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CONNECTED COMPONENTS OF POSITIVE SOLUTIONS OF BIHARMONIC EQUATIONS WITH THE CLAMPED PLATE CONDITIONS IN TWO DIMENSIONS

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In memory of Professor Alan C. Lazer

Abstract. This article concerns the clamped plate equation

$$\Delta^2 u = \lambda a(x) f(u), \quad \text{in } \Omega,$$
$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^2$ of class $C^{4,\alpha}$, $a \in C(\bar{\Omega}, (0,\infty))$, $f : [0,\infty) \to [0,\infty)$ is a locally Hölder continuous function with exponent α, and λ is a positive parameter. We show the existence of S-shaped connected component of positive solutions under suitable conditions on the nonlinearity. Our approach is based on bifurcation techniques.

1. Introduction

Let Ω denote a bounded domain in $\mathbb{R}^2$ of class $C^{4,\alpha}$. We consider the clamped plate problem

$$\Delta^2 u = \lambda \tilde{f}(x,u) \quad \text{in } \Omega, \tag{1.1}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\partial/\partial\nu$ is the outward normal derivative, $\alpha \in (0,1]$, $\tilde{f} : \bar{\Omega} \times [0,\infty) \to [0,\infty)$ is a locally Hölder continuous function with exponent α. (1.1), (1.2) forms a model for the clamped plate where $\tilde{f}$ is the load and u the deviation of the plate Ω. Boggio [2, 3] and Hadamard [16, 17] extensively studied this model when $\lambda \tilde{f}(x,u) = e(x)$ and $\tilde{f}(x,u) = u$, respectively.

Dalmasso [7] used the Schauder fixed point theorem to study the existence of positive solutions of nonlinear boundary-value problem of elliptic equation of order $2m$ under the assumptions

(1) for $x \in \Omega$, $\tilde{f}(x,s)$ is nondecreasing in s;
(2) $\lim_{s\to 0} \min_{x\in\bar{\Omega}} \tilde{f}(x,s)/s = \infty$, $\lim_{s\to\infty} \max_{x\in\bar{\Omega}} \frac{\tilde{f}(x,s)}{s} = 0$,

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and considered the following domains: the unit ball $B = \{x \in \mathbb{R}^N : \|x\| < 1\}$, $N \geq 1$, and a bounded domain of class $C^{2m,\alpha}$ close in $C^{2m,\alpha}$-sense to a ball. Mâagli, Toumi, and Zribi [20] also used the Schauder fixed point theorem to show the existence of positive continuous solution (in the sense of distributions), when Ω is the unit ball B in $\mathbb{R}^N$ and $N \geq 2$, and the nonlinearity $\tilde{f}$ satisfies appropriate conditions related to a Kato class of functions $K_{m,N}$. At most two radial positive solutions were obtained in above mentioned papers.

The aim of this article is to study the global structure of positive solutions for problem (1.1), (1.2) on $\Omega \subset \mathbb{R}^2$ when

$$\tilde{f}(x,s) = a(x)f(s), \quad x \in \bar{\Omega},\ s \in [0,\infty),$$

and to show that the positive solutions set contains an S-shaped connected component under suitable conditions; consequently, (1.1), (1.2) possesses at least three positive solutions for λ belonging to certain open interval.

We work on $\Omega \subset \mathbb{R}^2$ for the following two reasons:

(1) we need to assume that Ω is a bounded domain of class $C^{4,\alpha}(\bar{\Omega})$ which is ϵ_0-close in $C^{4,\alpha}$-sense to $B \subset \mathbb{R}^2$ for some $\epsilon_0 > 0$ (see Grunau and Sweers [13, 14] for the detail);

(2) Harnack inequalities are very important in study of the shape of connected components of positive solutions of second order elliptic problems, see Sim and Tanaka [23]. However, no general Harnack inequalities are available for the polyharmonic problems, see Gazzola, Grunau, and Sweers [11, P.146]. Caristi and Mitidieri [6, Theorem 3.6] proved a Harnack type inequalities for linear biharmonic equations containing a Kato potential when $N > 4$, which cannot be used to treat the biharmonic problem on $\Omega \subset \mathbb{R}^2$. To establish a Harnack inequality for biharmonic problems on $\Omega \subset \mathbb{R}^2$, we need (4.13) below. Notice that (4.13) need the restriction $N = m = 2$.

For earlier results on the existence and multiplicity of solutions to the mathematical models of nonlinearly supported bending beams see the well-known survey paper of Lazer and Mckenna [18].

2. Preliminaries

Let Y be the Banach space $C(\bar{\Omega})$ equipped with the supremum norm $\|\cdot\|_{C(\bar{\Omega})}$.

2.1. **Principal eigenvalue.** The biharmonic eigenvalue problem with Dirichlet boundary conditions has the form

$$\begin{gathered} \Delta^2\varphi = \lambda\varphi \quad \text{in } \Omega, \\ \varphi = \frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

The famous conjecture for this problem was as follows; by now it has numerous counterexamples.

Conjecture (Szegö, 1950) If Ω is a 'nice' domain (convex), then the first eigenfunction for (2.1) is of fixed sign.

This conjecture was proved to be wrong, see Duffin and others [8, 10, 19, 4, 22]. Coffman [4] proved that the first eigenfunction on a square changes sign. For the domains

$$A_\epsilon = \{(x,y) \in \mathbb{R}^2 : \epsilon^2 < x^2 + y^2 < 1\} \quad \text{with } 0 < \epsilon < 1.$$

Coffman, Duffin and Shaffer [5] proved the fundamental mode of vibration of a clamped annular plate A_ϵ is not of one-sign.

We first recall the definition of closeness of domain introduced by Grunau and Sweers [13].

Definition 2.1. Let $\epsilon > 0$, $\alpha \in (0,1]$, Ω is called ϵ-closed in $C^{k,\alpha}$-sense to Ω^*, if there exists a $C^{k,\alpha}$ mapping $g : \bar{\Omega}^* \to \bar{\Omega}$ such that $g(\bar{\Omega}^*) = \bar{\Omega}$ and

$$\|g - Id\|_{C^{k,\alpha}(\bar{\Omega}^*)} \le \epsilon.$$

Using Dalmasso [7, Lemma 3.1(2)] and Dalmasso [7, Theorem 2.2 (ii)], we may deduce the following result.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ and Ω is a bounded domain of class $C^{4,\alpha}$. Then there exists $\epsilon_0 > 0$ such that if Ω is ϵ-close in $C^{4,\alpha}$ sense to B for all $0 < \epsilon \le \epsilon_0$, then*

(1) the problem

$$\begin{gathered} \Delta^2 u = e \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{gathered}$$

with some $e \in C^{0,\alpha}(\bar{\Omega})$ has unique solution $u \in C^{4,\alpha}(\bar{\Omega})$.

(2) If $e \ge 0$ and $e \not\equiv 0$, then $\frac{\partial^2 u}{\partial \nu^2} > 0$ for $x \in \partial\Omega$.

In the following, we consider the eigenvalue problem

$$\begin{gathered} \Delta^2 u = \lambda a(x) u, \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.2}$$

where $a \in C(\bar{\Omega}, (0,\infty))$. The first eigenvalue of (2.2) is defined as

$$\lambda_1(a(\cdot)) = \min_{u \in H_0^2(\Omega)\setminus\{0\}} \frac{\|\Delta u\|^2_{H_0^2}}{\|a^{1/2} u\|^2_{L^2}},$$

where $H_0^2(\Omega)$ is the closure of $C_c^\infty(\Omega)$ with respect to the normal $\|\cdot\|_{W^{2,2}}$, and $C_c^\infty(\Omega)$ is the space of $C^\infty(\Omega)$-functions having compact support in Ω.

Applying Lemma 2.2 and the standard Krein-Rutman type argument, we may obtain the following result.

Lemma 2.3. *Let ϵ_0 be the constant as given in Lemma 2.2. If $\Omega \subset \mathbb{R}^2$ and Ω is a bounded domain of class $C^{4,\alpha}(\bar{\Omega})$ which is ϵ_0-close in $C^{4,\alpha}$-sense to B, then*

1. *the first eigenvalue $\lambda_1(a(\cdot))$ of (2.2) is simple;*
2. *the corresponding eigenfunction ψ is of one sign;*
3. $\frac{\partial^2 \psi}{\partial \nu^2} > 0, \quad x \in \partial\Omega.$

2.2. **Shape of positive solutions.** We will make the following assumptions:

- (H0) $f : [0,\infty) \to [0,\infty)$ is a Hölder continuous function with exponent α, and $f(s) > 0$ for $s > 0$;
- (H1) $a \in C(\bar{\Omega}, (0,\infty))$;
- (H2) there exist $\beta > 0$, $f_0 > 0$ and $f_1 > 0$ such that

$$\lim_{s\to 0^+} \frac{f(s) - f_0 s}{s^{1+\beta}} = -f_1;$$

(H3)

$$f_\infty := \lim_{s\to\infty} \frac{f(s)}{s} = 0.$$

Remark 2.4. It is easy to show that if (H2) holds, then

$$\lim_{s\to 0^+} \frac{f(s)}{s} = f_0.$$

Moreover, if (H3) holds, then there exists $\tilde{s} > 0$, $f^* > 0$ and $\gamma^* > 0$ such that

$$f(s) \le f^* s, \quad \forall s \ge 0; \quad f(s) \ge \gamma^* s, \quad \forall s \in [0, \tilde{s}]. \tag{2.3}$$

Lemma 2.5. *Let* (H0)–(H2) *hold. Let* $s_0 \in (0,\infty)$ *be a constant and let* (λ, u) *be the nonnegative solution of*

$$\begin{gathered} \Delta^2 u = \lambda a(x) f(u) \quad x \in \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 \quad x \in \partial\Omega \end{gathered} \tag{2.4}$$

with $\max\{u(x) : x \in \bar{\Omega}\} = u(x_0) = s_0$. *Then*

$$\lambda \in (0, M_1]$$

for some positive constant $M_1 > 0$*, which is independent of* u *and* λ*.*

Proof. Assume on the contrary that there exists a sequence $\{(\mu_n, u_n)\}$ of positive solutions of (2.4) with

$$\|u_n\|_{C(\bar{\Omega})} = s_0, \quad \mu_n \to \infty \quad \text{as } n \to \infty. \tag{2.5}$$

Let $y_n := u_n / \|u_n\|_{C(\bar{\Omega})}$. Then

$$\begin{gathered} \Delta^2 y_n = \mu_n a(x) \frac{f(u_n(x))}{u_n(x)} y_n \quad x \in \Omega, \\ y_n = \frac{\partial y_n}{\partial \nu} = 0 \quad x \in \partial\Omega. \end{gathered} \tag{2.6}$$

Since (H0) and (H2) imply that $f(s)/s \ge \rho_0$ for $s \in (0, s_0]$ for some $\rho_0 > 0$, we let $\psi : \psi(x) > 0$ in Ω, be the eigenfunction corresponding $\lambda_1(a(\cdot))$, i.e.

$$\begin{gathered} \Delta^2 \psi = \lambda_1(a(\cdot)) a(x) \psi, \qquad \text{in } \Omega, \\ \psi = \frac{\partial \psi}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.7}$$

Multiplying the equation in (2.6) by ψ and multiplying the equation in (2.7) by y_n, integrating over Ω by parts and using that

$$\int_\Omega \psi \, \Delta^2 y_n dx = \int_\Omega \Delta y_n \Delta \psi \, dx, \tag{2.8}$$

we deduce from $\mu_n \to \infty$ that y_n must change its sign in Ω if n is large enough. However, this is a contradiction. □

Lemma 2.6. *Let* (H0)–(H2) *hold. Let* $s_0 \in (0,\infty)$ *be a constant and let* $\Lambda := \big[0, \max\{M_1, \lambda_1(a(\cdot))/f_0 + 1\}\big]$ *be a compact interval. Let* (λ, u) *be the nonnegative solution of*

$$\Delta^2 u = \lambda a(x) f(u) \quad x \in \Omega, \tag{2.9}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad x \in \partial\Omega, \tag{2.10}$$

with $\lambda \in \Lambda$ *and* $\max\{u(x) : x \in \bar{\Omega}\} = u(x_0) = s_0$. *Then*

$$x_0 \in \Omega_\delta := \{x \in \Omega : d(x, \partial\Omega) \geq \delta\} \tag{2.11}$$

for some positive constant $\delta = \delta(s_0)$, *which is independent of* $\lambda \in \Lambda$.

Proof. Assume on the contrary that there exists a sequence $\{(\mu_k, y_k)\}$ of nonnegative solutions of (2.9), (2.10) with $\mu_k \in \Lambda$, $\|y_k\|_{C(\bar{\Omega})} = s_0$ and

$$d(x_{0,k}, \partial\Omega) \to 0 \quad \text{as } k \to \infty,$$

where $y_k(x_{0,k}) = \max\{y_k(x) : x \in \bar{\Omega}\}$. Since $\{\mu_k a(\cdot) f(y_k(\cdot))\}$ is uniformly bounded in $C(\bar{\Omega})$, it follows that

$$\|\mu_k a(\cdot) f(y_k(\cdot))\|_{L^p(\Omega)} \leq M_2 \tag{2.12}$$

for some constant $M_2 > 0$.

By Agmon-Douglis-Nirenberg estimates in [1], for any $p > 1$,

$$\|u_k\|_{W^{4,p}(\Omega)} \leq C_p \|\mu_k a(\cdot) f(y_k(\cdot))\|_{L^p(\Omega)} \leq C_p M_2, \tag{2.13}$$

where C_p is a positive constant. By the embedding theorem [11, Theorem 2.6],

$$W^{4,p}(\Omega) \hookrightarrow C^{3,\alpha}(\bar{\Omega})$$

for all $p > \frac{2}{4-3} = 2$ and $\alpha \in (0, 1 - \frac{2}{p}] \cap (0, 1)$. Thus

$$\|u_k\|_{C^{3,\alpha}(\bar{\Omega})} \leq M_3 \tag{2.14}$$

for some constant $M_3 > 0$. Since $C^{3,\alpha}(\bar{\Omega}) \hookrightarrow\hookrightarrow C(\bar{\Omega})$ is a compact embedding, it follows that after taking a subsequence if necessary, y_k converges to $\hat{y}$ in $C(\bar{\Omega})$. Moreover,

$$\|\hat{y}\|_{C(\bar{\Omega})} = s_0. \tag{2.15}$$

Since $\bar{\Omega} \subset \mathbb{R}^2$ is bounded and closed, we may assume that $x_{0,k} \to x^*$, and consequently, $\hat{y}(x^*) = s_0$. On the other hand, $x^* \in \partial\Omega$, which together with the fact $y_n(x) = 0$ on $\partial\Omega$ imply $\hat{y}(x^*) = 0$. However, this contradicts (2.15). □

2.3. **Global solutions branches for positive mappings.** Suppose that E is a real Banach space with norm $\|\cdot\|$. Let K be a cone in E. A nonlinear mapping $A : [0, \infty) \times K \to E$ is said to be *positive* if $A([0, \infty) \times K) \subseteq K$. It is said to be K-*completely continuous* if A is continuous and maps bounded subsets of $[0, \infty) \times K$ to precompact subset of E. If L is a continuous linear operator on E, denote $r(L)$ the spectral radius of L. Define

$$c_K(L) = \{\lambda \in [0, \infty) : \text{there exists } x \in K \text{ with } \|x\| = 1 \text{ and } x = \lambda L x\}.$$

The following Lemma will play a very important role in the proof of our main results, which is essentially a consequence of Dancer [9, Theorem 2] .

Lemma 2.7. *Assume that*

(i) K *has nonempty interior and* $E = \overline{K - K}$;
(ii) $A : [0, \infty) \times K \to E$ *is* K-*completely continuous and positive,* $A(\lambda, 0) = 0$ *for* $\lambda \in \mathbb{R}$, $A(0, u) = 0$ *for* $u \in K$ *and*

$$A(\lambda, u) = \lambda L u + F(\lambda, u),$$

where $L : E \to E$ *is a strongly positive linear compact operator on* E *with* $r(L) > 0$, $F : [0, \infty) \times K \to E$ *satisfies* $\|F(\lambda, u)\| = \circ(\|u\|)$ *as* $\|u\| \to 0$ *locally uniformly in* λ.

Then there exists an unbounded connected subset $\mathcal{C}$ of

$$\mathcal{D}_K(A) = \{(\lambda, u) \in [0, \infty) \times K : u = A(\lambda, u),\ u \neq 0\} \cup \{(r(L)^{-1}, 0)\}$$

such that $(r(L)^{-1}, 0) \in \mathcal{C}$.

3. Main results

Let $\tilde{s}$ be a positive constant. In the rest of this paper we will take δ to be the constant in Lemma 2.6 with $\Lambda = [0, \max\{M_1, \lambda_1(a(\cdot))/f_0 + 1\}]$. To study the multiplicity of positive solutions of (2.9),(2.10), we need the following assumption

(H4)

$$\min_{\frac{\tilde{s}}{C} \leq s \leq \tilde{s}} \frac{f(s)}{s} > \frac{C f_0}{\lambda_1(a(\cdot)) \min_{\Omega_{\delta/2}} G_{2,2,\Omega}(x, y) a_0 |B_{\delta/2}|}, \tag{3.1}$$

where $a_0 = \min_{\bar{\Omega}} a(\cdot)$, $|B_{\delta/2}| = \operatorname{meas} B_{\delta/2}$,

$$\Omega_r := \{x \in \Omega : \ d(x, \partial\Omega) > r\}, \quad B_r := \{x \in B : \ d(x, \partial B) > r\},$$

and C is the constant satisfying

$$\frac{1}{C}(d(x))^2 G_{2,2,B}(0, y) \leq G_{2,2,B}(x, y) \leq C G_{2,2,B}(0, y) \quad x, y \in B, \tag{3.2}$$

where $d(x) = d(x, \partial\Omega)$, $G_{2,2,B}$ is the Green function of Δ^2 for the Dirichlet problem in B, see Mâagli, Toumi and Zribi [20, P.3] for the details.

Using a similar idea to show the existence of three positive solutions of one-dimensional p-Laplacian problem and arguing the shape of bifurcation as in Sim and Tanaka [23], we have the following results for

$$\Delta^2 u = \lambda a(x) f(u) \quad \text{in } \Omega, \tag{3.3}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega. \tag{3.4}$$

Theorem 3.1. *Let ϵ_0 be the constant in Lemma 2.2. Let $\Omega \subset \mathbb{R}^2$ is a bounded domain of class $C^{4,\alpha}(\bar{\Omega})$ which is ϵ_0-close in $C^{4,\alpha}$-sense to B. Let* (H0)–(H4) *hold. Then there exist $\lambda_* \in (0, \lambda_1(a(\cdot))/f_0)$ and $\lambda^* \in (\lambda_1(a(\cdot))/f_0, \infty)$ such that*

- (i) (3.3), (3.4) *has at least one positive solution if $\lambda = \lambda_*$;*
- (ii) (3.3),(3.4) *has at least two positive solutions if $\lambda_* < \lambda \leq \lambda_1(a(\cdot))/f_0$;*
- (iii) (3.3), (3.4) *has at least three positive solutions if $\lambda_1(a(\cdot))/f_0 < \lambda < \lambda^*$;*
- (iv) (3.3), (3.4) *has at least two positive solutions if $\lambda = \lambda^*$;*
- (v) (3.3), (3.4) *has at least one positive solution if $\lambda > \lambda^*$.*

See illustrations in Figure 1.

Remark 3.2. From Grunau and Sweers [14, 15], the Green function in (3.2) is

$$G_{2,2,B}(x, y) = k_{2,2}|x - y|^2 \int_1^{\left||x|y - \frac{x}{|x|}\right|/|x-y|} (v^2 - 1)v^{-1} dv, \quad x, y \in B, \tag{3.5}$$

and satisfies

$$G_{2,2,B}(x, y) \sim d(x)d(y) \min\big\{1, \frac{d(x)d(y)}{|x - y|^2}\big\}, \tag{3.6}$$

where $k_{2,2}$ is a known constant. By combining (3.5), (3.6) and doing numerical calculation, the exact value of C in (H4) can be obtained, denoted as $C^\diamond$.

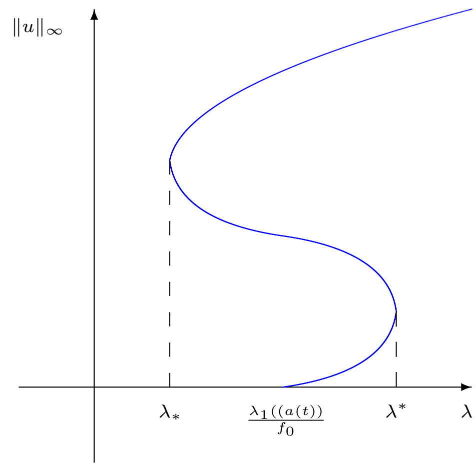

FIGURE 1. Connected component of the solution set of (3.3), (3.4)

Remark 3.3. For the general case $\Omega \neq B$, we may transform (3.3), (3.4) into a new problem in B using the holomorphic mapping from Ω to B, see Grunau and Sweers [15]. By (3.5) and some simple computations, we may obtain a constant $C^* > 0$ such that the Green function $G_{2,2,\Omega}(x,y)$ of (3.3), (3.4) and $G_{2,2,B}(x,y)$ satisfy

$$\frac{1}{C^*} G_{2,2,B}(x,y) \le G_{2,2,\Omega}(x,y) \le C^* G_{2,2,B}(x,y).$$

Remark 3.4. We may provide an example to illustrate the application of Theorem 3.1 in the case $\Omega = B$. Take

$$K = \max\Big\{\frac{1}{2}, \frac{C^\diamond}{\lambda_1(1)\,\tilde{G}_{\delta/2}\,|B_{\delta/2}|}\Big\} + 1$$

and $\tilde{G}_{\delta/2} := \min_{B_{\delta/2}} G_{2,2,B}(x,y)$. Let us consider the boundary value problem

$$\begin{gathered} \Delta^2 u = \hat{f}(u), \quad \text{in } B, \\ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B, \end{gathered} \tag{3.7}$$

with

$$\hat{f}(s) = \begin{cases} s - s^2, & \text{if } s \in [0, 1/2), \\ (2K - \frac{1}{2})s - K + \frac{1}{2}, & \text{if } s \in [1/2, 1), \\ Ks^2, & \text{if } s \in [1, C^\diamond], \\ K(C^\diamond)^{3/2}\sqrt{s}, & \text{if } s \in (C^\diamond, \infty). \end{cases}$$

Obviously, $\hat{f}$ is a continuous, non-decreasing function with $f(0) \ge 0$, from [11, Theorem 7.1] the solution u of (3.7) is radially symmetric. So, we may take $\delta = 1/4$.

Obviously, $\hat{f}$ satisfies (H2) and (H3) with $\beta = 1, f_1 = 1, f_0 = 1$; (H4) with $\tilde{s} = C^\diamond$ is satisfied since

$$\min_{\frac{\tilde{s}}{C^\diamond} \le s \le \tilde{s}} \frac{f(s)}{s} = \min_{1 \le s \le C^\diamond} Ks > K > \frac{C^\diamond}{\lambda_1(1)\tilde{G}_{1/8}\,|B_{1/8}|}.$$

Thus, we are in the position to use Theorem 3.1.

4. Bounds of solutions

4.1. A priori estimation. Let

$$X=\big\{u\in C^{2,\alpha}(\bar\Omega): u \text{ satisfies (3.4), and there exists } \gamma\in(0,\infty) \text{ such that } -\gamma\psi(x)\le u(x)\le\gamma\psi(x),\ x\in\Omega\big\}. \tag{4.1}$$

Then X is a Banach space under the norm

$$\|u\|_X:=\inf\{\gamma: -\gamma\psi(x)\le u(x)\le\gamma\psi(x) \text{ for } x\in\Omega\}.$$

Let

$$P:=\{u\in X: u(x)\ge 0,\ x\in\Omega\}. \tag{4.2}$$

Then P is normal, has a nonempty interior, and $X=\overline{P-P}$.

Lemma 4.1. *Let Ω be as in Theorem 3.1. Let* (H0)–(H3) *hold. Let $J:=[a_1,b_1]\subset[0,\infty)$. Assume that $\{(\mu_n,y_n)\}$ be a sequence of solutions of* (3.3),(3.4) *with*

$$\mu_n\in J,\quad \|y_n\|_{C(\bar\Omega)}\le M \tag{4.3}$$

for some constant M, independent of n. Then $y_n\in C^4(\bar\Omega)\cap X$ and $\{y_n\}$ is bounded in X.

Proof. It follows from (2.3),

$$\Delta^2 y_n=\mu_n a(x)f(y_n)\quad \text{in } \Omega,$$
$$y_n=\frac{\partial y_n}{\partial\nu}=0\quad \text{on } \partial\Omega,$$

and Grunau and Sweers [14, P.620], that for any $p>1$,

$$\|y_n\|_{W_0^{4,p}(\Omega)}\le M_4$$

for some positive constant M_4, independent of n. Thus, the Sobolev imbedding theorem [12, Corollary 7.1] guarantees that

$$\|y_n\|_{C^3(\bar\Omega)}\le M_5,$$

and consequently, $\|y_n\|_{C^{0,\alpha}(\bar\Omega)}\le M_6$ for some positive constant M_6, independent of n. Thus

$$\|\mu_n a f(y_n)\|_{C^{0,\alpha}(\bar\Omega)}\le M_7$$

for some positive constant M_7, independent of n. Combining this with (3.3), (3.4) and using [7, Lemma 3.1], it follows that

$$\|y_n\|_{C^{4,\alpha}(\bar\Omega)}\le M_8$$

for some positive constant M_8, independent of n. Therefore,

$$|y_n(x)|\le C_8\psi(x)\quad x\in\Omega$$

for some positive constant C_8, independent of n. Therefore, $\|y_n\|_X\le M_9$ for some positive constant M_9, independent of n. □

Let $h : B \to \Omega$ be a bijection such that

$$h(x_1 + ix_2) = h_1(x_1, x_2) + ih_2(x_1, x_2)$$

is a holomorphic mapping. Then $\Delta(u \circ h) = \frac{1}{2}|\nabla h|^2(\Delta u) \circ h$. We write

$$g(x) = 2|(\nabla h)(x)|^{-2}. \tag{4.4}$$

If $\partial\Omega$ is sufficiently smooth, then a Theorem of Kellogg-Warschawski (see [21]) implies that h is sufficiently smooth and that there exist $c_i > 0$ such that $c_1 \leq |(\nabla h)(x)|^{-2} \leq c_2$. The problem (3.3), (3.4) can be transformed into

$$(g(\cdot)\Delta)^2(u \circ h) = (\lambda a(\cdot)f(u) \circ h) \quad \text{in } B, \tag{4.5}$$

$$(u \circ h) = \frac{\partial(u \circ h)}{\partial\nu} = 0 \quad \text{on } \partial B, \tag{4.6}$$

which can also be written as

$$\big((-\Delta)^2 + \mathcal{A}\big)(u \circ h) = g^{-2}\big((\lambda a(\cdot)f(u)) \circ h\big) \quad \text{in } B, \tag{4.7}$$

$$(u \circ h) = \frac{\partial(u \circ h)}{\partial\nu} = 0 \quad \text{on } \partial B, \tag{4.8}$$

where for some $\mathcal{A}$ of the form

$$\mathcal{A} = \sum_{|\alpha|<4} a_\alpha(x)D^\alpha, \quad a_\alpha \in C(\bar{B}). \tag{4.9}$$

And Ω is close to the disk B means that $\|h - Id\|_{C^3(\bar{B})}$ sufficiently small. For example this holds for an ellipse that is close to a circle, see Grunau and Sweers[13].

Lemma 4.2. *Let Ω be as in Theorem 3.1 and $N = 2$. Let $I \subset (0, \infty)$ be a compact interval. Assume that* (H0)–(H3) *hold. Then there exists $M_{10} > 0$, such that for any positive solutions of* (3.3), (3.4) *with $\lambda \in I$, we have*

$$\|u\|_{C(\bar{\Omega})} \leq M_{10}. \tag{4.10}$$

Proof. Suppose on the contrary that there exists a sequence $\{(\mu_n, u_n)\}$ of positive solutions of (3.3), (3.4), such that

$$\mu_n \in I, \quad \|u_n\|_{C(\bar{\Omega})} \to \infty. \tag{4.11}$$

This together with the fact $h : B \to \Omega$ is a bijection and $\|h - Id\|_{C^3(\bar{B})}$ is sufficiently small that

$$\|u_n \circ h\|_{C(\bar{B})} \to \infty. \tag{4.12}$$

By Mâagli, Toumi and Zribi [20, P.3], $N = m = 2$ implies

$$\frac{1}{C}(d(x))^2 G_{2,2,B}(0, y) \leq G_{2,2,B}(x, y) \leq CG_{2,2,B}(0, y) \quad x, y \in B, \tag{4.13}$$

where $d(x) := \text{dist}(x, \partial B) > 0$ in B. From this and (4.11), (4.12), it follows that for $x \in B$,

$$\begin{aligned}
(u_n \circ h)(x) &= \lambda \int_B G_{2,2,B}(x,y) a f((u_n \circ h)(y)) dy \\
&\geq \lambda \int_B \frac{1}{C}(d(x))^2 G_{2,2,B}(0,y) a f((u_n \circ h)(y)) dy \\
&\geq \lambda \int_B \frac{1}{C}(d(x))^2 \frac{1}{C} G_{2,2,B}(x_u,y) a f((u_n \circ h)(y)) dy \\
&= \big(\frac{1}{C}\big)^2 (d(x))^2 \int_B \lambda G_{2,2,B}(x_u,y) a f((u_n \circ h)(y)) dy \\
&= \frac{1}{C^2}(d(x))^2 \|u_n \circ h\|_{C(\bar{B})},
\end{aligned} \tag{4.14}$$

where $(u \circ h)(x_u) = \|u \circ h\|_{C(\bar{\Omega})}$. Thus, for any $\sigma > 0$,

$$\lim_{n\to\infty} (u_n \circ h)(x) = \infty \quad \text{uniformly for } x \in \Omega_\sigma. \tag{4.15}$$

Let

$$y_n := \frac{u_n \circ h}{\|u_n \circ h\|_{C(\bar{B})}}.$$

Then by (4.11), (4.12) and standard compact argument, we deduce that after taking a subsequence if necessary, $y_n \to y^*$ for some y^* with $\|y^*\|_{C(\bar{B})} = 1$.

On the other hand, combining (4.11), (4.12), and using $f_\infty = 0$, $I \subset [0, \infty)$, and (4.15), it follows that $\|y^*\|_{C(\bar{B})} = 0$. However, this is a contradiction. □

Using a similar argument for (4.14), we obtain the following Harnack type inequalities.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^2$ be as in Theorem 3.1. Let β_1 and $\beta_2 \in (0,\infty)$ be two positive constants. Let $V \in C(\bar{\Omega})$ with*

$$\beta_1 \leq V(x) \leq \beta_2 \quad x \in \Omega.$$

If u is a nonnegative weak solution of

$$\begin{gathered}
\Delta^2 u = V(x) u \quad x \in \Omega, \\
u = \frac{\partial u}{\partial \nu} = 0 \quad x \in \partial\Omega,
\end{gathered}$$

then for any $\sigma > 0$, there exists $C = C(\beta_1, \beta_2)$ such that we have

$$\sup_{\bar{\Omega}} u \leq C \inf_{\Omega_\sigma} u,$$

where C is independent of u and $V \in \{w \in Y : \beta_1 \leq w(x) \leq \beta_2 \text{ for } x \in \Omega\}$.

5. Rightward bifurcation

Define $L : D(L) \to Y$ by

$$Lu := \Delta^2 u,$$

on the domain

$$D(L) = \{u \in C^{2,\alpha}(\bar{\Omega}) \cap C^4(\Omega) : u \text{ satisfies } (3.4)\}.$$

It is easy to check that $L^{-1} : Y \to Y$ is compact.

It follows from Dalmasso [7, Theorem 2.3] that if for any $z \in Y$ with $z \geq 0$ and $z(x_0) > 0$ for some $x_0 \in \bar{\Omega}$ with

$$Lu - z = 0. \tag{5.1}$$

Then $u \in \operatorname{int} P$.

Let $\zeta, \xi \in C([0,\infty))$ be such that

$$f(u) = f_0 u + \zeta(u),$$
$$f(u) = f_\infty u + \xi(u)$$

with

$$\lim_{u\to 0} \frac{\zeta(u)}{u} = 0, \quad \lim_{u\to\infty} \frac{\xi(u)}{u} = 0.$$

Let

$$\tilde{\xi}(r) = \max\{|\xi(u)| : 0 \leq u \leq r\}. \tag{5.2}$$

Then $\tilde{\xi}$ is nondecreasing and

$$\lim_{r\to\infty} \frac{\tilde{\xi}(r)}{r} = 0. \tag{5.3}$$

Let us consider

$$Lu(x) = \lambda f_0 a(x) u(x) + \lambda a(x) \zeta(u(x)), \quad x \in \bar{\Omega} \tag{5.4}$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

Combining this with Lemma 2.7, we can conclude that there exists an unbounded connected subset $\mathcal{C}$ of the set

$$\{(\lambda, u) \in (0,\infty) \times P : (\lambda, u) \text{ satisfies (5.4)},\ u \in \operatorname{int} P\} \cup \{(\lambda_1(a(\cdot))/f_0, 0)\}$$

such that $(\lambda_1(a(\cdot))/f_0, 0) \in \mathcal{C}$.

By the method used by Sim and Tanaka to prove [23, Lemma 2.3], with obvious changes, we obtain the following result.

Lemma 5.1. *Let Ω be as in Theorem 3.1. Let* (H0)–(H2) *hold. Let $\{(\eta_j, u_j)\}$ be a sequence of positive solutions to* (3.3), (3.4) *which satisfies $\|u_j\|_{C(\bar{\Omega})} \to 0$ and $\eta_j \to \lambda_1(a(\cdot))/f_0$. Let ψ be the eigenfunction corresponding to $\lambda_1(a(\cdot))$, which satisfies $\|\psi\|_{C(\bar{\Omega})} = 1$. Then there exists a subsequence of $\{u_j\}$, again denoted by $\{u_j\}$, such that $u_j/\|u_j\|_{C(\bar{\Omega})}$ converges uniformly to ψ on $\bar{\Omega}$.*

Lemma 5.2. *Let Ω be as in Theorem 3.1. Let* (H0)–(H2) *hold. Let $\mathcal{C}$ be as in Lemma 2.7. Then there exists $\hat{\delta} > 0$ such that $(\lambda, u) \in \mathcal{C}$ and $|\lambda - \lambda_1(a(\cdot))/f_0| + \|u\|_{C(\bar{\Omega})} \leq \hat{\delta}$ imply $\lambda > \lambda_1(a(\cdot))/f_0$.*

Proof. Assume on the contrary that there exists a sequence $\{(\eta_j, u_j)\}$ such that $(\eta_j, u_j) \in \mathcal{C}$, $\eta_j \to \lambda_1(a(\cdot))/f_0$, $\|u_j\|_{C(\bar{\Omega})} \to 0$ and $\eta_j \leq \lambda_1(a(\cdot))/f_0$. By the standard argument, we may get that there exists a subsequence of $\{u_j\}$, again denoted by $\{u_j\}$, such that $u_j/\|u_j\|_{C(\bar{\Omega})}$ converges uniformly to ψ on $\bar{\Omega}$, where $\psi > 0$ is the first eigenfunction of (2.2) which satisfies $\|\psi\|_{C(\bar{\Omega})} = 1$. Multiplying (3.3) with $(\lambda, u) = (\eta_j, u_j)$ by u_j and integrating it over Ω, we obtain

$$\eta_j \int_\Omega a(x) f(u_j(x)) u_j(x) dx = \int_\Omega (\Delta u_j(x))^2 dx.$$

Using the definition of $\lambda_1(a(\cdot))$, we obtain

$$\eta_j \int_\Omega a(x) f(u_j(x)) u_j(x) dx \geq \lambda_1(a(\cdot)) \int_\Omega a(x)(u_j(x))^2 dx.$$

It is easy to see that

$$\begin{aligned}&\int_\Omega a(x)\frac{f(u_j(x))-f_0u_j(x)}{|u_j(x)|^{1+\beta}}\Big|\frac{u_j(x)}{\|u_j\|_{C(\bar\Omega)}}\Big|^{2+\beta}dx\\ &\geq\frac{\lambda_1(a(\cdot))-f_0\eta_j}{\eta_j\|u_j\|^\beta_{C(\bar\Omega)}}\int_\Omega a(x)\Big|\frac{u_j(x)}{\|u_j\|_{C(\bar\Omega)}}\Big|^2dx.\end{aligned}$$

Lebesgue's dominated convergence theorem and (H2) imply that

$$\int_\Omega a(x)\frac{f(u_j(x))-f_0\,u_j(x)}{|u_j(x)|^{1+\beta}}\Big|\frac{u_j(x)}{\|u_j\|_{C(\bar\Omega)}}\Big|^{2+\beta}dx\to -f_1\int_\Omega a(x)|\psi(x)|^{2+\beta}dx<0$$

and

$$\int_\Omega a(x)\Big|\frac{u_j(x)}{\|u_j\|_{C(\bar\Omega)}}\Big|^2dx\to\int_\Omega a(x)|\psi(x)|^2\,dx>0.$$

This contradicts $\eta_j\leq\lambda_1(a(\cdot))/f_0$. □

6. Direction turn of bifurcation

In this section, we show that there is a direction turn of the bifurcation under assumptions (H3) and (H4).

Lemma 6.1. *Let Ω be as in Theorem 3.1. Let* (H0)–(H3) *hold. Let $u\in C^4(\bar\Omega)$ be the positive solution of* (3.3), (3.4) *with $u(x_0)=\|u\|_{C(\bar\Omega)}=s_0$ for some $s_0>0$, and $\lambda\in[0,\max\{M_1,\lambda_1(a(\cdot))/f_0+1\}]$. Then*

$$\frac{1}{C}\|u\|_{C(\bar B_{\delta/2}(x_0))}\leq u(x)\leq\|u\|_{C(\bar B_{\delta/2}(x_0))},\quad x\in B_{\delta/2}(x_0)\tag{6.1}$$

where C is the constant in (3.2).

Proof. Lemma 2.6 yields $x_0\in\Omega_\delta$. Thus the desired results is an immediate consequence of (4.13). □

Lemma 6.2. *Let Ω be as in Theorem 3.1. Assume that* (H0)–(H4) *hold. Let u be a positive solution of* (3.3),(3.4) *with $\|u\|_{C(\bar\Omega)}=s_0$. Then*

$$\lambda<\lambda_1(a(\cdot))/f_0,\quad or\quad \lambda>\lambda_1(a(\cdot))/f_0+1.$$

Proof. Let u be a positive solution of (3.3), (3.4). Then from Lemma 6.1 we have

$$\frac{1}{C}s_0\leq u(x)\leq s_0,\quad x\in B_{\delta/2}(x^*),$$

where $u(x^*)=\|u\|_{C(\bar\Omega)}$.

Assume on the contrary that $\lambda\geq\lambda_1(a(\cdot))/f_0$. Then from Lemma 2.6 and (H4), it follows that

$$\begin{aligned}s_0&=u(x^*)\\ &=\lambda\int_\Omega G_{2,2,\Omega}(x^*,y)a(y)f(u(y))dy\\ &\geq\lambda\int_{\Omega_{\delta/2}}G_{2,2,\Omega}(x^*,y)a(y)f(u(y))dy\\ &\geq\lambda\int_{B_{\delta/2}(x^*)}G_{2,2,\Omega}(x^*,y)a(y)f(u(y))dy\end{aligned}$$

$$
\begin{aligned}
&\geq \lambda \int_{B_{\delta/2}(x^*)} G_{2,2,\Omega}(x^*,y)a(y)\frac{f(u(y))}{u(y)}(u(y))dy \\
&\geq \frac{\lambda_1(a(\cdot))}{f_0} \min_{\Omega_{\delta/2}} G_{2,2,\Omega}(x,y)a_0 \operatorname{meas} B_{\delta/2} \min_{\frac{s_0}{C}\leq s\leq s_0} \frac{f(s)}{s}\frac{s_0}{C} \\
&> s_0.
\end{aligned}
$$

This is a contradiction. Therefore, $\lambda < \frac{\lambda_1(a(\cdot))}{f_0}$. □

7. Second turn and proof of Theorem 3.1

In this section, we give a block for a parameter and a priori estimate and finally a proof of Theorem 3.1.

Lemma 7.1. *Let Ω be as in Theorem 3.1. Assume that* (H0)—(H4) *hold. Let (λ, u) be a positive solution of* (3.3),(3.4)*. Then there exists $C_1 > 0$ independent of u such that $\lambda \underline{f}(\|u\|_{C(\bar{\Omega})}) < C_1$, where*

$$
\underline{f}(s) := \min_{\frac{s}{C}\leq t\leq s} f(t)/t. \tag{7.1}
$$

Proof. Let $u(x_u) = \|u\|_{C(\bar{\Omega})}$. Then

$$
\begin{aligned}
u(x_u) &= \lambda \int_\Omega G_{2,2,\Omega}(x_u,y)a(y)f(u(y))dy \\
&\geq \lambda \int_{B_\delta(x_u)} G_{2,2,\Omega}(x_u,y)a(y)f(u(y))dy \\
&\geq \lambda \min_{\Omega_{\delta/2}} G_{2,2,\Omega}(x,y)|B_\delta| a_0 \underline{f}(\|u\|_{C(\bar{\Omega})})\frac{1}{C}\|u\|_{C(\bar{\Omega})},
\end{aligned}
$$

which implies $\lambda \underline{f}(\|u\|_{C(\bar{\Omega})}) < C_1$ for some $C_1 > 0$. □

Proof of Theorem 3.1. By Lemma 5.2, $\mathcal{C}$ is bifurcating from $(\lambda_1(a(\cdot))/f_0, 0)$ and goes rightward.

We claim that there exists a sequence $\{(\beta_j, u_j)\} \subset \mathcal{C}$ satisfying

$$
\beta_j \to +\infty, \quad \|u_j\|_{C(\bar{\Omega})} \to \infty. \tag{7.2}
$$

Assume on the contrary that there exists $\beta^* > 0$, such that

$$
\|u\|_{C(\bar{\Omega})} \leq M_{11} \quad \text{for all } (\lambda, u) \in \mathcal{C} \text{ with } \lambda > \beta^*. \tag{7.3}
$$

Then $0 \leq \|u\|_{C(\bar{\Omega})} \leq M_{11}$ implies $\underline{f}(\|u\|_{C(\bar{\Omega})}) \geq \delta_0$ for some constant $\delta_0 > 0$, and consequently

$$
\lambda \underline{f}(\|u\|_{C(\bar{\Omega})}) \to \infty \quad \text{as } \lambda \to \infty. \tag{7.4}
$$

However, this contradicts Lemma 7.1. Therefore, (7.2) holds.

Thus, there exists $(\beta_0, u_0) \in \mathcal{C}$ such that $\|u_0\|_{C(\bar{\Omega})} = s_0$. Lemma 6.2 implies that $\beta_0 < \lambda_1(a(\cdot))/f_0$. By Lemmas 5.2, 6.2 and 4.3, $\mathcal{C}$ passes through some points $(\lambda_1(a(\cdot))/f_0, v_1)$ and $(\lambda_1(a(\cdot))/f_0, v_2)$ with $\|v_1\|_{C(\bar{\Omega})} < s_0 < \|v_2\|_{C(\bar{\Omega})}$. By Lemmas 5.2 and 6.2 and the fact $\mathcal{C} \cap (\{0\} \times P) = \{(0,0)\}$, there exist $\bar{\lambda}$ and $\underline{\lambda}$ which satisfy $0 < \underline{\lambda} < \lambda_1(a(\cdot))/f_0 < \bar{\lambda}$ and both (i) and (ii):

(i) if $\lambda \in (\lambda_1(a(\cdot))/f_0, \bar{\lambda}]$, then there exists u and v such that (λ, u), $(\lambda, v) \in \mathcal{C}$ and $\|u\|_{C(\bar{\Omega})} < \|v\|_{C(\bar{\Omega})} < s_0$;

(ii) if $\lambda \in (\underline{\lambda}, \lambda_1(a(\cdot))/f_0]$, then there exists u and v such that (λ, u), $(\lambda, v) \in \mathcal{C}$ and $\|u\|_{C(\bar{\Omega})} < s_0 < \|v\|_{C(\bar{\Omega})}$.

Define $\lambda^* = \sup\{\bar{\lambda} : \bar{\lambda}$ satisfies (i)$\}$ and $\lambda_* = \inf\{\underline{\lambda} : \underline{\lambda}$ satisfies (ii)$\}$. Then by the standard argument, (3.3), (3.4) has a positive solution at $\lambda = \lambda_*$ and $\lambda = \lambda^*$, respectively. Since $\mathcal{C}$ passes through $(\lambda_1(a(\cdot))/f_0, v_2)$ and (β_j, u_j), Lemma 6.2 and 2.7 imply that, for each $\lambda > \lambda_1(a(\cdot))/f_0$, there exists w such that $(\lambda, w) \in \mathcal{C}$ and $\|w\|_{C(\bar{\Omega})} > s_0$. This completes the proof. □

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