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A THIRD LOOK AT THE FIRST RESULT OF LANDESMAN-LAZER TYPE

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Dedicated with great admiration to Alan Lazer

Abstract. We review some of the Landesman-Lazer-Leach results and provide elementary proofs by means of shooting type arguments. An appropriate extension of a first result by Alan Lazer to systems can be regarded as a generalization of the fundamental theorem of algebra.

1. Introduction

In the celebrated paper [10], Landesman and Lazer gave a sufficient condition for the existence of solutions to a nonlinear elliptic equation under resonance at a simple eigenvalue. To simplify, we consider the Dirichlet problem

$$\Delta u(x) + \lambda u(x) + g(u(x)) = p(x), \quad u|_{\partial\Omega} = 0,$$

where $\Omega \subset \mathbb{R}^d$ is a smooth bounded domain, $g \in C(\mathbb{R})$ has finite limits at $\pm\infty$, $p \in L^2(\Omega)$, and λ is a simple eigenvalue of the operator $Lu := -\Delta u$ under Dirichlet conditions. Let ψ be an eigenfunction associated with λ and define Ω^+ and Ω^- as the subsets of Ω in which $\psi > 0$ and $\psi < 0$ respectively. In this setting, the Landesman-Lazer condition reads

$$\begin{aligned} &g(+\infty)\int_{\Omega^+}\psi(x)\,dx + g(-\infty)\int_{\Omega^-}\psi(x)\,dx \\ &> \int_{\Omega} p(x)\psi(x)\,dx \\ &> g(+\infty)\int_{\Omega^-}\psi(x)\,dx + g(-\infty)\int_{\Omega^+}\psi(x)\,dx. \end{aligned} \tag{1.1}$$

As shown in [10], this condition is sufficient for the existence of at least one solution and, furthermore, if

$$g(u) \neq g(\pm\infty) \quad u \in \mathbb{R},$$

then (1.1) is also necessary. The latter property is immediate and is left for the reader as an exercise. From now on, we shall focus only on the sufficiency part of the result.

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A few years after [10] was published, a nice application of elementary critical point theory allowed obtaining a more general conditions, well-known Ahmad-Lazer-Paul conditions [1]. Since then, several extensions of the Landesman-Lazer theorem have been obtained; for a survey see for example [16]. In 2000, Alan Lazer published a delightful article in EJDE [12], in which the original result and some variants for ODEs were discussed, including the Lazer-Leach [13] and the so-called Frederickson-Lazer theorems. In this paper, we shall review some of these results and give simple proofs for the classical ODE cases. Moreover, we shall present some extensions to systems and more general situations.

The paper is organized as follows. In the next section, we shall give a very elementary proof of the ODE analogue of the original Landesman-Lazer result by means of a shooting argument. Section 3 is devoted to the periodic problem: in the first place, a non-asymptotic extension for a systems under resonance at the first eigenvalue is given and its connection with the fundamental theorem of algebra is explored. In the second place, it shall be shown that, when resonance occurs at a higher order eigenvalue, the shooting method also allows to give an elementary proof of the original Lazer-Leach theorem.

2. The one-dimensional Landesman-Lazer theorem: easy proof by the shooting method

In this section, we shall give an elementary proof of the Landesman-Lazer theorem for an ODE, namely the Dirichlet problem

$$u''(t) + n^2 u(t) + g(u(t)) = p(t), \quad u(0) = u(\pi) = 0, \tag{2.1}$$

where g has finite limits $g(\pm\infty)$ and $p \in L^2(0,\pi)$. Here, it is readily seen that n^2 is a simple eigenvalue of $Lu := -u''$, with associated eigenfunction subspace spanned by $\psi(t) := \sin(nt)$.

Theorem 2.1 (Landesman-Lazer)**.** *In the above situation, if* (1.1) *holds then problem* (2.1) *admits at least one solution.*

For a proof, let us firstly assume that g is smooth, so the initial value problem

$$u''(t) + n^2 u(t) + g(u(t)) = p(t), \quad u(0) = 0,\, u'(0) = s$$

has a unique solution u_s. By standard results, u_s is defined up to $t = \pi$ and the shooting operator $s \mapsto u_s(\pi)$ is continuous; thus, it suffices to show the existence of numbers $s^+, s^- \in \mathbb{R}$ such that $u_{s^-}(\pi) < 0 < u_{s^+}(\pi)$. By variation of parameters, it is verified that

$$u_s(t) = -\frac{1}{n}\int_0^t \sin(n\tau)\xi_s(\tau)\,d\tau\cos(nt) + \frac{1}{n}\Big(s + \int_0^t \cos(n\tau)\xi_s(\tau)\,d\tau\Big)\sin(nt)$$

where $\xi_s(\tau) := p(\tau) - g(u_s(\tau))$. Because g is bounded, it follows that

$$u_s(t) = \frac{s}{n}\sin(nt) + B_s(t),$$

with $|B_s(t)| \le B$ for some constant B independent of s. Next, observe that

$$u_s(\pi) = \frac{(-1)^{n+1}}{n}\Big(\int_0^\pi \sin(n\tau)p(\tau)\,d\tau - \int_0^\pi \sin(n\tau)g(u_s(\tau))\,d\tau\Big)$$

and, by the dominated convergence,

$$\int_0^\pi \sin(n\tau)g(u_s(\tau))\,d\tau \to g(\pm\infty)\int_{\Omega^+}\sin(n\tau)\,d\tau + g(\mp\infty)\int_{\Omega^-}\sin(n\tau)\,d\tau,$$

as $s \to \pm\infty$, where Ω^+ and Ω^- are, as before, the positive and negative sets the function $\psi(t) = \sin(nt)$ over $\Omega := (0,\pi)$. Thus, it is deduced from (1.1) that if $s \gg 0$ then $u_s(\pi)$ and $u_{-s}(\pi)$ have different signs and the conclusion follows.

Remark 2.2. It is not difficult to see that the result is still valid if g is bounded and (1.1) is replaced by the following weaker condition: there exist $u_0, \varepsilon > 0$ such that

$$\begin{aligned} &g(u)\int_{\Omega^+}\psi(x)\,dx + g(-u)\int_{\Omega^-}\psi(x)\,dx - \varepsilon \\ &> \int_\Omega p(x)\psi(x)\,dx \\ &> g(u)\int_{\Omega^-}\psi(x)\,dx + g(-u)\int_{\Omega^+}\psi(x)\,dx + \varepsilon. \end{aligned} \tag{2.2}$$

for all $u \geq u_0$. Using this fact, it is possible to deduce the proof when g is only continuous by an approximation argument(details are left to the reader).

3. Periodic problem: from the scalar equation to systems

A particular situation occurs when the boundary conditions are periodic, because all the eigenvalues of the operator $Lu := -u''$ are multiple, except for the first one $\lambda_0 = 0$, whose corresponding eigenfunction subspace is the set of constant functions. The nonlinear problem related to this case reads

$$u''(t) + g(u(t)) = p(t) \tag{3.1}$$

where $p \in L^2_{\rm loc}(\mathbb{R})$ is T-periodic and $g \in C(\mathbb{R})$ is bounded. It is well known that the eigenfunction associated to the first eigenvalue does not change sign; here, this fact is particularly obvious and hence the Landesman-Lazer condition takes a very simple form:

$$g(+\infty) > \overline{p} := \frac{1}{T}\int_0^T p(t)\,dt > g(-\infty). \tag{3.2}$$

However, shortly before [10] was published, Lazer [11] employed the Schauder fixed point theorem to obtain T-periodic solutions of (3.1) under a slightly different assumption:

(L) There exists $R_0 > 0$ such that

$$g(u) \geq \overline{p} \geq g(-u) \quad \text{for all } u \geq R_0.$$

It is clear that this condition is more general than the Landesman-Lazer condition, because (L) is non-asymptotic, in the sense that, besides the sign, no specific behaviour for g is prescribed as $u \to \pm\infty$. In particular, it might happen that $g(u)$ tends to $\overline{p}$, a situation which is described in the literature as a *vanishing nonlinearity*. Furthermore, observe that, differently to (3.2), the inequalities in (L) are non-strict, although another approximation argument shows that it suffices to prove the result for the strict case, taking for instance $g_k(u) := g(u) + \frac{1}{k}\arctan(u)$.

It is worth mentioning also that the original result in [11] required that g be sublinear instead of bounded; this extension is easily deduced from the fact that sublinearity, together with (L), is sufficient to obtain a priori bounds for the solutions and allows a truncation argument. Summarizing, we have:

Theorem 3.1. *In the above situation, if* (L) *holds then* (3.1) *admits at least one* T*-periodic solution.*

In 1972, Mawhin [14] presented an extension of Lazer's theorem when (3.1) is a system of N equations and $g \in C(\mathbb{R}^N, \mathbb{R}^N)$ is sublinear and satisfies, for each coordinate $j = 1, \dots, N$, the assumption

(A1) There exists $R_0 > 0$ such that $g_j(u) \geq \overline{p}_j \geq g_j(-u)$ for all $u \in \mathbb{R}^N$ such that $u_j \geq R_0$.

As in the scalar case, it may be assumed that g is bounded and smooth. Moreover, without loss of generality, from now on we may also assume that $\overline{p} = 0$. Moreover, it shall follow from the proof that the inequalities in (A1) can be reversed for some (or all) coordinates.

Proposition 3.2. *In the above situation, if* (A1) *holds then system* (3.1) *admits at least one* T*-periodic solution.*

Proof. Let us verify that a shooting type argument can be applied also in this case, although the periodic conditions yield a $2N$-dimensional fixed point problem for the associated Poincaré operator defined by

$$(x, y) \mapsto (u_{xy}(T), u'_{xy}(T))$$

where u_{xy} is the unique solution of (3.1) satisfying the initial conditions

$$u_{xy}(0) = x, \quad u'_{xy}(0) = y \tag{3.3}$$

for arbitrary $x, y \in \mathbb{R}^N$. Equivalently, we define

$$\varphi(x, y) := (y - u'_{xy}(T), u_{xy}(T) - x)$$

and apply the Poincaré-Miranda theorem. Indeed, fix $M > \|p\|_{L^1(0,T)} + T\|g\|_\infty$ and write

$$u'_{xy}(t) - y = \int_0^t [p(s) - g(u_{xy}(s))]\, ds$$

to deduce that

$$|u'_{xy}(t) - y|_\infty < M$$

for all t, where $|\cdot|_\infty$ stands for the maximum norm of $\mathbb{R}^N$. In particular, writing $\varphi = (\varphi_1, \varphi_2)$ it is seen, when $y_j = M$, that $\left(u'_{xy}\right)_j(t) > 0$ for all t and hence $\left(u_{xy}(T)\right)_j > x_j$; that is, $(\varphi_2)_j(x, y) > 0$. In the same way, it follows that if $y_j = -M$ then $(\varphi_2)_j(x, y) < 0$.

Next, observe that if $|y_j| \leq M$ for all j then

$$|u_{xy}(t) - x|_\infty = \Big| \int_0^t u'_{xy}(s)\, ds \Big|_\infty < \int_0^t (M + |y|_\infty)\, ds \leq 2TM := r.$$

Set $R := R_0 + r$ and assume that $x_j = R$, then $(u_{xy})_j(t) \geq R - r \geq R_0$ for all t. In turn, this implies $g_j(u_{xy}(t)) \geq 0$, and consequently,

$$(\varphi_1)_j(x, y) = y_j - \left(u'(T)\right)_j = \int_0^T g_j(u_{xy}(t))\, dt \geq 0.$$

Analogously, if $x_j = -R$ then $(\varphi_1)_j(x, y) \leq 0$. This completes the proof. $\square$

To generalize Mawhin's result, it may be noticed that, if (A1) is strict, then the Brouwer degree of g is well defined and satisfies $\deg_B(g, B_R(0), 0) = 1$ for all R sufficiently large, or eventually -1 if some (an odd number) of the inequalities are reversed. Thus, one might wonder if T-periodic solutions still exist when more rotation is allowed, that is, when the winding number of the field g over large balls is an arbitrary nonzero integer,

$$\deg_B(g, B_R(0), 0) \neq 0 \tag{3.4}$$

for all $R \gg 0$. In particular, this is the case in a result by Nirenberg [17], which represents an accurate extension of the original Landesman-Lazer condition. In order to understand this statement, let us observe that (3.2) can be regarded as two separate conditions:

(1) $g(\pm\infty) \neq \overline{p}$.
(2) The mapping $\phi : S^0 := \{-1, 1\} \to \mathbb{R}$ given by $\phi(\pm 1) := g(\pm\infty)$ wraps around $\overline{p}$, that is, $\phi(-1) - \overline{p}$ and $\phi(1) - \overline{p}$ have different signs.

Recalling that $\overline{p} = 0$, the previous conditions are generalized to a system of N equations as follows. Let S^{N-1} be the unit sphere of $\mathbb{R}^N$ and assume that the (finite) radial limits

$$g_v := \lim_{s\to+\infty} g(sv)$$

exist uniformly for $v \in S^{N-1}$. This implies that g_v is a continuous function of v. In this setting, Nirenberg's conditions read

(A2) $g_v \neq 0$ for all $v \in S^{N-1}$.
(A3) $\deg(\phi) \neq 0$, where $\phi : S^{N-1} \to S^{N-1}$ is defined by $\phi(v) := g_v/|g_v|$.

For the reader's convenience, let us recall that the degree of a continuous mapping $\phi : S^{N-1} \to S^{N-1}$ can be simply defined as the Brouwer degree $\deg_B(\hat{\phi}, B_1(0), 0)$, where $\hat{\phi} : \overline{B_1(0)} \to \mathbb{R}^N$ is an arbitrary continuous extension of ϕ. The following proposition follows from the result in [17]:

Proposition 3.3. *In the above situation, if* (A2) *and* (A3) *hold, then system* (3.1) *admits at least one T-periodic solution.*

It is clear that (A2) and (A3) imply (3.4); however, it was shown in [18] that, unlike for the scalar case, the latter condition alone does not guarantee the existence of T-periodic solutions: more precisely, an example was found of a continuous bounded function g such that $g(u) \neq 0$ for $|u|$ large and satisfying (3.4), for which such solutions do not exist. The authors proposed then a weaker version of Nirenberg's conditions that allows the non-existence of the radial limits g_v, as well as vanishing nonlinearities. Namely, if g is bounded with $g(u) \neq 0$ for $|u|$ large, the existence result in [18] replaces g_v by the limits

$$\hat{g}_v := \lim_{s\to+\infty} \frac{g(sv)}{|g(sv)|},$$

which are assumed to be uniform in $v \in S^{N-1}$. As before, this implies that the mapping $\hat{g} : v \mapsto \hat{g}_v$ is continuous. In this context, the sufficient conditions for existence in [18] read

(A4) $\hat{g}_v \neq 0$ for $v \in S^{N-1}$.
(A5) $\deg(\hat{g}) \neq 0$.

This yields the following result.

Proposition 3.4. *In the above situation, if* (A4) *and* (A5) *hold, then system* (3.1) *admits at least one T-periodic solution.*

Although (A4) and (A5) constitute an improvement with respect to (A2) and (A3), they are not implied by (A1). Remarkably, a geometric non-asymptotic condition was given in [19] which can be seen as an extension of both results and, as before, admits a shooting type argument. For simplicity, assume that $g \in C(\mathbb{R}^N, \mathbb{R}^N)$ is bounded and fix constants M and r as in the previous proof. Set $R > r$ and consider the following condition

(A6) $0 \notin \operatorname{co}(g(B_r(x)))$ for all $x \in \mathbb{R}^N$ such that $|x| = R$, where $\operatorname{co}(A)$ denotes the convex hull of an arbitrary set $A \subset \mathbb{R}^N$.

It is an easy exercise to prove that both (A4) and the non-strict version of (A1) imply (A6). Let us adapt the shooting argument to prove that solutions exist when (A6) and (3.4) are assumed. As a consequence, the existence results in [17] and [18] are deduced.

Proposition 3.5. *In the above situation, if* (A6) *and* (3.4) *hold then system* (3.1) *admits at least one T-periodic solution.*

Proof. We may assume that g is smooth and define the mapping $\varphi(x, y)$ exactly as before. On the one hand, it follows from the previous computations that if $|y_j| = M$ then

$$s\big(u_{xy}(T) - x\big)_j + (1 - s)y_j \neq 0$$

for all $s \in [0, 1]$. On the other hand, it is also seen that if $|y_j| \leq M$ for all j, then $|u_{xy}(t) - x| < r$ for all t which, in turn, implies that if also $|x| = R$, then the mapping

$$s \mapsto \int_0^T g(x + s(u_{xy}(t) - x))\, dt$$

does not vanish for $s \in [0, 1]$. Indeed, this is due to the mean value theorem for vector integrals, namely

$$\overline{\gamma} \in \operatorname{co}(\operatorname{Im}(\gamma)),$$

where $\gamma : [0, T] \to \mathbb{R}^N$ is a continuous curve and $\operatorname{Im}(\gamma)$ denotes the range of γ. In our context, it suffices to consider $\gamma(t) = g(x + s(u_{xy}(t) - x))$ and the conclusion follows from (A6). Summarizing, we have proven that if $U := B_R(0) \times (-M, M)^n \subset \mathbb{R}^{2N}$ then the homotopy

$$h(x, y, s) := \Big(\int_0^T g(x + s(u_{xy}(t) - x))\, dt, s\,(u_{xy}(T) - x) + (1 - s)y \Big)$$

does not vanish on ∂U. This implies that the degree of $h(\cdot, s)$ over U is well defined and does not depend on s. In particular, observe that $\varphi = h(\cdot, 1)$ and hence

$$\deg_B(\varphi, U, 0) = \deg_B(h(\cdot, 0), U, 0)$$

Finally, observe that

$$h(x, y, 0) = (Tg(x), y)$$

and the result is immediately deduced from condition (3.4). $\square$

At this point, it is worth noticing that, if one uses the Leray-Schauder degree theory, then more powerful results can be obtained. For instance, the original problem studied in [14] included first order terms, namely

$$u''(t) + \frac{d}{dt}\nabla F(u(t)) + g(u(t)) = p(t) \tag{3.5}$$

for a C^1 mapping $F : \mathbb{R}^N \to \mathbb{R}$. In order to treat this case, it is useful to adapt the standard continuation method (see e. g. [7]) to this specific situation.

Lemma 3.6. *In the above situation, assume there exist $M, R > 0$ such that*

(1) *If*
$$u''(t) = s\big[p(t) - \frac{d}{dt}\nabla F(u(t)) - g(u(t))\big] \tag{3.6}$$
for some $s \in (0,1)$, then $\|u'\|_{L^2} < M$ and $|\overline{u}| \neq R$.

(2) *$\phi(x) \neq 0$ for $x \in \mathbb{R}^N$ with $|x| = R$, where*
$$\phi(x) := \frac{1}{T}\int_0^T p(t)\,dt - g(x).$$

(3) $\deg(\phi, B_R(0), 0) \neq 0$.

Then system (3.5) *has at least one T-periodic solution.*

Again, for simplicity we may assume that g is bounded and $\overline{p} = 0$, then multiplying (3.6) by $u(t) - \overline{u}$ it is readily seen that all possible T-periodic solutions with $s \in (0,1)$ satisfy

$$\|u'\|_{L^2}^2 < (\|p\|_{L^2} + \sqrt{T}\|g\|_\infty)\|u - \overline{u}\|_{L^2}$$

because

$$\int_0^T \langle \frac{d}{dt}\nabla F(u(t)), u(t) - \overline{u}\rangle\,dt = -\int_0^T \langle \nabla F(u(t)), u'(t)\rangle\,dt = 0.$$

From the Wirtinger and Sobolev inequalities, we deduce that

$$\|u'\|_{L^2} < \frac{T}{2\pi}(\|p\|_{L^2} + \sqrt{T}\|g\|_\infty) := M,$$
$$\|u - \overline{u}\|_\infty < \sqrt{\frac{T}{12}}M := r.$$

Moreover, integrating (3.6) yields

$$\int_0^T g(u(t))\,dt = 0.$$

Thus, the next result, which extends all the preceding ones, is directly deduced from Lemma 3.6.

Theorem 3.7. *In the above situation, if* (A6) *and* (3.4) *hold, then system* (3.5) *has at least one T-periodic solution.*

The result is easily generalized for sublinear g, after appropriately redefining the constant r. However, it is interesting to observe that no growth restrictions are needed in the case of a gradient system, namely, when $g = \nabla G$ for some C^1 mapping $G : \mathbb{R}^n \to \mathbb{R}$. Indeed, assume that F is of class C^2 and strictly convex or strictly concave, that is, the hessian $HF(x)$ is uniformly strictly positive or negative definite. In other words, assume there exists a constant $\alpha > 0$ such that

$$\inf_{v\in S^{N-1}} |\langle HF(x)v, v\rangle| \geq \alpha > 0 \tag{3.7}$$

for all $x \in \mathbb{R}^N$. If u is a T-periodic solution of (3.6), then multiplying the equation by $u'(t)$ we obtain, upon integration,

$$0 = \int_0^T \langle u''(t), u'(t)\rangle\, dt = s\int_0^T \langle p(t) - \frac{d}{dt}\nabla F(u(t)) - g(u(t)), u'(t)\rangle\, dt.$$

When $g = \nabla G$, the integral of the last term in the right-hand side is equal to 0, whence

$$\int_0^T \langle \frac{d}{dt}\nabla F(u(t)), u'(t)\rangle\, dt = \int_0^T \langle p(t), u'(t)\rangle\, dt$$

Employing (3.7) we deduce, from the Cauchy-Schwarz inequality,

$$\alpha\int_0^T |u'(t)|^2\, dt \leq \|p\|_{L^2}\|u'\|_{L^2};$$

that is

$$\|u'\|_{L^2} \leq \frac{\|p\|_{L^2}}{\alpha}.$$

A particular instance of the above case is the complex equation

$$z''(t) + az'(t) + g(\overline{z}(t)) = p(t)$$

where $a \in \mathbb{R}\backslash\{0\}$, g is an analytic function and $\overline{z}$ denotes the conjugate of a number $z \in \mathbb{C}$. The fact that the latter equation is a gradient system for $N = 2$ is due to the Cauchy-Riemann conditions. Alternatively, one may multiply by $\overline{z}'(t)$ the homotopy equation

$$z''(t) = s[p(t) - az'(t) - g(\overline{z}(t))]$$

to obtain

$$\int_0^{2\pi} z''(t)\overline{z}'(t)\, dt + a\int_0^{2\pi} |z'(t)|^2\, dt = \int_0^{2\pi} p(t)\overline{z}'(t)\, dt.$$

Taking now into account that $\frac{d}{dt}|z'(t)|^2 = 2\Re(z''(t)\overline{z}'(t))$, it follows that

$$a\int_0^{2\pi} |z'(t)|^2\, dt = \Re\Big(\int_0^{2\pi} p(t)\overline{z}'(t)\, dt\Big)$$

and hence

$$\|z'\|_{L^2} \leq \frac{\|p\|_{L^2}}{|a|}.$$

A similar situation occurs with the problem

$$z''(t) + a\overline{z}'(t) + g(z(t)) = p(t), \tag{3.8}$$

where g is an analytic function and $a \in \mathbb{C}\backslash\{0\}$. Here, if $z \in C^2(\mathbb{R}, \mathbb{C})$ is T-periodic and satisfies

$$z''(t) = s[p(t) - a\overline{z}'(t) - g(z(t))]$$

for $s \in (0, 1]$, then multiplying by $z'(t)$ we obtain

$$a\int_0^T |z'(t)|^2\, dt = \int_0^T p(t)z'(t)\, dt,$$

whence

$$\|z'\|_{L^2} \le \frac{\|p\|_{L^2}}{|a|}.$$

In both cases, it is clear that (A6) cannot hold for arbitrary g; for instance, if $p = 0$ then all possible T-periodic solutions are constant and no solutions exist when $g(z) = e^z$. However, it is readily verified that (A4) and (A5) are fulfilled when g is a polynomial of degree $k \ge 1$ with leading coefficient a_k, since

$$\frac{g(sz)}{|g(sz)|} \to \frac{a_k z^k}{|a_k|} := \psi(z), \quad \text{as } s \to +\infty$$

uniformly for $|z| = 1$ and $\deg(\psi) = k$. In particular, letting $p = 0$ the solutions are the roots of g and, according to Mawhin in [15], the previous result may be interpreted as a generalization of the fundamental theorem of algebra. It is easy to extend the result for arbitrary $p = p(t, z)$ continuous and T-periodic in t, provided for example that p is sublinear with respect to z.

Higher order eigenvalues: the Lazer-Leach theorem. This section is devoted to the case in which resonance occurs at a higher order eigenvalue, leading to the also celebrated Lazer-Leach case introduced in [13]. If for convenience we fix the period $T = 2\pi$, then the eigenvalues of the operator $Lu = -u''$ are simply given by $\lambda_n = n^2$ and, when $n > 0$, the eigenfunctions form the subspace spanned by $\{\cos(nt), \sin(nt)\}$. The original result in [13] states, analogously to the Landesman-Lazer result, that if the projection of p to this space is small, then 2π-periodic solutions exist, where the smallness assumption is related to the (finite) limits $g(\pm\infty)$. In more precise terms, consider the problem

$$u''(t) + n^2 u(t) + g(u(t)) = p(t) \tag{3.9}$$

where p is 2π-periodic and $g \in C(\mathbb{R}, \mathbb{R})$ is bounded. The projection of p can be expressed in terms of its nth order Fourier coefficients. For convenience, we may use the complex notation and set $z_p := \int_0^{2\pi} p(t)e^{-int}\, dt$, then the Lazer-Leach condition reads

$$|z_p| < 2|g(+\infty) - g(-\infty)|. \tag{3.10}$$

Theorem 3.8 (Landesman-Lazer)**.** *In the above situation, if* (3.10) *holds, then problem* (3.9) *has at least one* 2π*-periodic solution.*

Again, the problem admits a simple approach by using of the Poincaré operator. The following sketch is a simplified version of the argument introduced in [8]. Assume that g is smooth and define as before $u = u_{xy}$ as the solution of (3.9) with initial condition (3.3), then

$$u_{xy}(t) = C_{xy}(t)\cos(nt) + S_{xy}(t)\sin(nt),$$

where

$$C_{xy}(t) := x - \int_0^t \frac{\sin(ns)}{n}[p(s) - g(u_{xy}(s))]\, ds,$$
$$S_{xy}(t) := \frac{y}{n} + \int_0^t \frac{\cos(ns)}{n}[p(s) - g(u_{xy}(s))]\, ds.$$

Next, we write $(x, \frac{y}{n})$ in polar coordinates (ρ, θ) to obtain

$$u_{xy}(t) = \rho\cos(nt - \theta) + \int_0^t \frac{\sin(n(t-s))}{n}[p(s) - g(u_{xy}(s))]\, ds,$$

$$u'_{xy}(t) = -n\rho\sin(nt-\theta) + \int_0^t \cos(n(t-s))[p(s)-g(u_{xy}(s))]\,ds$$

Thus, we can define the complex function for $z = x+iy$ given by

$$F(z) = [u'_{xy}(2\pi) - u'_{xy}(0)] + in[u_{xy}(0) - u_{xy}(2\pi)]$$

then

$$F(\rho,\theta) = \int_0^{2\pi} [p(s)-g(u_{xy}(s))]e^{ins}\,ds = z_p - \int_0^{2\pi} g(u_{xy}(s))e^{ins}\,ds.$$

It is convenient to write the latter term as

$$\begin{aligned}\int_0^{2\pi} g(u_{xy}(s))e^{ins}\,ds &= e^{i\theta}\int_0^{2\pi} g(u_{xy}(s))e^{i(ns-\theta)}\,ds\\ &= e^{i\theta}\int_0^{2\pi} g(\rho\cos(ns-\theta)+B(s))e^{i(ns-\theta)}\,ds\end{aligned}$$

where, since g is bounded, $B(t)$ is bounded independently of r and θ. Now a straightforward application of the dominated convergence theorem shows that

$$F(\rho,\theta) \to z_p - e^{i\theta}\Big(\int_{\Omega_\theta^+} g(+\infty)e^{i(ns-\theta)}\,ds + \int_{\Omega_\theta^-} g(-\infty)e^{i(ns-\theta)}\,ds\Big)$$

as $\rho \to +\infty$, uniformly on θ, where $\Omega_\theta^\pm \subset [0,2\pi]$ are, respectively, the positive and negative sets of the function $\cos(nt-\theta)$. Since it is clear that

$$\begin{gathered}\int_{\Omega_\theta^+} \cos(ns-\theta)\,ds = -\int_{\Omega_\theta^-} \cos(ns-\theta)\,ds = 2,\\ \int_{\Omega_\theta^+} \sin(ns-\theta)\,ds = \int_{\Omega_\theta^-} \sin(ns-\theta)\,ds = 0,\end{gathered}$$

we conclude that

$$F(\rho,\theta) \to z_p - 2e^{i\theta}[g(+\infty)-g(-\infty)]$$

as $\rho \to +\infty$, uniformly on θ. Together with (3.10), this implies that if we fix $\rho = R \gg 0$, then z_p belongs to the interior of the disk of radius $2|g(+\infty)-g(-\infty)|$. In other words, the index of the curve $\gamma(\theta) := F(R,\theta)$ is equal to 1 which, in turn, implies that F has at least one zero z with $|z| < R$. As before, the general result follows now by an approximation argument.

Different extensions of the Lazer-Leach result were also introduced by several authors; for instance, it is immediate to verify that the result is still true if in (3.10) the quantity $|g(+\infty)-g(-\infty)|$ is replaced by

$$\liminf_{u\to+\infty} g(u) - \limsup_{u\to-\infty} g(u)$$

or

$$\liminf_{u\to-\infty} g(u) - \limsup_{u\to+\infty} g(u).$$

More general situations are analyzed in [4, 6, 9], among other works. It is worth noticing, however, that non-asymptotic conditions are more scarce; see e.g. [2, 5]. Extensions to systems are also more difficult than the case of resonance at the first eigenvalue; a quite general account of this situation for systems of delayed differential equations was recently given in [3].

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References

[1] S. Ahmad A. Lazer, J. Paul; *Elementary critical point theory and perturbations of elliptic boundary value problems at resonance*, Indiana Univ. Math. J. 25 (1976), 933–944.

[2] P. Amster, P. De Nápoli; *Non-asymptotic Lazer-Leach type conditions for a nonlinear oscillator*, Discrete and Continuous Dynamical Systems, Series A 29 No. 3, (2011), 757–767.

[3] P. Amster, J. Epstein, A. Sanjuán; *Periodic Solutions for Systems of Functional-Differential Semilinear Equations at Resonance*. To appear in Topological Methods in Nonlinear Analysis.

[4] C. Fabry, A. Fonda; *Nonlinear resonance in asymmetric oscillators*, J. Differential Equations 147 (1998), 58–78.

[5] C. Fabry, C. Franchetti; *Nonlinear equations with growth restrictions on the nonlinear term*. J. Differential Equations 20 (1976), No. 2, 283–291.

[6] C. Fabry, J. Mawhin; *Oscillations of a forced asymmetric oscillator at resonance*. Nonlinearity 13 (2000), No. 3, 493–505.

[7] R. Gaines, J. Mawhin; *Coincidence Degree and Nonlinear Differential Equations*. vol. 568 of Lecture Notes in Mathematics, Springer, 1977.

[8] S. Hastings, B. McLeod; *Short proofs of results by Landesman, Lazer, and Leach on problems related to resonance*, Differential and Integral Equations 24 No. 5/6 (2011), 435–441.

[9] A. M. Krasnosel'skii, J. Mawhin; *Periodic Solutions of Equations with Oscillating Nonlinearities*, Mathematical and Computer Modelling 32, No. 11 (2000), 1445–1455.

[10] E. Landesman, A. Lazer; *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech. 19 (1970), 609–623.

[11] A. Lazer; *On Schauder's Fixed point theorem and forced second-order nonlinear oscillations*, J. Math. Anal. Appl. 21 (1968) 421–425.

[12] A. Lazer; *A second look at the first result of Landesman-Lazer type*, Electron. J. Differ. Equ. Conf. 5 (2000), 113–119.

[13] A. Lazer, D. Leach; *Bounded perturbations of forced harmonic oscillators at resonance*, Ann. Mat. Pura Appl. 82 (1969), 49–68.

[14] J. Mawhin; *An extension of a theorem of A. C. Lazer on forced nonlinear oscillations*, Journal of Mathematical Analysis and Applications 40 No. 1 (1972), 20–29.

[15] J. Mawhin: *Periodic solutions of some planar non-autonomous polynomial differential equations*. Differential and Integral Equations 7, No. 4 (1994), 1055–1061.

[16] J. Mawhin; *Landesman-Lazer conditions for boundary value problems: A nonlinear version of resonance*. Bol. de la Sociedad Española de Mat. Aplicada 16 (2000), 45–65.

[17] L. Nirenberg; *Generalized degree and nonlinear problems, Contributions to nonlinear functional analysis*, Ed. E. H. Zarantonello, Academic Press New York (1971), 1–9.

[18] R. Ortega, L. Sánchez; *Periodic solutions of forced oscillators with several degrees of freedom*, Bull. London Math. Soc. 34 (2002), 308–318.

[19] D. Ruiz, J. R. Ward Jr.; *Some notes on periodic systems with linear part at resonance*, Discrete and Continuous Dynamical Systems Series A 11, No. 2–3 (2004), 337–350.

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