. First we show the existence of a positive solution. To this end, let $K(z,x)=\int_0^x k(z,s)ds$ and consider the C^1-functional $\tau: W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\tau(u)=\frac{1}{p}\|Du\|_p^p+\frac{1}{q}\|Du\|_q^q-\int_\Omega K(z,u)dz \quad \text{for all } u\in W_0^{1,p}(\Omega).$$

It is clear from (3.2) that $\tau(\cdot)$ is coercive. Also using the Sobolev embedding theorem, we see that $\tau(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $\underline{u}\in W_0^{1,p}(\Omega)$ such that

$$\tau(\underline{u})=\min\big\{\tau(u): u\in W_0^{1,p}(\Omega)\big\}. \tag{3.4}$$

Let $u\in\operatorname{int} C_+$ and choose $t\in(0,1)$ small such that $0\le tu(z)\le\vartheta$ for all $z\in\overline{\Omega}$. Then on account of (3.2) we have

$$\tau(tu)=\frac{t^p}{p}\|Du\|_p^p+\frac{t^q}{q}\|Du\|_q^q+\frac{t^rC_2}{r}\|u\|_r^r-\frac{t^\mu C_0}{\mu}\|u\|_\mu^\mu.$$

Since, by hypothesis $1<\mu<q<p<r$, then choosing $t\in(0,1)$ even smaller if necessary, we have

$$\begin{gathered} \tau(tu)<0\Rightarrow\tau(\underline{u})<0=\tau(0) \text{ (see (3.4))},\\ \Rightarrow \underline{u}\ne 0. \end{gathered}$$

From (3.4) we have $\tau'(\underline{u})=0$ which implies

$$\langle A_p(\underline{u}),h\rangle+\langle A_q(\underline{u}),h\rangle=\int_\Omega k(z,\underline{u})h\,dz \quad \text{for all } h\in W_0^{1,p}(\Omega). \tag{3.5}$$

In (3.5) first we choose $h=-\underline{u}^-\in W_0^{1,p}(\Omega)$. We obtain

$$\begin{gathered} \|D\underline{u}^-\|_p^p\le 0 \quad \text{(see (3.2))},\\ \Rightarrow \underline{u}\ge 0, \quad \underline{u}\ne 0. \end{gathered}$$

Next, in (3.5) we choose $h=[\underline{u}-\vartheta]^+\in W_0^{1,p}(\Omega)$. Then we have

$$\begin{aligned} &\langle A_p(\underline{u}),(\underline{u}-\vartheta)^+\rangle+\langle A_q(\underline{u}),(\underline{u}-\vartheta)^+\rangle\\ &=\int_\Omega\big[C_0\vartheta^{\mu-1}-C_2\vartheta^{r-1}\big](\underline{u}-\vartheta)^+dz \quad \text{(see (3.2))}\\ &\le\int_\Omega f(z,\vartheta)(\underline{u}-\vartheta)^+dz \quad \text{(see (3.1))}\\ &\le 0 \quad \text{(see (H1)(iv))}\\ &=\langle A_p(\vartheta),(\underline{u}-\vartheta)^+\rangle+\langle A_q(\vartheta),(\underline{u}-\vartheta)^+\rangle,\\ &\Rightarrow \underline{u}\le\vartheta \quad \text{(see Proposition 2.1)}. \end{aligned}$$

So, we have proved that

$$\underline{u} \in [0, \vartheta], \quad \underline{u} \neq 0. \tag{3.6}$$

From (3.6), (3.2), (3.5) it follows that $\underline{u} \in W_0^{1,p}(\Omega)$ is a solution of problem (3.3). By Theorem 7.1 of Ladyzhenskaya and Uraltseva [10, p. 286], we have that $\underline{u} \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [12] implies that $\underline{u} \in C_+ \setminus \{0\}$. We have

$$\begin{aligned}
&\Delta_p \underline{u} + \Delta_q \underline{u} \le C_2 \|\underline{u}\|_\infty^{r-p} \underline{u}^{p-1} \quad \text{in } \Omega \\
&\Rightarrow \underline{u} \in \operatorname{int} C_+ \quad \text{(see Pucci and Serrin [24, pp. 111, 120]).}
\end{aligned}$$

Next, we show that this positive solution is unique. To this end, we consider the integral functional $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\|Du^{1/\mu}\|_p^p + \frac{1}{q}\|Du^{1/\mu}\|_q^q, & \text{if } u \ge 0, u^{1/\mu} \in W_0^{1,p}(\Omega) \\ \infty, & \text{otherwise.} \end{cases}$$

From Lemma 1 (and its proof) of Diaz and Saa [4], we have that the functional $j(\cdot)$ is convex. Let $\tilde{u} \in W_0^{1,p}(\Omega)$ be another positive solution of (3.3). Again we show that $\tilde{u} \in [0, \vartheta] \cap \operatorname{int} C_+$. Using Proposition 4.1.22 of Papageorgiou, Rădulescu and Repovš [18, p. 274], we have

$$\frac{\underline{u}}{\tilde{u}} \in L^\infty(\Omega) \quad \text{and} \quad \frac{\tilde{u}}{\underline{u}} \in L^\infty(\Omega).$$

If $\operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < \infty\}$ (the effective domain of $j(\cdot)$) and $h = \underline{u}^\mu - \tilde{u}^\mu \in W_0^{1,p}(\Omega)$, then for $|t| < 1$ small we have

$$\underline{u}^\mu + th \in \operatorname{dom} j \quad \text{and} \quad \tilde{u}^\mu + th \in \operatorname{dom} j.$$

So, from the convexity of $j(\cdot)$, we obtain that it is Gâteaux differentiable at $\underline{u}^\mu$ and $\tilde{u}^\mu$ in the direction h. Using the nonlinear Green identity (see Corollary 1.5.17 of Papageorgiou, Rădulescu and Repovš [18, p. 35]), we have

$$\begin{aligned}
j'(\underline{u}^\mu)(h) &= \frac{1}{\mu}\int_\Omega \frac{-\Delta_p \underline{u} - \Delta_q \underline{u}}{\underline{u}^{\mu-1}}(\underline{u}^\mu - \tilde{u}^\mu)dz \\
&= \frac{1}{\mu}\int_\Omega [C_0 - C_2 \underline{u}^{r-\mu}](\underline{u}^\mu - \tilde{u}^\mu)dz, \\
j'(\tilde{u}^\mu)(h) &= \frac{1}{\mu}\int_\Omega \frac{-\Delta_p \tilde{u} - \Delta_q \tilde{u}}{\tilde{u}^{\mu-1}}(\underline{u}^\mu - \tilde{u}^\mu)dz \\
&= \frac{1}{\mu}\int_\Omega [C_0 - C_2 \tilde{u}^{r-\mu}](\underline{u}^\mu - \tilde{u}^\mu)dz.
\end{aligned}$$

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. Therefore

$$0 \le \int_\Omega C_2[\tilde{u}^{r-\mu} - \underline{u}^{r-\mu}](\underline{u}^\mu - \tilde{u}^\mu)dz \le 0, \Rightarrow \underline{u} = \tilde{u}.$$

This proves the uniqueness of the positive solution $\underline{u} \in \operatorname{int} C_+$ of (3.3).

Let $\hat{d}(z) = d(z, \partial\Omega)$ for all $z \in \overline{\Omega}$. It follows by Lemma 14.16 of Gilbarg and Trudinger [9, p. 355] that we can find $\delta_0 > 0$ small such that $\hat{d} \in C^2(\Omega_{\delta_0})$, where

$$\Omega_{\delta_0} = \{z \in \overline{\Omega} : d(z, \partial\Omega) < \delta_0\}.$$

Then it follows that $\hat{d} \in \operatorname{int} C_+$ and so using Proposition 4.1.22 of Papageorgiou, Rădulescu and Repovš [18, p. 274], we can find $C_3, C_4 > 0$ such that $C_3\hat{d} \le \underline{u} \le C_4\hat{d}$.

Let $s > 1$. Then we have

$$\begin{aligned}
\int_\Omega \big(a(z)\underline{u}^{-\eta}\big)^s\,dz &= \int_\Omega (\underline{u}^{1-\eta})^s \Big(\frac{a(z)}{\underline{u}}\Big)^s dz \\
&\le C_4 \int_\Omega \Big(\frac{a(z)}{\underline{u}}\Big)^s dz \quad \text{for some } C_4 > 0, \text{ since } \underline{u} \in \operatorname{int} C_+. \\
&\le C_5 \int_\Omega \big(\frac{a(z)}{\hat{d}}\big)^s dz \quad \text{for some } C_5 > 0 \\
&\le C_5 \|Da\|_s^s \quad \text{(by Hardy's inequality, see Brezis [3, p. 313])},
\end{aligned}$$

which implies

$$\|a\underline{u}^{-\eta}\|_s \le C_6 \quad \text{for some } C_6 > 0,\ \text{all } s > 1,$$

Therefore, $a(\cdot)\underline{u}(\cdot)^{-\eta} \in L^\infty(\Omega)$. The proof is now complete. $\square$

4. Positive solutions

In this section, using $\underline{u} \in \operatorname{int} C_+$ from Proposition 3.1, we are able to bypass the singularity and deal with C^1-functionals on which we can use the minimax theorems of critical point theory.

So, with $\underline{u} \in \operatorname{int} C_+$ from Proposition 3.1, we introduce the Carathéodory functions $g, \hat{g} : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$g(z,x) = \begin{cases} a(z)\underline{u}(z)^{-\eta} + f(z,\underline{u}(z)), & \text{if } x \le \underline{u}(z) \\ a(z)x^{-\eta} + f(z,x), & \text{if } \underline{u}(z) < x, \end{cases} \tag{4.1}$$

$$\hat{g}(z,x) = \begin{cases} g(z,x), & \text{if } x \le \vartheta \\ g(z,\vartheta), & \text{if } \vartheta < x, \end{cases} \tag{4.2}$$

recall that $\underline{u} \le \vartheta$.

We set $G(z,x) = \int_0^x g(z,s)ds$, $\hat{G}(z,x) = \int_0^x \hat{g}(z,s)ds$ and consider the C^1-functionals $\Psi, \hat{\Psi} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\begin{gathered}
\Psi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{q}\|Du\|_q^q - \int_\Omega G(z,u)dz, \\
\hat{\Psi}(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{q}\|Du\|_q^q - \int_\Omega \hat{G}(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).
\end{gathered}$$

Proposition 4.1. *If hypotheses* (H0), (H1) *hold, then problem* (1.1) *admits a positive solution*

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\underline{u}, \vartheta].$$

Proof. From (4.2) it is clear that $\hat{\Psi}$ is coercive. Also by the Sobolev embedding theorem $\hat{\Psi}(\cdot)$ is sequentially weakly lower semicontinuous. So, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\hat{\Psi}(u_0) = \inf\{\hat{\Psi}(u) : u \in W_0^{1,p}(\Omega)\},$$

which implies $\hat{\Psi}'(u) = 0$, and

$$\langle A_p(u_0), h\rangle + \langle A_q(u_0), h\rangle = \int_\Omega \hat{g}(z,u_0)h\,dz \text{ for all } h \in W_0^{1,p}(\Omega). \tag{4.3}$$

In (4.3) first we choose $h = [\underline{u} - u_0]^+ \in W_0^{1,p}(\Omega)$. We have

$$
\begin{aligned}
&\langle A_p(u_0), (\underline{u} - u_0)^+\rangle + \langle A_q(u_0), (\underline{u} - u_0)^+\rangle \\
&= \int_\Omega [a(z)\underline{u}^{-\eta} + f(z, \underline{u})](\underline{u} - u_0)^+ dz \quad \text{(see (4.1), (4.2))} \\
&\geq \int_\Omega f(z, \underline{u})(\underline{u} - u_0)^+ dz \quad \text{(see hypothesis (H0))} \\
&\geq \int_\Omega [C_0\underline{u}^{\mu-1} - C_2\underline{u}^{r-1}](\underline{u} - u_0)^+ dz \quad \text{(see (3.1))} \\
&= \langle A_p(\underline{u}), (\underline{u} - u_0)^+\rangle + \langle A_q(\underline{u}), (\underline{u} - u_0)^+\rangle \quad \text{(see Proposition 3.1)}, \\
&\Rightarrow \underline{u} \leq u_0.
\end{aligned}
$$

Next, in (4.3) we choose $h = [u_0 - \vartheta]^+ \in W_0^{1,p}(\Omega)$. We have

$$
\begin{aligned}
&\langle A_p(u_0), (u_0 - \vartheta)^+\rangle + \langle A_q(u_0), (u_0 - \vartheta)^+\rangle \\
&= \int_\Omega [a(z)\vartheta^{-\eta} + f(z, \vartheta)](u_0 - \vartheta)^+ dz \text{ (see (4.1), (4.2))} \\
&\leq 0 \quad \text{(from (H1)(iv))} \\
&= \langle A_p(\vartheta), (u_0 - \vartheta)^+\rangle + \langle A_q(\vartheta), (u_0 - \vartheta)^+\rangle, \\
&\Rightarrow u_0 \leq \vartheta.
\end{aligned}
$$

So, we have proved that

$$
u_0 \in [\underline{u}, \vartheta]. \tag{4.4}
$$

From (4.4), (4.1), (4.2) and (4.3), it follows that u_0 is a positive solution of (1.1) and, as before, the nonlinear regularity theory of Lieberman [12] implies that

$$
u_0 \in [\underline{u}, \vartheta] \cap \operatorname{int} C_+. \tag{4.5}
$$

Since $1 < \mu < q < p < r$, we can find $\tilde{\xi}_\vartheta > 0$ such that the function $x \mapsto C_0x^{\mu-1} - C_2x^{r-1} + \tilde{\xi}_\vartheta x^{p-1}$ is nondecreasing on $[0, \vartheta]$. So, we have

$$
\begin{aligned}
&- \Delta_p u_0 - \Delta_q u_0 + \tilde{\xi}_\vartheta u_0^{p-1} \\
&= a(z)u_0^{-\eta} + f(z, u_0) + \tilde{\xi}_\vartheta u_0^{p-1} \\
&\geq C_0u_0^{\mu-1} - C_2u_0^{r-1} + \tilde{\xi}_\vartheta u_0^{p-1} \quad \text{(see (3.1) and (H0))} \\
&\geq C_0\underline{u}^{\mu-1} - C_2\underline{u}^{r-1} + \tilde{\xi}_\vartheta \underline{u}^{p-1} \quad \text{(see (4.5))} \\
&= -\Delta_p\underline{u} - \Delta_q\underline{u} + \tilde{\xi}_\vartheta \underline{u}^{p-1} \quad \text{(see Proposition 3.1)}.
\end{aligned} \tag{4.6}
$$

Since $u_0 \in \operatorname{int} C_+$ and $a(z) > 0$ for all $z \in \Omega$ (see hypothesis (H0)), it follows that $0 \preceq a(\cdot)u_0(\cdot)^{-\eta}$. Hence from (4.6) and Proposition 3.2 of Gasiński and Papageorgiou [6], we infer that

$$
u_0 - \underline{u} \in \operatorname{int} C_+. \tag{4.7}
$$

On the other hand, let $\hat{\xi}_\vartheta > 0$ be as postulated by hypothesis $H_1(v)$. We have

$$
\begin{aligned}
&- \Delta_p u_0 - \Delta_q u_0 + \hat{\xi}_\vartheta u_0^{p-1} - a(z)u_0^{-\eta} \\
&= f(z, u_0) + \hat{\xi}_\vartheta u_0^{p-1} \\
&\leq f(z, \vartheta) + \hat{\xi}_\vartheta \vartheta^{p-1} \quad \text{(see (4.5) and (H1)(v))} \\
&\leq -\Delta_p\vartheta - \Delta_q\vartheta + \hat{\xi}_\vartheta \vartheta^{p-1} - a(z)\vartheta^{-\eta} \quad \text{(see hypothesis (H1)(iv))}.
\end{aligned} \tag{4.8}
$$

We know that
$$\vartheta^{-\eta} + f(z,\vartheta) \le -\hat{C} < 0 \quad \text{for a.a. } z \in \Omega.$$
So, from (4.8) and Proposition 6 of Papageorgiou, Rădulescu and Repovš [17] we have
$$u_0(z) < \vartheta \quad \text{for all } z \in \overline{\Omega}. \tag{4.9}$$
From (4.7) and (4.9), we conclude that $u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\underline{u}, \vartheta]$. The proof is complete. □

Proposition 4.2. *If hypotheses* (H0), (H1) *hold, then* $u_0 \in \operatorname{int} C_+$ *is a local minimizer of the functional* $\Psi(\cdot)$.

Proof. From (4.1) and (4.2) it is clear that
$$\Psi\big|_{[\underline{u},\vartheta]} = \hat{\Psi}\big|_{[\underline{u},\vartheta]}. \tag{4.10}$$
From the proof of Proposition 4.1 we know that u_0 is a minimizer of $\hat{\Psi}$. Since $u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\underline{u}, \vartheta]$, it follows from (4.10) that
$$\begin{gathered} u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \Psi(\cdot) \\ \Rightarrow u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \Psi(\cdot) \end{gathered}$$
(see Papageorgiou and Rădulescu [15, Proposition 2.12]). This completes the proof. □

Using (4.1) and the nonlinear regularity theory, we have
$$K_\Psi \subseteq [\underline{u}) \cap \operatorname{int} C_+. \tag{4.11}$$
From (4.1) and (4.11), we see that we may assume that
$$K_\Psi \text{ is finite.} \tag{4.12}$$
Otherwise we already have an infinity of positive smooth solutions of (1.1) and so we are done.

Combining Proposition 5, relation (4.12) and Theorem 5.7.6 of Papageorgiou, Rădulescu and Repovš [18, p. 449], we deduce that we can find $\rho \in (0,1)$ small such
$$\Psi(u_0) < \inf\{\Psi(u) : \|u - u_0\| = \rho\} = \overline{m}. \tag{4.13}$$

Proposition 4.3. *If hypotheses* (H0), (H1) *hold, then* $\Psi(t\hat{u}_1(p)) \to -\infty$ *as* $t \to +\infty$.

Proof. We have
$$\begin{aligned} \frac{d}{dx}\Big[\frac{F(z,x)}{x^p}\Big] &= \frac{f(z,x)x^p - pF(z,x)x^{p-1}}{x^{2p}} \\ &= \frac{f(z,x)x - pF(z,x)}{x^{p+1}} \quad \text{for a.a. } z \in \Omega, \text{ all } x > 0. \end{aligned} \tag{4.14}$$
On account of hypothesis (H1)(iii), we can find $M > 0$ and $\beta_1 \in (0, \beta_0)$ such that
$$f(z,x) - pF(z,x) \le -\beta_1 x^\tau \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M. \tag{4.15}$$
We return to (4.14) and use (4.15) to obtain
$$\frac{d}{dx}\Big[\frac{F(z,x)}{x^p}\Big] \le \frac{-\beta_1 x^\tau}{x^{p+1}} = \frac{-\beta_1}{x^{p-\tau+1}} \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M,$$

which implies

$$\frac{F(z,v)}{v^p} - \frac{F(z,x)}{x^p} \le \frac{\beta_1}{p-\tau}[\frac{1}{v^{p-\tau}} - \frac{1}{x^{p-\tau}}] \quad \text{for a.a. } z \in \Omega, \text{ all } v \ge x \ge M.$$

We let $v \to +\infty$ and using (H1)(ii) we obtain

$$\begin{aligned}
&\frac{\hat{\lambda}_1(p)}{p} - \frac{F(z,x)}{x^p} \le -\frac{\beta_1}{p-\tau}\frac{1}{x^{p-\tau}},\\
&\Rightarrow \frac{\hat{\lambda}_1(p)x^p - pF(z,x)}{x^\tau} \le -\frac{p\beta_1}{p-\tau} = -\beta_2 \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge M,. \qquad (4.16)\\
&\Rightarrow \limsup_{x\to+\infty} \frac{\hat{\lambda}_1(p)x^p - pF(z,x)}{x^\tau} \le -\beta_2 < 0 \quad \text{uniformly for a.a. } z \in \Omega
\end{aligned}$$

Then we have

$$\begin{aligned}
\Psi(t\hat{u}_1(p)) &= \frac{t^p}{p}\hat{\lambda}_1(p) + \frac{t^q}{q}\|D\hat{u}_1(p)\|_q^q - \int_\Omega G(z, t\hat{u}_1(p))dz\\
&= \frac{t^p}{p}\hat{\lambda}_1(p) + \frac{t^q}{q}\|D\hat{u}_1(p)\|_q^q - \int_{\{t\hat{u}_1(p)\le \underline{u}\}} [a(z)\underline{u}^{-\eta} + f(z,\underline{u})](t\hat{u}_1(p))dz\\
&\quad - \frac{1}{1-\eta}\int_{\{t\hat{u}_1(p)>\underline{u}\}} a(z)^{1-\eta}[(t\hat{u}_1(p))^{1-\eta} - \underline{u}^{1-\eta}]dz\\
&\quad - \int_{\{t\hat{u}_1(p)>\underline{u}\}} [F(z,t\hat{u}_1(p)) - F(z,\underline{u})]dz\\
&\le \frac{1}{p}\int_\Omega \frac{\hat{\lambda}_1(p)(t\hat{u}_1(p))^p - pF(z,t\hat{u}_1(p))}{t^\tau} t^\tau dz + C_7 \quad \text{for some } C_7 > 0.
\end{aligned}$$

Passing to the limit as $t \to -\infty$ and using (4.16), we obtain $\Psi(t\hat{u}_1(p)) \to -\infty$ as $t \to +\infty$. The proof is now complete. □

Remark 4.4. From the above proof we see that

$$pF(z,x) - \hat{\lambda}_1(p)x^p \to +\infty$$

uniformly for a.a. $z \in \Omega$, as $x \to +\infty$. Hence the resonance is from the right of the principal eigenvalue, making our problem noncoercive. This means that the direct method of the calculus of variations cannot be used and we need to appeal to the minimax theorems of critical point theory.

Proposition 4.5. *If hypotheses* (H0), (H1)*, hold, then the functional* $\Psi(\cdot)$ *satisfies the* C*-condition.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\Psi(u_n)| \le \hat{M} \quad \text{for some } \hat{M} > 0, \text{ all } n \in \mathbb{N}, \qquad (4.17)$$

$$(1 + \|u_n\|)\Psi'(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) \text{ as } n \to \infty. \qquad (4.18)$$

From (4.18) we have

$$|\langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle| - \int_\Omega g(z,u_n)h\,dz| \le \frac{\varepsilon_n\|h\|}{1+\|u_n\|} \qquad (4.19)$$

for all $h \in W_0^{1,p}(\Omega)$, with $\varepsilon_n \to 0^+$. In (4.19) we let $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\begin{gathered}\|Du_n^-\|_p \le C_8 \quad \text{for some } C_8 > 0, \text{ all } n \in \mathbb{N}, \\ \Rightarrow \{u_n^-\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}\end{gathered} \tag{4.20}$$

Next we show that $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. Arguing by contradiction, assume that at least for a subsequence, we have

$$\|u_n^+\| \to \infty \text{ as } n \to \infty. \tag{4.21}$$

We set $y_n = u_n^+/\|u_n^+\|$ for all $n \in \mathbb{N}$. We have that $\|y_n\| = 1$ and $y_n \ge 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^p(\Omega) \text{ as } n \to \infty,\ y \ge 0. \tag{4.22}$$

From (4.19) and (4.20) we have

$$\big|\langle A_p(u_n^+), h\rangle + \langle A_q(u_n^+), h\rangle - \int_\Omega g(z, u_n^+) h\, dz\big| \le C_9\|h\|$$

for some $C_9 > 0$, all $h \in W_0^{1,p}(\Omega)$, all $n \in \mathbb{N}$. Then

$$\big|\langle A_p(y_n), h\rangle + \frac{1}{\|u_n^+\|^{p-q}}\langle A_q(y_n), h\rangle - \int_\Omega \frac{g(z, u_n^+)}{\|u_n^+\|^{p-1}} h\, dz\big| \le \frac{C_9\|h\|}{\|u_n^+\|^{p-1}} \tag{4.23}$$

for all $h \in W_0^{1,p}(\Omega)$, all $n \in \mathbb{N}$.

From (4.1), Proposition 3.1 and hypothesis (H1)(i), we see that

$$\Big\{\frac{g(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}}\Big\}_{n\in\mathbb{N}} \subseteq L^{p'}(\Omega) \text{ is bounded.} \tag{4.24}$$

So, if in (4.23) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (4.22), (4.21), (4.24), then we obtain

$$\begin{gathered}\lim_{n\to\infty}\langle A_p(y_n), y_n - y\rangle = 0, \\ \Rightarrow y_n \to y \quad \text{in } W_0^{1,p}(\Omega), \text{ hence } \|y\| = 1,\ y \ge 0 \quad \text{(see Proposition 2.1).}\end{gathered} \tag{4.25}$$

From (4.24), (4.1) and (H1)(ii), we see that at least for a subsequence, we have

$$\frac{g(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}} \xrightarrow{w} \hat\eta(\cdot) y(\cdot)^{p-1} \quad \text{in } L^{p'}(\Omega) \tag{4.26}$$

with $\hat\eta \in L^\infty(\Omega)$, $\hat\lambda_1(p) \le \hat\eta(z)$ for a.a. $z \in \Omega$ (see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 16).

So, if in (4.23) we pass the limit as $n \to \infty$ and use (4.25), (4.21) and (4.26), we obtain

$$\begin{gathered}\langle A_p(y), h\rangle = \int_\Omega \hat\eta(z) y^{p-1} h\, dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow -\Delta_p y = \hat\eta(z) y^{p-1} \quad \text{in } \Omega,\ y\big|_{\partial\Omega} = 0.\end{gathered} \tag{4.27}$$

If $\hat\eta \not\equiv \hat\lambda_1(p)$ (see (4.26)), then by Proposition 2.2 we have

$$\tilde\lambda_1(\hat\eta, p) < \tilde\lambda_1(\hat\lambda_1(p), p) = 1.$$

Then form (4.27) we infer that y must be nodal, which contradicts (4.25).

Now suppose that $\hat\eta(z) = \hat\lambda_1(p)$ for a.a. $z \in \Omega$. Then from (4.27) it follows that

$$y = \mu\hat u_1(p) \in \operatorname{int} C_+, \quad \text{with } \mu > 0.$$

This means that $u_n^+(z) \to +\infty$ for a.a. $z \in \Omega$, which implies

$$\liminf_{n\to\infty} \int_\Omega \frac{pF(z,u_n^+) - f(z,u_n^+)u_n^+}{(u_n^+)^\tau} dz \geq \hat{\beta} > 0 \tag{4.28}$$

(by Fatou's lemma and hypothesis (H1)(iii)). From (4.17) and (4.20) we have

$$-\|Du_n^+\|_p^p - \frac{p}{q}\|Du_n^+\|_q^q + \int_\Omega pF(z,u_n^+)dz \leq M_1 \tag{4.29}$$

for some $M_1 > 0$, all $n \in \mathbb{N}$. Also from (4.19) with $h = u_n^+ \in W_0^{1,p}(\Omega)$, we have

$$\|Du_n^+\|_p^p + \|Du_n^+\|_q^q - \int_\Omega f(z,u_n^+)u_n^+ dz \leq M_2 \tag{4.30}$$

for some $M_2 > 0$, all $n \in \mathbb{N}$.

We add (4.29) and (4.30) to obtain

$$\int_\Omega [pF(z,u_n^+) - f(z,u_n^+)u_n^+]dz \leq \Big(\frac{p}{q} - 1\Big)\|Du_n^+\|_q^q + M_3$$

with $M_3 = M_1 + M_2 > 0$, for all $n \in \mathbb{N}$. Then

$$\begin{aligned}
&\int_\Omega \frac{pF(z,u_n^+) - f(z,u_n^+)u_n^+}{(u_n^+)^\tau} y_n^\tau dz \leq \Big(\frac{p}{q} - 1\Big)\frac{\|Dy_n\|_q^q}{\|u_n^+\|^{\tau-q}} + \frac{M_3}{\|u_n^+\|^\tau} \\
&\Rightarrow \limsup_{n\to\infty} \int_\Omega \frac{pF(z,u_n^+) - f(z,u_n^+)u_n^+}{(u_n^+)^\tau} y_n^\tau dz \leq 0
\end{aligned} \tag{4.31}$$

(since $q < \tau$ and use (4.21)).

Comparing (4.31) and (4.28), we have a contradiction. This proves that $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. It follows that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded (see (4.20))}.$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \in L^p(\Omega). \tag{4.32}$$

Now we return to (4.19), choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (4.32). Then

$$\begin{aligned}
&\lim_{n\to\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A_q(u_n), u_n - u\rangle] = 0, \\
&\Rightarrow \limsup_{n\to\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A_q(u), u_n - u\rangle] \leq 0 \quad \text{(since } A_q(\cdot) \text{ is monotone)} \\
&\Rightarrow \limsup_{n\to\infty} \langle A_p(u_n), u_n - u\rangle \leq 0, \\
&\Rightarrow u_n \to u \quad \text{in } W_0^{1,p}(\Omega) \text{ (see Proposition 1)}.
\end{aligned}$$

This proves that $\Psi(\cdot)$ satisfies the C-condition. □

Proposition 4.6. *If hypotheses* (H0), (H1) *hold, then problem* (1.1) *admits a second positive solution* $\hat{u} \in \operatorname{int} C_+$*, with* $\hat{u} \neq u_0$*.*

Proof. Propositions 4.3, 4.5 and relation (4.13), permit the use of the mountain pass theorem. So, we can find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\begin{gathered}
\hat{u} \in K_\Psi \subseteq [\underline{u}) \cap \operatorname{int} C_+ \quad \text{(see (4.11))}, \\
\Psi(u_0) < \overline{m} \leq \Psi(\hat{u}) \quad \text{(see (4.13))}.
\end{gathered}$$

It follows that $\hat{u} \in \operatorname{int} C_+$ is a positive solution of (1.1) (see (4.1)) and $\hat{u} \neq u_0$. □

Summarizing our findings, we can state the following multiplicity theorem for problem (1.1).

Theorem 4.7. *If hypotheses* (H0), (H1) *hold, then problem* (1.1) *admits at least two positive solutions*

$$u_0, \hat{u} \in \operatorname{int} C_+, \quad u_0 \neq \hat{u}, \quad u_0(z) < \vartheta \quad \textit{for all } z \in \overline{\Omega}.$$

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