INFINITELY MANY RADIAL SOLUTIONS FOR A p-LAPLACIAN PROBLEM WITH NEGATIVE WEIGHT AT THE ORIGIN

ALFONSO CASTRO, JORGE COSSIO, SIGIFREDO HERRÓN, CARLOS VÉLEZ

ABSTRACT. We prove the existence of infinitely many sign-changing radial solutions for a Dirichlet problem in a ball defined by the p-Laplacian operator perturbed by a nonlinearity of the form $W(|x|)g(u)$, where the weight function W changes sign exactly once, $W(0) < 0$, $W(1) > 0$, and function g is p-superlinear at infinity. Standard phase plane analysis arguments do not apply here because the solutions to the corresponding initial value problem may blow up in the region where the weight function is negative. Our result extend those in [2], where W is assumed to be positive at 0 and negative at 1.

1. INTRODUCTION

We study the quasilinear Dirichlet problem

$$
\begin{aligned}
\Delta_p\, u + W(|x|)g(u) = 0 &\quad \text{in } B_1(0) \subset \mathbb{R}^N, \\
u = 0 &\quad \text{on } \partial B_1(0),
\end{aligned}
\tag{1.1}
$$

where $N \geq 2$, $p > 1$, $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator, and $B_1(0)$ denotes the unit ball in $\mathbb{R}^N$ centered at the origin.

We assume that g is a non-decreasing function which is also locally Lipschitz continuous and satisfies

$$
sg(s) > 0 \text{ for } s \neq 0, \quad \lim_{|s|\to\infty} \frac{g(s)}{s|s|^{p-2}} = \infty.
\tag{1.2}
$$

We also assume that there exists $C > 0$ such that

$$
|g(s)| \leqslant C\,|s|^{p-1}, \quad \text{for all } s \in [-1, 1].
\tag{1.3}
$$

Note that the inequality in (1.2) implies $g(0) = 0$ and $G(s) := \int_0^s g(t)dt > 0$ for all $s \in \mathbb{R} \smallsetminus \{0\}$. The function W is assumed to be of class $C^1[0, 1]$ and such that there exists $0 < X < 1$ with

$$
\begin{aligned}
W(s) < 0 &\quad \text{for } s \in [0, X), \\
W(X) = 0, \quad W'(X) > 0, \quad W(s) > 0 &\quad \text{for } s \in (X, 1].
\end{aligned}
\tag{1.4}
$$

The solutions to

$$\big(|u'|^{p-2}\, u'\big)'(r) + \frac{N-1}{r}|u'(r)|^{p-2}u'(r) + W(r)g\big(u(r)\big) = 0, \quad 0 < r < 1, \quad (1.5)$$

subject to

$$u'(0) = 0, \quad u(1) = 0, \tag{1.6}$$

give the radial solutions to (1.1) in the sense of distributions. More exactly, if $v : \overline{B_1(0)} \to \mathbb{R}$ is a radial function and the function $u : [0,1] \to \mathbb{R}$ defined by $u(\sqrt{x_1^2 + \cdots + x_N^2}) := v(x_1, \ldots, x_N)$ satisfies (1.5)-(1.6), then v is a solution to (1.1) in the sense of distributions, see Theorem 2.10 below. Because of the singularity given by the zeros of u' the solutions to (1.1) need not be of class C^2. In fact, regularity theory for quasilinear problems indicates that the distributional solutions to (1.1) may only be expected to be in the Holder space $C^{1,\mu}$ for some $\mu \in (0,1)$, see [9, 14]. Thus, from now on, *solution* stands for *distributional solution*. Our main result is the following theorem.

Theorem 1.1. *If $p \leqslant N$, (1.2), (1.3) and (1.4) hold, then for each odd positive integer k problem (1.1) has a radial solution with k zeros. In particular, (1.1) has infinitely many radial solutions.*

For $p = 2$, W a positive constant, and g satisfying a growth condition such as

$$\text{(subcritical):} \quad \lim_{|u| \to +\infty} \frac{g(u))}{|u|^{q-1}u} \in (0,\infty) \text{ with } q \in \big(1, \frac{N+2}{N-2}\big), \quad \text{or}$$

$$\text{(subcritical on } (0,+\infty)) : \quad \lim_{u \to +\infty} \frac{g(u))}{|u|^{q-1}u} \in (0,\infty) \text{ with } q \in \big(1, \frac{N}{N-2}\big), \tag{1.7}$$

Theorem 1.1 was proven in [4]. This result was extended to all $p > 1$ in [10]. In [5] the existence of infinitely many solutions with W constant and $p = 2$ was extended to sub-supercritical nonlinearities; that is, nonlinearities satisfying

$$\text{(subcritical in } (0,+\infty)) : \quad \lim_{u \to +\infty} \frac{g(u)}{|u|^{q_1-1}u} \in (0,\infty) \text{ with } q_1 \in \big(1, \frac{N+2}{N-2}\big), \quad \text{or}$$

$$\text{(supercritical in } (-\infty,0)) : \quad \lim_{u \to +\infty} \frac{g(u))}{|u|^{q_2-1}u} \in (0,\infty) \text{ with } q_2 \in \big(\frac{N+2}{N-2}, +\infty\big). \tag{1.8}$$

In turn, the results in [5] were extended in [3] to all $p > 1$. The approach in [4, 10, 5, 3] combines the continuous dependence of solutions to (1.5) with initial value at $r = 0$ with phase plane analysis in order to apply the intermediate value theorem. More recently, in [2], the authors considered the case where W is positive at zero, negative at 1 and changes sign only once. In this case, the approach in [4, 10, 5, 3] fails because the solutions to (1.5) with initial value at $r = 0$ may blow up in $(X, 1]$. Such a difficulty was overcome in [2] by figuring out a subclass of initial conditions at X such that the solutions to the corresponding initial value problems do not blow up and depend continuously on those initial conditions.

Our assumption $W < 0$ in $[0, X)$ causes some solutions to (1.5) with initial condition at $r = 0$ to blow up in that interval. We bypass this difficulty by figuring out initial conditions at X for which the solutions to initial value problems are defined in $[0, X]$, depend continuously on such initial conditions, and yield solutions to (1.1). In establishing that such solutions exist the additional assumption $p \leq N$ is needed due to the singularity at $r = 0$, see Lemma 2.7. On the other hand, since

the coefficient $(N - 1)/r$ is bounded away from zero in $[X, 1]$, shooting from $r = 1$ towards X allows for g not to have growth restrictions of the type (1.7) or (1.8).

For examples of applications to problems with indefinite weight the reader is referred to [11]. For recent results on quasilinear problems with weight see [1, 2, 7, 13, 16]. For related results on the existence of infinitely many radial solutions to quasilinear problems see [3, 8, 12].

This article is organized as follows: in Section 2 we show that all solutions to (2.1) below are defined in $[X, 1]$ and figure out the initial conditions at $r = X$ for which the solutions to (1.5) do not blow up in $[0, X]$. More precisely, we find initial conditions (a, η_a), $a > 0$, for the initial value problem (2.9) such that $u \equiv u_{a, \eta(a)}$ is defined on $[0, X]$, positive, increasing, and $u'(0) = 0$ (see Lemmas 2.6 and 2.7 below). In Section 3 we prove that, for any $T \in (X, 1)$, if $u(r, d)$ is the solution to (2.1) below then $\lim_{d \to -\infty} (u^2(r, d) + (u'(r, d))^2) = +\infty$ uniformly for $r \in [T, 1]$. We also present in Section 3 the phase plane analysis of the solutions to (2.1) in $[X, 1]$. The arguments in Section 3 may be traced back to work by Professor Alan C. Lazer and one the authors in [6]. In Section 4 we prove the main result by connecting at X solutions to the regular initial value problem at $r = 1$ with those that do not blow up in $[0, X]$. Namely, we prove the existence of infinitely many values of d such that the solution to (2.1) satisfies $u(X) = a, u'(X) = \eta_a$, for some $a > 0$. Hence, by Theorem 2.10, they give infinitely many solutions to (1.1).

2. INITIAL VALUE PROBLEM AND PRELIMINARIES

We consider the initial value problem

$$\left(r^{N-1}|u'|^{p-2}u'\right)' + r^{N-1}W(r)g(u(r)) = 0, \quad 0 < r < 1,$$
$$u(1) = 0, \quad u'(1) = d. \tag{2.1}$$

Let $\Phi_p(x) = x|x|^{p-2}$ for $x \in \mathbb{R}$. We observe that, for each $d \in \mathbb{R}$, a continuous function u satisfies the integral equation

$$u(r) = -\int_r^1 \Phi_p^{-1}\left(\Phi_p(d)t^{1-N} + \int_t^1 \left(\frac{s}{t}\right)^{N-1} W(s)g(u(s))ds\right)dt \tag{2.2}$$

if and only if it is a solution to (2.1). In general, for any $r_0 \in (0, 1], a \in \mathbb{R}, b \in \mathbb{R}$, a continuous function u satisfies

$$u(r) = a - \int_r^{r_0} \Phi_p^{-1}\left(\left(\frac{r_0}{t}\right)^{N-1}\Phi_p(b) + \int_t^{r_0} \left(\frac{s}{t}\right)^{N-1} W(s)g(u(s))ds\right)dt, \tag{2.3}$$

if and only if it satisfies

$$\left(r^{N-1}|u'|^{p-2}u'\right)' + r^{N-1}W(r)g(u(r)) = 0, \quad 0 < r < r_0,$$
$$u(r_0) = a, \quad u'(r_0) = b. \tag{2.4}$$

For $d_0 \in \mathbb{R} - \{0\}$, using the Contraction Mapping Principle and the fact that g is a locally Lipschitzian function, we see there exists $\gamma \in (0, 1]$ such that for each $d \in [d_0 - \gamma, d_0 + \gamma]$, equation (2.2) has a unique solution $u(\cdot, d)$ in the space of continuous functions defined on $[1 - \gamma, 1]$. This and the continuity of the right hand side in (2.2) on (d, u) imply the continuous dependence of $u(\cdot, d)$ on d. When $\gamma = 1$ such a solution is a solution to (2.1). If $\gamma \in (0, 1)$ the solution may be extended to $[1 - \gamma_1, 1]$ for some $\gamma_1 > \gamma$ by applying a similar argument to (2.3) with $a = u(1 - \gamma, d)$ and $b = u'(1 - \gamma, d)$. Hence, the function $u(\cdot, d)$ may be extended to a maximal

interval which is either $[0,1]$ or $(\hat{\theta}(d), 1]$ with $\lim_{t\to\hat{\theta}(d)^+}[u^2(t)+(u'(t))^2] = +\infty$. We remark that from the results in [15], because of hypothesis (1.3), no solution to (2.4) satisfies $\lim_{t\to\hat{\theta}(d)^+}[u^2(t) + (u'(t))^2] = 0$ when $(a, b) \neq (0, 0)$. For a comprehensive study of existence, uniqueness and continuous dependence, we refer the reader to [15].

In our next lemma we prove that $\hat{\theta}(d) \leq X$. Since $d_0 \in \mathbb{R} - \{0\}$ is arbitrary, this shows the existence of a unique solution to (2.1) on $[X, 1]$ that depends continuously on d.

Lemma 2.1. *For each $d \neq 0$, the solution to* (2.1) *is defined in* $[X, 1]$.

Proof. Let $u(r) := u(r, d)$ be a solution to (2.1) and

$$\mathcal{E}(r, d) \equiv \mathcal{E}(r) := \frac{p-1}{p}|u'(r)|^p + W(r)G(u(r)). \qquad (2.5)$$

Let $C_1 = p(N-1)/(p-1)$. Since $W'(X) > 0$, $\lim_{r\to X^+} W'(r)/W(r) = +\infty$. Thus, there exists $C_2 > 0$ such that $W'(r)/W(r) \geq -C_2$ for $r \in (X, 1]$. We also let $C_3 = C_1/X + C_2$. Assuming that $\hat{\theta}(d) \geq X$, there exists $s \in (X, 1)$ such that

$$\mathcal{E}(s) > e^{C_3}\mathcal{E}(1). \qquad (2.6)$$

Since $|x|^p, x|x|^{p-2}$ and $|x|^{p/(p-1)}$ are differentiable functions, and

$$(|x|^{p/(p-1)})' = \frac{p}{p-1}|x|^{(2-p)/(p-1)}x,$$

the function $|u'|^{p-2}u'$ is differentiable in $(s, 1)$ (see (2.1)). Therefore, $\mathcal{E}$ is differentiable on $(s, 1]$ and for each $r \in (s, 1]$,

$$
\begin{aligned}
\mathcal{E}'(r) &= \Big(\frac{p-1}{p}\big||u'(r)|^{p-2}u'(r)\big|^{p/(p-1)}\Big)' \\
&\quad + W'(r)G(u(r)) + W(r)g(u(r))u'(r) \\
&= \big||u'(r)|^{p-2}u'(r)\big|^{(2-p)/(p-1)} |u'(r)|^{p-2}u'(r)\big(|u'(r)|^{p-2}u'(r)\big)' \\
&\quad + W'(r)G(u(r)) + W(r)g(u(r))u'(r) \\
&= |u'(r)|^{2-p}|u'(r)|^{p-2}u'(r)\big(|u'(r)|^{p-2}u'(r)\big)' \\
&\quad + W'(r)G(u(r)) + W(r)g(u(r))u'(r).
\end{aligned} \qquad (2.7)
$$

This and (2.1) yield

$$
\begin{aligned}
\mathcal{E}'(r) &= u'(r)\Big(-\frac{N-1}{r}|u'(r)|^{p-2}u'(r) - W(r)g(u(r))\Big) + W'(r)G(u(r)) \\
&\quad + W(r)g(u(r))u'(r) \\
&= -\frac{N-1}{r}|u'(r)|^p + W'(r)G(u(r)) \\
&= -\frac{p(N-1)}{(p-1)r}\mathcal{E}(r) + G(u(r))\Big[\frac{p(N-1)}{(p-1)r}W(r) + W'(r)\Big] \\
&\leq W'(r)G(u(r)),
\end{aligned} \qquad (2.8)
$$

where we used (2.5). Hence, from (2.7),

$$\mathcal{E}'(r) \geq -\frac{p(N-1)}{(p-1)r}\mathcal{E}(r) + G(u(r))W'(r)$$

$$\geq -\frac{p(N-1)}{(p-1)r}\mathcal{E}(r) + [W(r)G(u(r))]W'(r)/W(r)$$

$$\geq -\frac{p(N-1)}{(p-1)r}\mathcal{E}(r) - C_2 W(r)G(u(r))$$

$$\geq -\frac{p(N-1)}{(p-1)r}\mathcal{E}(r) - C_2 \mathcal{E}(r)$$

$$= \left(-\frac{C_1}{r} - C_2\right)\mathcal{E}(r)$$

$$\geq \left(-\frac{C_1}{X} - C_2\right)\mathcal{E}(r) := -C_3 \mathcal{E}(r).$$

Integrating on $[s,1]$, we have $e^{C_3}\mathcal{E}(1) - e^{C_3 s}\mathcal{E}(s) \geq 0$. Since this inequality contradicts (2.6) we have proven that $\hat{\theta}(d) \leq X$, and hence the lemma follows. $\qquad\square$

For $a > 0$ and $b \in \mathbb{R}$, we consider the initial value problem

$$\left(r^{N-1}|u'(r)|^{p-2}u'(r)\right)' + r^{N-1}W(r)g\big(u(r)\big) = 0, \quad 0 < r < X$$
$$u(X) = a, \quad u'(X) = b. \tag{2.9}$$

Because of our assumptions on g, the initial value problem (2.9) has a unique solution $u_{a,b}$ on a maximal interval $(r_{a,b}, X]$.

Lemma 2.2. *Let $r_0 \in (r_{a,b}, X]$. If $u_{a,b}(r_0) > 0$ and $u'_{a,b}(r_0) \leq 0$, then $u'_{a,b} < 0$ in $(r_{a,b}, r_0)$.*

Proof. Let $u := u_{a,b}$. Let $\varepsilon > 0$ be such that $u(r) > u(r_0)/2$ for all $r \in (r_0 - \varepsilon, r_0]$. From (2.9), for $r \in (r_0 - \varepsilon, r_0]$ we have

$$r^{N-1}|u'(r)|^{p-2}u'(r)$$
$$= r_0^{N-1}|u'(r_0)|^{p-2}u'(r_0) + \int_r^{r_0} s^{N-1}W(s)g(u(s))ds < 0. \tag{2.10}$$

Let $\hat{r} = \inf\{r \in (r_{a,b}, r_0) : u' < 0 \text{ in } (r, r_0)\}$. Hence $\hat{r} \leq r_0 - \varepsilon$. Assuming that $\hat{r} > r_{a,b}$ we have $u(\hat{r}) > 0$ and, by (2.10), $u'(\hat{r}) < 0$. Arguing as before, there exists $\delta > 0$ such that $u' < 0$ in $(\hat{r} - \delta, \hat{r})$ which contradicts the definition of $\hat{r}$ and proves that $u' < 0$ in $(r_{a,b}, r_0)$. $\qquad\square$

Lemma 2.3. *If $0 < b < \tilde{b}$, $y > \max\{r_{a,b}, r_{a,\tilde{b}}\}$, then $u_{a,b}(r) > u_{a,\tilde{b}}(r)$ for all $r \in [y, X)$. Moreover, $u'_{a,b}(r) < u'_{a,\tilde{b}}(r)$ for all $r \in [y, X)$.*

Proof. Let $u = u_{a,b}$ and $v = u_{a,\tilde{b}}$. Since $b < \tilde{b}$ there exists $\varepsilon > 0$ such that $u(r) > v(r)$ for all $r \in (X - \varepsilon, X)$. Assuming that $u(r) \not> v(r)$ for all $r \in [y, X)$, due to the continuity of u and v there exists $z \in [y, X)$ such that $u(z) = v(z)$ and

$u(r) > v(r)$ for all $r \in (z, X)$. On the other hand,

$$z^{N-1}|u'(z)|^{p-2}u'(z)$$

$$= X^{N-1}|u'(X)|^{p-2}u'(X) + \int_z^X s^{N-1}W(s)g(u(s))ds$$

$$\leqslant X^{N-1}|u'(X)|^{p-2}u'(X) + \int_z^X s^{N-1}W(s)g(v(s))ds \qquad (2.11)$$

$$< X^{N-1}|v'(X)|^{p-2}v'(X) + \int_z^X s^{N-1}W(s)g(v(s))ds$$

$$= z^{N-1}|v'(z)|^{p-2}v'(z),$$

contradicting that $v'(z) \leq u'(z)$ since $v(r) < u(r)$ for all $r \in (z, X)$. A similar argument proves the second assertion of the lemma. Thus, the lemma is proven. $\square$

Lemma 2.4. *For each $a > 0$ there exists $\varepsilon > 0$ such that if $b \in (0, \varepsilon)$ then there exists $r_b \in (r_{a,b}, X]$ such that $u'_{a,b}(r_b) = 0$ and $u'_{a,b}(r) > 0$ for all $r \in [r_b, X]$.*

Proof. Let $y = r_{a,0}$ and $u = u_{a,0}$. By Lemma 2.2 we have $u'((y + X)/2)) < 0$. This and continuous dependence on initial conditions imply the existence of $\varepsilon > 0$ such that if $b \in (0, \varepsilon)$ then $u'_{a,b}((y + X)/2) < 0$. Hence, by the intermediate value theorem, for each $b \in (0, \varepsilon)$ there exists $r_b \in ((y + X)/2, X)$ such that $u'_{a,b}(r_b) = 0$. By Lemma 2.2, r_b is unique. Hence $u'_{a,b}(r)$ does not change sign in $(r_b, X]$. Since $u'_{a,b}(X) > 0$, it follows that $u'_{a,b}(r) > 0$ for all $r \in (r_b, X]$. This proves the lemma. $\square$

Lemma 2.5. *For each $a > 0$ there exists $b > 0$ such that $u_{a,b}(r_1) = 0$ for some $r_1 \in (r_{a,b}, X]$.*

Proof. Let $b > 0$ be such that

$$b^{p-1} > \max\{-\inf\{W(r); r \in (0, X]\}2g(a), 2^p a^{p-1}/X^{p-1}\}$$

and $u := u_{a,b}$. Since $b > 0$ there exists $\varepsilon > 0$ such that $0 < u(s) \leq a$ for all $s \in (X - \varepsilon, X]$. Let $r \in (0, X)$ be such that $0 < u(s) \leq a$ for $s \in (r, X]$. Hence

$$|u'(r)|^{p-2}u'(r) = r^{1-N}\Big(X^{N-1}b^{p-1} + \int_r^X s^{N-1}W(s)g(u(s))ds\Big)$$

$$> r^{1-N}\Big(X^{N-1}b^{p-1} + \frac{X^N}{N} \cdot \inf\{W(r); r \in (0, X]\}g(a)\Big) \qquad (2.12)$$

$$\geq b^{p-1} + \inf\{W(r); r \in (0, X]\}g(a)\frac{X}{N}$$

$$> \frac{b^{p-1}}{2} > \frac{2^{p-1}a^{p-1}}{X^{p-1}} > 0.$$

Therefore, $u'(r) > 2a/X$. Let

$$r_1 = \inf\{r \in (0, X); 0 < u(s) \leq a \quad \text{for all } s \in (r, X]\} \leq X - \varepsilon.$$

By the Mean Value Theorem, $a \geq a - u(r_1) > (X - r_1)\frac{2a}{X}$. Hence

$$r_1 > \frac{X}{2}. \qquad (2.13)$$

Since $u' > 0$ on $[r_1, X]$, we have $u(r_1) < a$. If $u(r_1) > 0$, by the continuity of u, there exists $r_2 < r_1$ such that $u(s) \in (0, a]$ for all $s \in [r_1, X]$ contradicting the definition of r_1. Thus $u(r_1) = 0$ and the lemma is proven. $\qquad\square$

For $a > 0$ we define

$$\hat{b} = \hat{b}(a) = \sup\{b > 0; u_{a,c}(r) > 0 \text{ for all } r \in (r_{a,c}, X] \text{ and all } c \in (0, b)\} \qquad (2.14)$$
$$\equiv \sup B_a.$$

From Lemma 2.4 and Lemma 2.5, $0 < \hat{b} < +\infty$.

Lemma 2.6. *For all $a > 0$, $r_{a,\hat{b}} = 0$, $u_{a,\hat{b}}(r) > 0$ for all $r \in (0, X]$, and $u_{a,\hat{b}}(r)$ is monotonically increasing.*

Proof. Let $z = u_{a,\hat{b}}$. First we prove that $z(r) > 0$ for all $r \in (r_{a,\hat{b}}, X]$. Suppose there exists $r_1 \in (0, X]$ such that $z(r_1) = 0$. Without loss of generality, we may assume that $z(r) > 0$ for all $r \in (r_1, X]$. By uniqueness of solutions to initial value problems $z'(r_1) > 0$. Hence there exists $\delta > 0$ such that $z(r) < 0$ for all $r \in (r_1 - \delta, r_1)$. Let $\{b_j\}_j$ be an increasing sequence in B_a converging to $\hat{b}$. By existence of solutions to initial value problems, there exists J such that if $j \geq J$ then $r_{a,b_j} < r_1 - \delta/2$. By continuous dependence on initial conditions, $\lim_{j \to +\infty} u_{a,b_j}(r_1 - \delta/2) = z(r_1 - \delta/2) < 0$ contradicting that $u_{a,b_j}(r) > 0$ for all $r \in (r_{a,b_j}, X]$. Hence $z(r) > 0$ for all $r \in (r_{a,\hat{b}}, X]$.

Now we prove that $z'(r) > 0$ in $(r_{a,\hat{b}}, X]$. Assuming that z is not monotonically increasing, there exists $r_2 \in (r_{a,\hat{b}}, X)$ such that $z'(r) > 0$ for $r \in (r_2, X]$ and $z'(r) < 0$ for all $r \in (r_{a,\hat{b}}, r_2)$. By continuous dependence on initial conditions, there exists $\eta > 0$ such that if $|\hat{b} - b| < \eta$ then $r_{a,b} < (r_2 + r_{a,\hat{b}})/2$. Also, there exists $\rho \in (0, \eta)$ such that if $|\hat{b} - b| < \rho$ then $u_{a,b}((r_2 + r_{a,\hat{b}})/2) > 0$ and $u'_{a,b}((r_2 + r_{a,\hat{b}})/2) < 0$. Hence $u_{a,b}(r) > 0$ for all $b < \hat{b} + \rho$ and all $r \in (r_{a,b}, X]$. Since this contradicts the definition of $\hat{b}$ we conclude that $z = u_{a,\hat{b}}$ is a monotonically increasing function.

Since z is monotonically increasing and positive, if $r_{a,\hat{b}} > 0$ then z may be extended to an interval of the form $(r_{a,\hat{b}} - \varepsilon, X]$ contradicting the definition of $r_{a,\hat{b}}$. This proves that $r_{a,\hat{b}} = 0$. $\qquad\square$

Lemma 2.7. *If $p \leq N$, $u \in C^1(0, 1]$ satisfies*

$$\left(r^{N-1}|u'|^{p-2}u'\right)' + r^{N-1}W(r)g(u(r)) = 0, \quad 0 < r < 1, \qquad (2.15)$$

and $u > 0$ and bounded in $(0, X)$, then

$$\zeta(r) := |u'(1)|^{p-2}u'(1) + \int_r^1 s^{N-1}W(s)g(u(s))ds = r^{N-1}|u'(r)|^{p-2}u'(r) \qquad (2.16)$$

is non-negative and non-decreasing. Moreover,

$$\lim_{r \to 0^+} \zeta(r) = \lim_{r \to 0^+} u'(r) = 0. \qquad (2.17)$$

Proof. For $s \in (0, X)$, $s^{N-1}W(s)g(u(s)) < 0$. Hence ζ increases on $(0, X)$ and $\lim_{r \to 0^+} \zeta(r)$ exists. Assuming there is $r_0 \in (0, X)$ such that

$$r_0^{N-1}|u'(r_0)|^{p-2}u'(r_0) =: c < 0,$$

from the monotonicity of ζ, we have $r^{N-1}|u'(r)|^{p-2}u'(r) < c$ for all $r \in (0, r_0)$. Therefore, since $p \leq N$, we have $-u'(r) \geq (-c)^{1/(p-1)}r^{-1}$ for all $r \in (0, r_0]$. Integrating $-u'$ in $[r, r_0]$ we see that $\lim_{r\to0+} u(r) = +\infty$ which contradicts that u is bounded. Hence $\zeta(r) \geq 0$ for all $r \in (0, X]$.

To prove (2.17), we assume to the contrary that there exists $\kappa > 0$ such that

$$r^{N-1}|u'(r)|^{p-2}u'(r) = |u'(1)|^{p-2}u'(1) + \int_r^1 s^{N-1}W(s)g(u(s))ds \geq \kappa > 0, \quad (2.18)$$

for all $r \in (0, 1]$. Solving for u' in (2.18),

$$u'(s) \geq \frac{\kappa^{1/(p-1)}}{s^{(N-1)/(p-1)}} \geq \frac{\kappa^{1/(p-1)}}{s} \quad \text{for all } s \in (0, 1] \text{ and } p \leq N. \quad (2.19)$$

Therefore,

$$u(r) = u(1) - \int_r^1 u'(s)ds \leq u(1) - \int_r^1 \frac{\kappa^{1/(p-1)}}{s}ds = u(1) + \kappa^{1/(p-1)}(\ln r) \to -\infty,$$

as $r \to 0^+$. This contradiction proves that $\lim_{r\to0^+} \zeta(r) = 0$.

Finally, by (2.16) and L'Hôspital's rule we obtain

$$\begin{aligned}
\lim_{r\to0^+} (u'(r))^{p-1} &= \lim_{r\to0^+} \frac{|u'(1)|^{p-2}u'(1) + \int_r^1 s^{N-1}W(s)g(u(s))ds}{r^{N-1}} \\
&= \lim_{r\to0^+} \frac{-rW(r)g(u(r))}{N-1} = 0,
\end{aligned} \quad (2.20)$$

and thus the lemma is proven. $\qquad\square$

Remark 2.8. The conclusion of Lemma 2.7 remains valid when 1 is replaced by $R \in (0, 1)$, under the assumptions that $u \in C^1(0, R]$ satisfies the differential equation in (2.15) on $(0, R]$, $u > 0$ and bounded on some interval $(0, x_0) \subset (0, R)$.

Lemma 2.9. *If $\eta : [0, +\infty) \to [0, +\infty)$ is defined by $\eta(a) = \hat{b}$ and $\eta(0) = 0$, then η is a continuous function.*

Proof. First we prove that the function η is a non-decreasing function. Let $u = u_{a_0,\eta(a_0)}$ and $v = u_{a,\eta(a_0)}$ with $a > a_0 \geq 0$. Suppose that there is $r_1 \in (r_{a,\eta(a_0)}, X)$ such that $u(r_1) = v(r_1)$. Without loss of generality we may also assume that $u(r) < v(r)$ for all $r \in (r_1, X]$. By the uniqueness of solutions to the initial value problem (2.9) we have $v'(r_1) > u'(r_1)$. On the other hand, from (2.9) it follows that

$$\begin{aligned}
r_1^{N-1}(|v'(r_1)|^{p-2}v'(r_1) &- |u'(r_1)|^{p-2}u'(r_1)) \\
&= \int_{r_1}^X s^{N-1}W(s)(g(v(s)) - g(u(s)))ds < 0.
\end{aligned} \quad (2.21)$$

Therefore $v'(r_1) \leq u'(r_1)$, which contradicts $v'(r_1) > u'(r_1)$. Thus $v(r) > u(r)$ for all $r \in (r_{a,\eta(a_0)}, X]$. This and the definition of $\hat{b}(a)$ prove that $\eta(a) \geqslant \eta(a_0)$. So, we have proved η is a non decreasing function.

Let $\{a_n\}_n$ be an increasing sequence that converges to $a_0 > 0$. Hence $\{\eta(a_n)\}$ is an increasing sequence bounded above by $\eta(a_0)$. Let $u = u_{a_0,\eta(a_0)}$ and $u_n = u_{a_n,\eta(a_n)}$. Suppose that $\eta(a_0) > \lim_{n\to\infty} \eta(a_n)$. For n sufficiently large, there exists $r_n \in (0, X)$ such that $u_n(r_n) = u(r_n)$, $\lim_{n\to+\infty} r_n = X$, $u_n'(r_n) < u'(r_n)$

and $u(r) < u_n(r)$ for all $r \in [0, r_n]$. Therefore, from Lemma 2.7 (see also Remark 2.8), for a fixed n large enough,

$$
\begin{aligned}
0 &= \lim_{r \to 0^+} r^{N-1} |u_n'(r)|^{p-2} u_n'(r) \\
&= \lim_{r \to 0^+} \left(r_n^{N-1} (u_n'(r_n))^{p-1} + \int_r^{r_n} s^{N-1} W(s) g(u_n(s)) ds \right) \\
&< \lim_{r \to 0^+} \left(r_n^{N-1} (u'(r_n))^{p-1} + \int_r^{r_n} s^{N-1} W(s) g(u(s)) ds \right) = 0.
\end{aligned}
\tag{2.22}
$$

This contradiction proves that η is continuous to the left. Similarly it is seen that η is continuous to the right. Thus the lemma is proven. $\qquad \square$

Theorem 2.10. *If $w : \overline{B_1(0)} \to \mathbb{R}$ is a radial function such that $w \in C^1(\overline{B_1(0)} \setminus \{0\}) \cap C(\overline{B_1(0)})$, and $U(r) = w(r, 0, \ldots, 0)$ satisfies the assumptions of Lemma 2.7, then w is a solution to (1.1) in the sense of distributions.*

Proof. Let φ be a function of class C^∞ and compact support in $B_1(0)$ and A the $(N-1)$-dimensional Lebesgue measure of the unit sphere in $\mathbb{R}^N$. Lemma 2.7 implies

$$
|U'(1)|^{p-2} U'(1) = - \int_0^1 s^{N-1} g(U(s)) W(s) ds.
\tag{2.23}
$$

Using that $\nabla w(x) = (U'(|x|)/|x|) x$ and (2.23), we have

$$
\begin{aligned}
&\int_{B_1(0)} \|\nabla w(x)\|^{p-2} \nabla w(x) \cdot \nabla \varphi(x) dx \\
&= \int_{B_1(0)} \|\nabla w(x)\|^{p-2} \nabla w(x) \cdot (\nabla \varphi(x) \cdot x) x / |x|^2 dx \\
&= \int_0^1 r^{N-1} |U'(r)|^{p-2} U'(r) \int_{\|\theta\|=1} (\nabla \varphi(r\theta) \cdot \theta) d\theta \, dr \\
&= \int_0^1 r^{N-1} |U'(r)|^{p-2} U'(r) \left(\frac{d}{dr} \int_{\|\theta\|=1} \varphi(r\theta) d\theta \right) dr \\
&= \int_0^1 |U'(1)|^{p-2} U'(1) \left(\frac{d}{dr} \int_{\|\theta\|=1} \varphi(r\theta) d\theta \right) dr \\
&\quad + \int_0^1 \left(\int_r^1 s^{N-1} W(s) g(U(s)) ds \right) \left(\frac{d}{dr} \int_{\|\theta\|=1} \varphi(r\theta) d\theta \right) dr \\
&= -A \left[|U'(1)|^{p-2} U'(1) + \left(\int_0^1 r^{N-1} W(r) g(U(r)) dr \right) \right] \varphi(0) \\
&\quad + \int_{\|\theta\|=1} \int_0^1 r^{N-1} W(r) g(U(r)) \varphi(r\theta) dr d\theta \\
&= \int_{B_1(0)} W(|x|) g(U(|x|)) \varphi(x) dx,
\end{aligned}
\tag{2.24}
$$

which proves the theorem. $\qquad \square$

3. ENERGY AND PHASE PLANE ANALYSIS

Let $T \in (X, 1]$. Since $W > 0$ on $(X, 1]$, there exists $W_0 = W_0(T) > 0$ such that $W(r) \geq W_0$ for each $r \in [T, 1]$.

We recall that, from Lemma 2.1, given $d < 0$ there is a unique solution $u(r, d)$ to

$$\left(r^{N-1}|u'|^{p-2} u'\right)' + r^{N-1}W(r)g\big(u(r)\big) = 0, \quad X \le r \le 1, \tag{3.1}$$
$$u(1) = 0, \quad u'(1) = d.$$

Lemma 3.1. *Let $T \in (X, 1)$ and $\mathcal{E}$ be as defined in (2.5). Then $\mathcal{E}(r) \to +\infty$ as $|d| \to +\infty$, uniformly for $r \in [T, 1]$.*

Proof. From (2.7) we have

$$\mathcal{E}'(r) \le W'(r)G\big(u(r)\big) = \frac{W'(r)}{W(r)}W(r)G\big(u(r)\big)$$
$$\le \frac{\max\{|W'(s)| : s \in [T, 1]\}}{\min\{W(s) : s \in [T, 1]\}}W(r)G\big(u(r)\big)$$
$$\le \tilde{C}W(r)G\big(u(r)\big) \le \tilde{C}\mathcal{E}(r),$$

for all $r \in [T, 1]$. Thus, $\left(e^{-\tilde{C}r}\mathcal{E}(r)\right)' \le 0$ for every $r \in [T, 1]$. Therefore,

$$\mathcal{E}(r) \ge \frac{p-1}{p}e^{\tilde{C}(T-1)}|d|^{p-1} \quad \text{for all } r \in [T, 1].$$

Thus, $\mathcal{E}(r) \to +\infty$ as $|d| \to +\infty$, uniformly for $r \in [T, 1]$. $\qquad\square$

Since $g(0) = 0$, by uniqueness of solutions to the initial value problem with initial data $(0, 0)$, we have

$$(u(r, d), u'(r, d)) \ne (0, 0)$$

for all $r \in [X, 1]$. Hence, there exists a continuous function $\phi(r, d)$, for $r \in [X, 1]$, such that $\phi(1, d) = -\pi/2$, and

$$u(r, d) = -\rho(r, d) \cos \phi(r, d), \tag{3.2}$$
$$u'(r, d) = \rho(r, d) \sin \phi(r, d),$$

where $\rho(r, d) = \sqrt{\big(u(r, d)\big)^2 + \big(u'(r, d)\big)^2}$. Moreover, $\phi(\cdot, d)$ is differentiable at $r \in [X, 1]$ provided $u'(r) \ne 0$.

Differentiating the first equation in (3.2) with respect to r, for $u'(r) \ne 0$,

$$u'(r) = -\rho'(r, d) \cos\big(\phi(r, d)\big) + \rho(r, d) \sin\big(\phi(r, d)\big) \cdot \phi'(r, d). \tag{3.3}$$

Since W is a continuous function, there exists $T \in (X, 1)$ such that

$$W(r) \ge \frac{W(1)}{2} =: \frac{m}{2}, \quad \text{for all } r \in [T, 1]. \tag{3.4}$$

Combining (3.2) and (3.1), for $r \in [X, 1]$ with $u'(r) \ne 0$, we have

$$\phi'(r, d) = \frac{(u'(r, d))^2}{\rho^2(r, d)} + \frac{W(r)u(r)g(u(r))}{(p-1)\rho^2(r, d)|u'(r)|^{p-2}} + \frac{(N-1)u(r)u'(r)}{r(p-1)\rho^2(r, d)}. \tag{3.5}$$

Remark 3.2.

(i) By Lemma 3.1, $\mathcal{E}(r, d) \to +\infty$ as $|d| \to +\infty$ uniformly for $r \in [T, 1]$, and therefore $\rho(r, d) \to +\infty$ as $|d| \to +\infty$ uniformly for $r \in [T, 1]$.

(ii) If $u(R, d) = 0$ with $R \in (X, 1)$, then $u'(R, d) \ne 0$. In addition, from (3.3) and the second equation in (3.2) it follows that $\phi'(R, d) = 1$.

(iii) If $u(R, d) = 0$ then $\phi(r, d) < \phi(R, d)$ for every $r \in [X, R]$. Suppose, by contradiction, there exists $R_1 \in [X, R]$ such that $\phi(R_1, d) = \phi(R, d)$. By the continuity of $\phi(\cdot, d)$ we can assume $\phi(r, d) < \phi(R, d)$ for all $r \in (R_1, R)$ (suffices choosing $R_1 = \inf\{r \in [X, R]: \phi(r, d) < \phi(R, d)\}$). Since $\phi(r, d) < \phi(R_1, d)$ for each $r \in (R_1, R)$, then $\phi'(R_1, d) \leq 0$. On the other hand, $\phi(R_1, d) = \phi(R, d) = j\pi + \pi/2$, for some $j \in \mathbb{Z}$. Thus $\phi'(R_1, d) = 1$, which is a contradiction.

Let k be a positive integer. For $x_0 > 0$, let us define

$$\tilde{m}(x_0) = \min\Big\{\frac{g(x)}{|x|^{p-2}x} : |x| \geq x_0\Big\}.$$

From the p-superlinearity of g we have $\tilde{m}(x_0) \to +\infty$ as $x_0 \to +\infty$. For $\rho > 0$ and $\eta > 0$ we define $\omega(\rho, \eta) := \tilde{m}(\rho \sin(\eta))M_1(\eta)/(p-1)$, where $M_1(\eta) := \min\{\sin^p(\eta), \sin^2(\eta)\}$. Let T be as in (3.4), let $\rho_0(k) := \rho_0 > 0$ and $\delta(k) := \delta \in (0, \pi/4)$ be such that

$$
\begin{aligned}
&(i) \quad 0 < \delta < \min\Big\{\frac{(p-1)T}{16(N-1)}, \Big(\frac{(1-T)(p-1)}{2}\Big)^{1/(p-1)}\Big\},\\
&(ii) \quad \omega(\rho_0, \delta) > \frac{2(N-1)}{m(p-1)T},\\
&(iii) \quad \tilde{m}(\rho_0/\sqrt{2}) \geq \frac{2^{(p/2)+5}k(p-1)}{m},\\
&(iv) \quad 16\delta + \frac{8\pi}{m\omega(\rho_0, \delta)} \leq \frac{1-T}{2k}.
\end{aligned}
\tag{3.6}
$$

By Remark 3.2-(i), there exists $d_0 < 0$ such that

$$\text{if } d < d_0, \text{ then } \rho(r, d) \geq \rho_0 \text{ for every } r \in [T, 1]. \tag{3.7}$$

Lemma 3.3. *If $T \leq r \leq 1$ and $\phi(r, d) \in [-j\pi/2 - \delta, -j\pi/2 + \delta]$ with $j > 0$ an odd integer, then $\phi'(r, d) > 1/4$.*

Proof. From (3.5),

$$\phi'(r, d) \geq \sin^2\phi + \frac{W(r)u(r, d)g(u(r, d))}{(p-1)\rho^2(r, d)|u'(r, d)|^{p-2}} - \frac{(N-1)|\cos\phi\sin\phi|}{r(p-1)}.$$

From (1.2) and (1.4), $W(r)u(r, d)g(u(r, d)) \geq 0$ for all $r \in [X, 1]$. This, the inequalities $|\sin(\phi(r, d))| \geq \cos\delta$, $|\cos(\phi(r, d))| \leq \sin\delta \leq \delta$ and (3.6)-(i) give

$$\phi'(r, d) \geq \cos^2\delta - \frac{(N-1)\delta}{(p-1)T} \geq \cos^2(\pi/4) - \frac{1}{16} \geq \frac{7}{16} > \frac{1}{4}.$$

Thus, the lemma is proven. $\square$

Lemma 3.4. *If $T \leq r \leq 1$ and $\phi(r, d) \in [-(j+1)\pi/2 + \delta, -j\pi/2 - \delta]$ with $j > 0$ an integer, then $\phi'(r, d) > m\omega(\rho_0, \delta)/4$.*

Proof. From (3.5),

$$
\begin{aligned}
\phi'(r, d) &> \frac{W(r)u(r, d)g(u(r, d))}{(p-1)\rho^2(r, d)|u'(r, d)|^{p-2}} - \frac{(N-1)}{2r(p-1)}\\
&\geq \frac{W(r)}{p-1}\frac{g(u(r, d))}{|u(r, d)|^{p-2}u(r, d)}\frac{|u(r, d)|^p}{\rho^2(r, d)|u'(r, d)|^{p-2}} - \frac{N-1}{2(p-1)T}
\end{aligned}
$$

$$\geq \frac{W(r)}{p-1} \frac{g(u(r,d))}{|u(r,d)|^{p-2}u(r,d)} \frac{|\cos\phi(r,d)|^p}{|\sin\phi(r,d)|^{p-2}} - \frac{N-1}{2(p-1)T}.$$

From $|\cos\phi(r,d)| \geq \sin\delta, |\sin\phi(r,d)| \geq \sin\delta$, and $\omega(\rho_0,\delta) > \frac{2(N-1)}{m(p-1)T}$ (see (3.6)-(ii)), it follows that

$$\phi'(r,d) > \frac{W(r)}{p-1} \frac{g(u(r,d))}{|u(r,d)|^{p-2}u(r,d)} M_1(\delta) - \frac{m\omega(\rho_0,\delta)}{4}.$$

Since $|u(r,d)| = \rho(r,d)|\cos\phi(r,d)| \geq \rho_0\sin\delta$ we obtain

$$g(u(r,d))/(|u(r,d)|^{p-2}u(r,d)) \geq \tilde{m}(\rho_0\sin\delta).$$

This and the definition of $\omega(\rho_0,\delta)$ yield

$$\phi'(r,d) > W(r)\omega(\rho_0,\delta) - \frac{m\omega(\rho_0,\delta)}{4} \geq \frac{m\omega(\rho_0,\delta)}{4}. \tag{3.8}$$

In the latter inequality we have used $W(r) \geq m/2$ for any $r \in [T,1]$. Thus, (3.8) proves the lemma. $\qquad\square$

Lemma 3.5. *If $T \leq r \leq 1$ and $\phi(r,d) \in (-j\pi, -j\pi+\delta] \cup [-j\pi-\delta, -j\pi)$ for some positive integer j, then*

$$\phi'(r,d) \geq 8k|\sin(\phi(r,d))|^{2-p}. \tag{3.9}$$

Proof. From $\delta < \pi/4$ and $|\cos\phi(r,d)| \geq \cos\delta$, it follows that

$$u^2(r,d) = \rho^2(r,d)(1-\sin^2(\delta)) \geq \rho^2(r,d)/2.$$

This, (3.6)-(i), (3.5) and (3.6)-(iii) imply that

$$\phi'(r,d)$$
$$\geq \frac{W(r)u(r,d)g(u(r,d))}{(p-1)\rho^p(r,d)|\sin(\phi(r,d))|^{p-2}} - \frac{(N-1)|\sin(\phi(r,d))|}{r(p-1)}$$
$$\geq \frac{W(r)u(r,d)g(u(r,d))|\sin(\phi(r,d))|^{2-p}}{2^{p/2}(p-1)|u(r,d)|^p} - \frac{(N-1)|\sin(\phi(r,d))|}{T(p-1)}$$
$$\geq \left(\frac{m}{2 \cdot 2^{p/2}(p-1)} \frac{g(u(r,d))}{|u(r,d)|^{p-2}u(r,d)} - \frac{N-1}{(p-1)T}|\sin(\phi(r,d))|^{p-1}\right)$$
$$\quad \times |\sin(\phi(r,d))|^{2-p} \tag{3.10}$$
$$\geq \left(\frac{m \cdot \tilde{m}(\rho_0/\sqrt{2})}{2^{(p/2)+1}(p-1)} - \frac{1}{16}\right)|\sin(\phi(r,d))|^{2-p}, \quad (\text{recall } |\sin\phi| \leq \sin\delta \leq \delta)$$
$$\geq \frac{m \cdot \tilde{m}(\rho_0/\sqrt{2})}{2^{(p/2)+2}(p-1)}|\sin(\phi(r,d))|^{2-p}$$
$$\geq 8k|\sin(\phi(r,d))|^{2-p},$$

which completes the proof of the lemma. $\qquad\square$

Proposition 3.6. $\lim_{d\to-\infty}\phi(X,d) = -\infty.$

Proof. Let $d < d_0$ be as in (3.7) and k as in (3.6). Because $\phi(1,d) \in [-\pi/2 - \delta, -\pi/2 + \delta]$, from Lemma 3.3 and (3.6)-(iv) there exists

$$r_1 \in [1-4\delta, 1) \subset [1-(1-T)/(8k), 1)$$

such that $\phi(r_1,d) = -\pi/2 - \delta$. By Lemma 3.4 and (3.6)-(iv) there exists

$$r_2 \in [r_1 - 2\pi/(m\omega(\rho_0,\delta)), r_1) \subset [r_1 - (1-T)/(8k), r_1)$$

such that $\phi(r_2, d) = -\pi + \delta$. By Lemma 3.5, if $p \geq 2$, there is

$$r_3 \in [r_2 - \delta/(8k), r_2) \subset [r_2 - (1-T)/(8k), r_2)$$

such that $\phi(r_3, d) = -\pi$. On the other hand, if $1 < p < 2$, from Lemma 3.5

$$\phi'(r, d) \geq 8k|\sin(\phi(r, d))|^{2-p}. \tag{3.11}$$

We claim that if $r < r_2$ and $\phi(r, d) > -\pi$ then $r_2 - r < (1-T)/(8k)$. Indeed, let $\phi(r, d) = -\pi + \theta(r, d)$; then $0 < \theta(r, d) \leq \delta$. From (3.11) we obtain $\theta'(r, d) \geqslant 8k(\sin\theta(r, d))^{2-p}$. Since $\sin(\theta)/\theta \to 1$ as $\theta \to 0$, we may assume, for δ sufficiently small, $\sin(\theta(r, d))/\theta(r, d) > 1/2$. Thus,

$$\theta'(r, d)(\theta(r, d))^{p-2} \geqslant \frac{8k}{2^{2-p}} > 4k.$$

Integrating on $[r, r_2]$ we get $4k(p-1)(r_2 - r) < \delta^{p-1}$. By using (3.6)-(i),

$$r_2 - r < \frac{\delta^{p-1}}{4k(p-1)} < \frac{1-T}{8k}.$$

From this the claim follows. Hence, there exists $r_3 \in [r_2 - (1-T)/(8k), r_2)$ such that

$$r_3 \in [1 - 3(1-T)/(8k), 1) \subset [1 - (1-T)/(2k), 1) \text{ and } \quad \phi(r_3, d) = -\pi.$$

Observe that $1 - r_3 \leq (1-T)/(2k)$ and $\phi(r_3, d) - \phi(1, d) = -\pi/2$. Repeating this argument $2k$ times it is shown that there is $\hat{r} \in [1 - 2k(1-T)/2k, 1) = [T, 1)$ such that $\phi(\hat{r}, d) - \phi(1, d) = -k\pi$, namely $\phi(\hat{r}, d) = -(2k+1)\pi/2$. Since $1 > \hat{r} \geq T > X$, Remark 3.2-(ii) implies $\phi(X, d) < \phi(\hat{r}, d) = -(2k+1)\pi/2$. This proves the proposition. $\square$

4. Proof of main theorem

Let $u(r, d)$ be the solution to (3.1) with $d < 0$ and $\phi(r, d)$ be the argument function defined by (3.2). By Proposition 3.6 ($\lim_{d \to -\infty} \phi(X, d) = -\infty$) and the continuous dependence of $\phi(X, d)$ on d, for each odd positive integer k there exist real numbers $\hat{d}_k$ and $\tilde{d}_k$ such that

$$\tilde{d}_k < \hat{d}_k, \quad \phi(X, \hat{d}_k) = -k\pi - \frac{\pi}{2}, \quad \phi(X, \tilde{d}_k) = -(k+1)\pi.$$

Since k is odd, $u(X, \hat{d}_k) = 0$ and $u'(X, \hat{d}_k) > 0$; also, $u(X, \tilde{d}_k) > 0$ and $u'(X, \tilde{d}_k) = 0$.

Because η is a continuous function (see Lemma 2.9) the set $\{(a, \eta(a)); a \geq 0\}$ separates $\{(0, y); y > 0\}$ from $\{(x, 0); x > 0\}$ in $\{(x, y); x \geq 0, y \geq 0\} - \{(0, 0)\}$, there exists $d_k \in (\tilde{d}_k, \hat{d}_k)$ such that $(u(X, d_k), u'(X, d_k)) = (a, \eta(a))$ for some $a > 0$. Hence defining $U_k(r) = u_{a,\eta(a)}(r)$ for $r \in [0, X]$ and $U_k(r) = u(r, d_k)$ for $r \in [X, 1]$ we have a solution to (1.1) (see Lemmas 2.6-2.7 and Theorem 2.10). Thus, the sequence $\{U_k(r)\}_k$ gives us infinitely many radially symmetric solutions to (1.1), which proves the theorem.

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Alfonso Castro
Department of Mathematics, Harvey Mudd College, Claremont, CA 91711, USA
Email address: castro@g.hmc.edu

Jorge Cossio
Escuela de Matemáticas, Universidad Nacional de Colombia Sede Medellín, Medellín, Colombia
Email address: jcossio@unal.edu.co

Sigifredo Herrón
Escuela de Matemáticas, Universidad Nacional de Colombia Sede Medellín, Medellín, Colombia
Email address: sherron@unal.edu.co

Carlos Vélez
Escuela de Matemáticas, Universidad Nacional de Colombia Sede Medellín, Medellín, Colombia
Email address: cauvelez@unal.edu.co