2021 UNC Greensboro PDE Conference,
Electronic Journal of Differential Equations, Conference 26 (2022), pp. 139–149.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

WELLPOSEDNESS OF KELLER-SEGEL SYSTEMS IN MIXED NORM SPACES

TIMOTHY ROBERTSON

Abstract. We study the well-posedness of the Cauchy problem for the Keller-Segal system in the setting of mixed norm spaces. We prove existence of mild solutions in scaling invariant spaces and uniqueness in a special case. These results allow for existence and uniqueness when the initial data has anisotropic properties. In particular, persistence of anisotropic properties under the evolution is demonstrated which could be of biological interest.

1. Introduction

In this article, we study the following Cauchy problem for the Keller-Segel system of parabolic equations

$$\begin{aligned} u_t &= \Delta u - \nabla \cdot (u \nabla v), \quad \text{in } \mathbb{R}^n \times (0,T), \\ v_t &= \Delta v - A(t)v + B(x,t)u \quad \text{in } \mathbb{R}^n \times (0,T), \\ u(0,\cdot) &= u_0(\cdot) \quad \text{in } \mathbb{R}^n, \\ v(0,\cdot) &= v_0(\cdot) \quad \text{in } \mathbb{R}^n, \end{aligned} \tag{1.1}$$

where in (1.1) the functions $u, v : \mathbb{R}^n \times (0,T) \to \mathbb{R}$ are unknown solutions with $T \in (0,\infty]$ and $n \in \mathbb{N}$, the functions $u_0, v_0 : \mathbb{R}^n \to (0,\infty)$ are given measurable initial data, and the functions $A : (0,T) \to [0,\infty)$ and $B : \mathbb{R}^n \times (0,T) \to (0,\infty)$ are given and measurable. The Keller-Segel system was proposed in [7] to describe chemotactic aggregation of cellular slime molds which move preferentially towards relatively high concentrations of a chemical secreted by the amoebae themselves. In this context, $u(x,t)$ represents the cell density of the slime molds at position x and time t, and similarly the concentration of the chemical substance at position x and time t is represented by $v(x,t)$.

Our goal is to develop the existence and uniqueness of solutions of (1.1) in the setting of anisotropic spaces. The motivation of the study comes from applications where initial data and solutions can concentrate and behave differently in different spatial variables. Mathematically, our results demonstrate the persistence of anisotropic properties under the evolution of the Keller-Segal system, which does not seem to be trivial. To put our study into perspective, we recall the definition of the anisotropic (mixed norm) space. For a given $\vec{p} = (p_1, \ldots, p_n) \in [1,\infty)^n$,

2020 *Mathematics Subject Classification.* 35A01, 35A02, 92C17.
Key words and phrases. Mixed norm spaces; Keller-Segel model; semigroup methods.

Published August 25, 2022.

the mixed norm Lebesgue space $L_{\vec{p}}(\mathbb{R}^n)$ is the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the mixed norm

$$\|f\|_{\vec{p}} = \Big(\int_{\mathbb{R}} \cdots \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f(x_1, x_2, \dots, x_n)|^{p_1} dx_1 \Big)^{\frac{p_2}{p_1}} dx_2 \Big)^{\frac{p_3}{p_2}} \cdots dx_n \Big)^{1/p_n} < \infty.$$

A similar definition can be formed if $p_i = \infty$ for some $i = 1, 2, \dots, n$. Similarly, for $q \in [1, \infty)$ and $\vec{p} = (p_1, \dots, p_n) \in [1, \infty)^n$, we write $L^q((0,T); L_{\vec{p}}(\mathbb{R}^n))$ to represent the space consisting of all $u : (0,T) \to L_{\vec{p}}(\mathbb{R}^n)$ such that

$$\|u\|_{T,q,\vec{p}} = \Big(\int_0^T \|u(\cdot, t)\|_{\vec{p}}^q dt \Big)^{1/q} < \infty.$$

The definition of mixed norm Sobolev space follows naturally from the definition of the mixed norm.

Definition 1.1. Let $\vec{I} = (I_1, \dots, I_n) \in [1, \infty)^n$. The Sobolev space $H^{\vec{I}}(\mathbb{R}^n)$ consists of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the weak derivative Df exists and

$$\|f\|_{H^{\vec{I}}(\mathbb{R}^n)} := \|f\|_{L_{\vec{I}}(\mathbb{R}^n)} + \|Df\|_{L_{\vec{I}}(\mathbb{R}^n)} < \infty.$$

Now, the mild solutions to (1.1) are defined as follows.

Definition 1.2. A pair of functions $(u, v) : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}^2$ is said to be a mild solution to (1.1) if

$$\begin{gathered} u(x,t) = e^{t\Delta u_0} - \int_0^t \nabla \cdot e^{(t-s)\Delta}(u \cdot \nabla v) ds, \\ v(x,t) = e^{-t} e^{t\Delta} v_0 + \int_0^t e^{-\bar{A}(t-s)} e^{(t-s)\Delta}(Bu(s)) ds \end{gathered}$$

for all $t \in (0, T)$, provided the integrals are well-defined.

We note that the following assumption on the coefficients A, B are used throughout this article: there is $\Lambda > 0$ such that The coefficients A, B are assumed to satisfy the condition that there is $\Lambda > 0$ such that

$$0 \le \bar{A}(t) := \int_0^t A(s)\, ds \quad \text{and} \quad 0 \le B(x,t) \le \Lambda, \quad \forall (x,t) \in \mathbb{R}^n \times (0,T). \tag{1.2}$$

We now state our main results. Our first result is on the local time existence of solutions in the mixed-norm space.

Theorem 1.3. *Let $\vec{I} = (I_1, \dots, I_n), \vec{J} = (J_1, \dots, J_n) \in (1, \infty)^n$ satisfy*

$$\frac{1}{2} \sum_{k=1}^n \frac{1}{I_k} = \sum_{k=1}^n \frac{1}{J_k} = 1.$$

Also let $\vec{r} = (r_1, \dots, r_n), \vec{R} = (R_1, \dots, R_n)$ be in $(1, \infty)^n$ and $1 < q < Q < \infty$ such that $1 < r_k < R_k < \infty$, $I_k < r_k$, $J_k < R_k$ for $k = 1, 2, \dots, n$ and

$$\frac{1}{q} + \frac{1}{2} \sum_{k=1}^n \frac{1}{r_k} = 1, \quad \frac{1}{Q} + \frac{1}{2} \sum_{k=1}^n \frac{1}{R_k} = \frac{1}{2}. \tag{1.3}$$

Then under assumption (1.2) and for a pair functions $u_0, v_0 : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$u_0 \in L_{\vec{I}}(\mathbb{R}^n), \quad v_0 \in H^{\vec{J}}(\mathbb{R}^n)$$

there exist $T > 0$ and a pair of functions

$$u \in L^q((0,T); L_{\vec{r}}(\mathbb{R}^n)) \quad and \quad v \in L^Q((0,T); H^{\vec{R}}(\mathbb{R}^n))$$

that solves (1.1) *in the mild sense.*

Observe that the conditions on $q, Q, \vec{r}$ and $\vec{R}$ in Theorem 1.3 gives invariance under the heat scaling. Precisely, for a given pair of solutions (u,v) and for $\lambda > 0$, let

$$u^\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^2 t), \quad v^\lambda(t,x) = v(\lambda x, \lambda^2 t), \quad x \in \mathbb{R}^n,\ t \in (0, T/\lambda).$$

Then

$$\|u^\lambda\|_{T/\lambda,q,\vec{r}} = \|u\|_{T,q,\vec{r}}, \quad \|\nabla v^\lambda\|_{T/\lambda,Q,\vec{R}} = \|\nabla v\|_{T,Q,\vec{R}}, \quad \forall \lambda > 0 \tag{1.4}$$

if and only if

$$1 = \frac{1}{q} + \frac{1}{2}\sum_{k=1}^n \frac{1}{r_k}, \quad \frac{1}{2} = \frac{1}{Q} + \frac{1}{2}\sum_{k=1}^n \frac{1}{R_k}.$$

Observe that the condition (1.4) must be the correct scaling condition since (u_λ, v_λ) is a solution of (1.1) when $A(t)$ is replaced by $A(\lambda^2 t)/\lambda^2$ and $B(x,t)$ is replaced by $B(\lambda x, \lambda^2 t)$. For a more extensive discussion of scaling see [5].

Our second result Theorem 1.4 shows uniqueness of mild solutions under a special case of Theorem 1.3.

Theorem 1.4. *Suppose $\vec{I} = (I_1, \dots, I_n), \vec{J} = (J_1, \dots, J_n)$ in $(1,\infty)^n$ satisfy $1 = \sum_{k=1}^n 1/J_k$ with $I_k = \frac{1}{2} J_k$ for $k = 1, \dots, n$ and*

$$u_0 \in L_{\vec{I}}(\mathbb{R}^n), \quad \nabla v_0 \in L_{\vec{J}}(\mathbb{R}^n).$$

If (u,v) is a mild solution of (1.1) *and satisfies*

$$u \in C((0,T); L_{\vec{I}}(\mathbb{R}^n)), \quad \nabla v \in C((0,T); L_{\vec{J}}(\mathbb{R}^n)) \tag{1.5}$$

then (u,v) is the only mild solution that satisfies (1.5) *on $\mathbb{R}^n \times (0,T)$.*

The wellposedness of the Keller-Segel model in its various forms has been extensively studied in the literature. Perhaps the most notable result thus far is the classical result that for $n = 2$ the Keller-Segel system has a solution if and only if the critical mass is less than 8π, as discussed in [3] and [4]. Unfortunately, the problem of existence is not so clear when $n \geq 3$. We do not present an exhaustive review here and refer interested readers to the recent review paper [1]. However, we do highlight several pertinent results. Global existence for small initial data $u_0 \in L_w^{n/2}$ and $v_0 \in BMO$ was shown in [8]. Small time existence was shown for initial data $u_0 \in L^{n/2}$ and $v_0 \in H^{1,n/2}$ in [9]. Our results extend the latter result to mixed norm spaces using techniques established in [12, 13].

The proof of Theorem 1.3 follows from the standard fixed point argument using Picard's iteration technique. To implement this method, we need to derive several estimates of the heat semi-group in mixed norm spaces which could be of independent interest. In Section 2 we introduce and review several analysis results and estimates on Sobolev imbedding theorems and semi-groups in mixed-norm spaces. In particular, in Lemma 2.2 below, the smoothing estimate of the heat semi-group in mixed-norm spaces is introduced, and this result seems to be new. The proof of Lemma 2.2 follows from the Marcinkiewicz interpolation theorem. The proof of Theorem 1.3 is then given in Section 3. To prove Theorem 1.4, we take the difference between the two solutions and then control it. The challenging part is to control the nonlinear terms in the mixed-norm spaces. Delicate details with algebra in mixed-norms are then required and all are presented in Section 4.

2. Preliminary inequalities and estimates in mixed-norm spaces

The proofs of Theorems 1.3, 3.1 rely on several technical lemmas stated below. The proof of Lemma 2.1 is given in [12] so it is omitted here.

Lemma 2.1. *Let $\vec{p} = (p_1, \dots, p_n)$ and $\vec{q} = (q_1, \dots, q_n)$ be in $[1,\infty]^n$ such that $p_k \le q_k$ for $k = 1, 2, \dots, n$. Then there exists $C = C(\vec{p}, \vec{q}, n) > 0$ such that*

$$\begin{gathered} \|e^{t\Delta} f\|_{\vec{q}} \le C t^{-\frac{1}{2}\sum_{k=1}^{n}(\frac{1}{p_k}-\frac{1}{q_k})} \|f\|_{\vec{p}}, \\ \|\nabla e^{t\Delta} f\|_{\vec{q}} \le C t^{-\frac{1}{2}-\frac{1}{2}\sum_{k=1}^{n}(\frac{1}{p_k}-\frac{1}{q_k})} \|f\|_{\vec{p}}, \end{gathered} \tag{2.1}$$

for all $f \in L_{\vec{p}}(\mathbb{R}^n)$ and for all $t > 0$.

Lemma 2.2 is an interpolation result yielding control of norms of semi-group operators. Here we follow the approach of the recent work [13, Lemma 3.2]. The proof is included for completeness.

Lemma 2.2. *Let $q \in (1,\infty)$, $\vec{p} = (p_1, \dots, p_n)$ and $\vec{r} = (r_1, \dots, r_n)$ be in $(1,\infty)^n$ satisfying $p_k < r_k$ for $k = 1, \dots, n$ and*

$$\frac{1}{q} = \frac{l}{2} + \frac{1}{2}\sum_{k=1}^{n}(\frac{1}{p_k} - \frac{1}{r_k}).$$

Then for $l = 0, 1$ there is $C_l = C(\vec{p}, q, n, l) > 0$ such that

$$\Big(\int_0^t \|D^l e^{s\Delta} f\|_{\vec{r}}^q ds\Big)^{1/q} \le C_l \|f\|_{\vec{p}}, \quad \text{for all } f \in L_{\vec{p}}(\mathbb{R}^n).$$

Proof. Let $\vec{p'} = (p_1, \dots, p_{n-1})$. To employ the Marcinkiewicz interpolation theorem consider the space

$$X = \{f : \text{measurable } f : \mathbb{R} \to L_{\vec{p'}}(\mathbb{R}^{n-1})\},$$

with $\|\cdot\|_X := \|\cdot\|_{L_{\vec{p'}}}$, and the operator $T_l : L_{\vec{r}}(X) \to L_w^\alpha$ defined by

$$T_l f : f \to \|D^l e^{t\Delta} f\|_{\vec{r}},$$

where L_w^α is the weak Lebesgue space with parameter α. For fixed l Lemma 2.1 yields

$$\|T_l(f)\|_{L_w^\alpha} < \infty$$

if

$$\frac{1}{\alpha} = \frac{l}{2} - \frac{1}{2}\sum_{k=1}^{n}\Big(\frac{1}{p_k} - \frac{1}{r_k}\Big).$$

We define

$$\alpha(p_n) = \Big(\frac{l}{2} - \frac{1}{2}\sum_{k=1}^{n-1}\Big(\frac{1}{p_k} - \frac{1}{r_k}\Big) + \frac{1}{2r_n} - \frac{1}{2p_n}\Big)^{-1}.$$

Note that $\hat{p}$ and $\tilde{p}$ may be chosen such that

$$1 < \hat{p} < p_n < \tilde{p} < r_n,$$

and by an appropriate choice of $\theta \in (0,1)$ the equality

$$\frac{1}{p_n} = \frac{1-\theta}{\hat{p}} + \frac{\theta}{\tilde{p}}$$

holds. Clearly,

$$\frac{1}{q} = \frac{1}{\alpha(p_n)} = \frac{1-\theta}{\alpha(\hat{p})} + \frac{\theta}{\alpha(\tilde{p})}.$$

Since T_l is of weak type $(\hat{p}, \alpha(\hat{p}))$ and $(\tilde{p}, \hat{\tilde{p}})$, by Marcinkiewicz interpolation T_l must be of strong type (p_n, q). Observing that

$$\|T_l\|_q \le C\|f\|_{L^{p_n}(X)} = C\|f\|_{\vec{p}}$$

completes the proof. □

Finally, this section concludes by recalling several function space inequalities that will be used in the uniqueness proof. Lemma 2.3 generalizes Hölder's inequality, while Lemma 2.4 generalizes Young's inequality to the Lorentz space setting. Recall that $\mathcal{L}^{p,q}$ is the Lorentz space with parameters p and q, with norm

$$\|f\|_{\mathcal{L}^{p,q}} = \Big(\int_0^\infty p^{1/q} s^{q-1} m(\{x : |f(x)| \ge s\})^{\frac{q}{p}} ds\Big)^{1/q}.$$

The proof for both lemmas is given in [10].

Lemma 2.3. *Let $q, Q \in [1,\infty]$ and $p, P \in (1,\infty)$ such that $\frac{1}{p} + \frac{1}{P} < 1$. Then there exists $C = C(p,P,q,Q) > 0$ such that*

$$\|fg\|_{\mathcal{L}^{\frac{1}{p}+\frac{1}{P},\min\{q,Q\}}} \le C\|f\|_{\mathcal{L}^{p,q}}\|g\|_{\mathcal{L}^{P,Q}}$$

for $f \in \mathcal{L}^{p,q}$ and $g \in \mathcal{L}^{P,Q}$.

Lemma 2.4. *Let $z, Z \in [1,\infty]$ and $p, q \in (1,\infty)$ such that $1 < \frac{1}{p} + \frac{1}{q} < 2$. Then for r defined by*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$$

there exists $C = C(p,q,z,Z) > 0$ such that

$$\|f * g\|_{\mathcal{L}^{r,\min\{z,Z\}}} \le C\|f\|_{\mathcal{L}^{p,z}}\|g\|_{\mathcal{L}^{p,Z}}.$$

for $f \in \mathcal{L}^{p,z}$ and $g \in \mathcal{L}^{q,Z}$.

The last lemma is a special case of the Sobolev Embedding Theorem in the mixed norm setting found in [2].

Lemma 2.5. *Let $\vec{I}, \vec{J} \in (1,\infty)^n$ and $\vec{p} = (p_1, \dots, p_n)$ be defined by $p_k = (\frac{1}{I_k} + \frac{1}{J_k})^{-1}$. Then there exists a constant $C = C(n, \vec{I}, \vec{J}) > 0$ such that*

$$\|f\|_{\vec{I}} \le C\Big[\|\nabla f\|_{\vec{p}} + \|f\|_{\vec{p}}\Big]$$

for all $f \in H^{\vec{p}}(\mathbb{R}^n)$.

3. Proof of Theorem 1.3

The proof of Theorem 1.3 combines Picard iteration with estimates of the heat semigroup. Concluding the iteration argument requires the following elementary lemma, the proof of which is provided in the appendix.

Lemma 3.1. *Let $a_1, b_1, C, K > 0$ be given real numbers. Suppose $0 < a_1 < \frac{(b_1C-1)^2}{4BC^2}$ and $0 < b_1 < \frac{1}{C}$. Then the sequences $\{a_n\}_n$ and $\{b_n\}_n$ defined recursively by*

$$\begin{aligned} a_{n+1} &= a_1 + Ca_nb_n \\ b_{n+1} &= b_1 + CKa_n \end{aligned} \tag{3.1}$$

converge.

We now turn to the proof of Theorem 1.3.

Proof. We define $\vec{p} = (p_1, \dots, p_n)$ by $p_k = (\frac{1}{r_k} + \frac{1}{R_k})^{-1}$ for $k = 1, \dots, n$ and z by $z = (\frac{1}{q} + \frac{1}{Q})^{-1}$. Let $a_1 = \|e^{t\Delta}u_0\|_{q,\vec{r}}$. Then it follows from Lemma 2.2 that

$$a_1 \le N\|u_0\|_{\vec{p}}$$

for $N = N(n, \vec{r}, q)$. Let

$$\begin{gathered} u_{m+1} = e^{t\Delta u_0} - \int_0^t \nabla \cdot e^{(t-s)\Delta}(u_m \cdot \nabla v_m)ds, \\ v_{m+1}(x,t) = e^{-t}e^{t\Delta}v_0 + \int_0^t e^{-\bar{A}(t-s)}e^{(t-s)\Delta}(Bu_m(s))ds. \end{gathered}$$

Note that the initial data u_0 and v_0 are the first terms of the sequences $\{u_m\}_{m=0}^\infty$ and $\{v_m\}_{m=0}^\infty$, respectively. Applying Minkowski's inequality and Lemma 2.1 yields

$$\begin{aligned} \|\int_0^t \nabla \cdot e^{(t-s)\Delta}(u_m\nabla v_m)ds\|_{\vec{r}} &\le \int_0^t \|\nabla \cdot e^{(t-s)\Delta}(u_m\nabla v_m)\|_{\vec{r}}ds \\ &\le \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2}\sum_{k=1}^n(\frac{1}{p_k}-\frac{1}{r_k})}\|u_m\nabla v_m\|_{\vec{p}}ds \\ &\le \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2}\sum_{k=1}^n\frac{1}{R_k}}\|u_m\|_{\vec{r}}\|\nabla v_m\|_{\vec{R}}ds \end{aligned}$$

Recalling that

$$\frac{1}{q} = \frac{1}{2} - \frac{1}{2}\sum_{k=1}^n\Big(\frac{1}{p_k} - \frac{1}{r_k}\Big),$$

the Hardy-Littlewood-Sobolev inequality and Hölder's inequality give

$$\begin{aligned} \Big\|\int_0^t \nabla \cdot e^{(t-s)\Delta}(u_m\nabla v_m)ds\Big\|_{q,\vec{r}} &\le \Big\|\int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2}\sum_{k=1}^n\frac{1}{R_k}}\|u_m\|_{\vec{r}}\|\nabla v_m\|_{\vec{R}}ds\Big\|_q \\ &\le C\|\|u_m\|_{\vec{r}}\|\nabla v_m\|_{\vec{R}}\|_z \\ &\le C\|u_m\|_{q,\vec{r}}\|\nabla v_m\|_{Q,\vec{R}} \end{aligned}$$

for some $C = C(n, q, Q, \vec{R})$. Letting $a_m = \|u_m\|_{q,\vec{r}}$ and $b_m = \|\nabla v_m\|_{Q,\vec{R}}$ it follows that

$$\|u_{m+1}\|_{q,\vec{r}} \le a_1 + C\|u_m\|_{q,\vec{r}}\|\nabla v_m\|_{Q,\vec{R}} \le a_1 + Ca_mb_m.$$

Similarly, we observe that

$$b_1 = \|\nabla e^{-\bar{A}t}e^{t\Delta}v_0\|_{Q,\vec{R}} = \big\|e^{-\bar{A}t}\nabla e^{t\Delta}v_0\big\|_{Q,\vec{R}} \le \|\nabla e^{t\Delta}v_0\|_{Q,\vec{R}} \le N\|v_0\|_{\vec{P}}.$$

Using the same estimation scheme as above we obtain that

$$\|v_{m+1}\|_{Q,\vec{R}} \le b_1 + C\Lambda \|u_m\|_{q,\vec{r}}.$$

In detail

$$\begin{aligned}
\|v_{m+1}\|_{Q,\vec{R}} &\le b_1 + \Big\| \int_0^t \|\nabla e^{-\bar{A}(t-s)} e^{(t-s)\Delta}(Bu_m)\|_{\vec{R}} ds \Big\|_Q \\
&\le b_1 + \Lambda C \Big\| \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2}\sum_{k=1}^n (\frac{1}{r_k}-\frac{1}{R_k})} \|u_m\|_{\vec{r}} ds \Big\|_Q \\
&\le b_1 + \Lambda C \|u_m\|_{q,\vec{r}} \\
&\le b_1 + \Lambda C a_m.
\end{aligned}$$

where the Hardy-Littlewood inequality justifies the second inequality. The constant C may not be the same constant as in the estimate for u_m, but we take C to be the maximum of the two constants.

Applying Lemma 3.1 gives the convergence of $\{\|u_m\|_{q,\vec{r}}\}_m$ and $\{\|\nabla v_m\|_{Q,\vec{R}}\}_m$. Let

$$X = \sup_m \|u_m\|_{T^*,q,\vec{r}}, \quad Y = \sup_m \|\nabla v_m\|_{T^*,Q,\vec{R}}.$$

Observe that for $T = T^* > 0$ small enough $X < \frac{1}{4C^2B}$ and $Y < \frac{1}{4C}$. Thus, for each $m \ge 2$,

$$\begin{aligned}
\|u_{m+1} - u_m\|_{T^*,q,\vec{r}} &= \| \int_0^t \nabla \cdot e^{(t-s)\Delta}(u_m \nabla v_m - u_{m-1}\nabla v_{m-1}) ds \|_{T^*,\vec{r}} \\
&\le CY\|u_m - u_{m-1}\|_{T^*,q,\vec{r}} + CX\|\nabla v_m - \nabla v_{m-1}\|_{T^*,Q,\vec{R}}) \\
&\le CY\|u_m - u_{m-1}\|_{T^*,q,\vec{r}} + C^2BX\|u_m - u_{m-1}\|_{T^*,q,\vec{r}}) \\
&\le \frac{1}{2}\|u_m - u_{m-1}\|_{T^*,q,\vec{r}}.
\end{aligned}$$

Clearly, $\{u_m\}_m$ is Cauchy in $L^q((0,T^*), L_{\vec{r}}(\mathbb{R}^n))$, which in turn implies that $\{v_m\}_m$ is Cauchy in $L^Q((0,T^*), L_{\vec{R}}(\mathbb{R}^n))$. Replacing u_m with its limit in the above estimate and using a similar strategy for v_m shows that the limits are indeed mild solutions. Recall that this scheme holds provided that $0 < b_1 < \frac{1}{C}$ and $0 < a_1 < \frac{(b_1C-1)^2}{4\Lambda C^2}$, which holds if T^* is chosen sufficiently small. Although there are many choices, it suffices for to T^* small enough that $b_1 < \frac{1}{2C}$ and $a_1 < \frac{1}{16\Lambda}$. □

To demonstrate the utility of this result we now give a short example.

Example 3.2. Choose $I_k > 1$ such that $\sum_{k=1}^n \frac{1}{I_k} < 4$ and $I_k \ne I_j$ for $k \ne j$. Then we have

$$q = \Big(1 - \frac{1}{4}\frac{1}{I_k}\Big)^{-1} > 1$$

and by Theorem 1.3 the existence of solutions

$$u \in L^q((0,T); L_{2\vec{I}}(\mathbb{R}^n)) \quad v \in L^{2q}((0,T); H^{4\vec{I}}(\mathbb{R}^n))$$

for initial data $u_0 \in L_{\vec{I}}(\mathbb{R}^n)$ and $v_0 \in H^{2\vec{I}}(\mathbb{R}^n)$. This example case compares naturally to [9, Theorem 1], but covers a distinctly different class of initial data.

4. Proof of uniqueness

In this section the second main result Theorem 1.4 is proved. The idea of the proof is to obtain an inequality of the form

$$\|u_2 - u_1\|_{T,p,\vec{I}} \le F(T)\|u_2 - u_1\|_{T,p,\vec{I}}$$

where $F(T)$ is a continuous function of T and u_1, u_2 are any two solutions to (1.1). Provided $F(T) < 1$ for small enough T, we obtain uniqueness for small time. Extending the local uniqueness to the entire interval of existence is then possible using a contradiction argument.

Proof. Our strategy is to estimate the difference between two solutions satisfying (1.5) and show that it must be zero. Namely, let (u_1, v_1) and (u_2, v_2) be two mild solutions that satisfy (1.5). It is convenient to define the following:

(1) $G(t; f, g) = \int_0^t \nabla e^{(t-s)\Delta}(fg)ds$ for $f : \mathbb{R}^n \times (0,T) \to \mathbb{R}$ and $g : \mathbb{R}^n \times (0,T) \to \mathbb{R}^n$
(2) $w_1(t) = \int_0^t e^{-\bar{A}(t-s)e^{(t-s)\Delta}}(\nabla(Bu_1))ds$
(3) $w_2(t) = -G(t; u_2, v_2)$
(4) $u = u_1 - u_2$
(5) $v = v_1 - v_2$

Since u_1 and u_2 are both mild solutions subtraction and substitution yields:

$$\begin{aligned} u(x,t) &= -G(t; u_1, \nabla v_1) + G(t; u_2, \nabla v_2) \\ &= -G(t; u_1, \nabla v_1) + G(t; u_2, \nabla v_1) - G(t; u_2, \nabla v_1) + G(t; u_2, \nabla v_2) \\ &= -G(t; u_1 - u_2, \nabla v_1) - G(t; u_2, \nabla v_1 - \nabla v_2) \\ &= -G(t; u, \nabla v_1) - G(t; u_2, \nabla v) \\ &= -G(t; u, e^{-\bar{A}t}e^{t\Delta}\nabla v_0 + w_1(t)) - G(t; e^{t\Delta}u_0 + w_2(t)) \\ &= -G(t; u, e^{-\bar{A}t}e^{t\Delta}\nabla v_0) - G(t; u, w_1(t)) \\ &\quad - G(t; e^{t\Delta}u_0, \nabla v) - G(t; w_2(t), \nabla v) \\ &:= T_1 + T_2 + T_3 + T_4. \end{aligned} \tag{4.1}$$

It remains to estimate each T_k. Formally, for $p \in (2, \infty)$ with $\vec{z} = (z_1, \dots, z_n)$ given by $z_k = (\frac{1}{I_k} + \frac{1}{J_k})^{-1}$ for $k = 1, \dots, n$:

$$\begin{aligned} \|(-\Delta)^{-1/2}(uw_1)\|_{T,p,\vec{I}} &\le C\|uw_1\|_{T,p,\vec{z}} \\ &\le C\|\|u\|_{\vec{I}}\|w_1\|_{\vec{J}}\|_{L^p} \\ &\le C\|u\|_{T,p,\vec{I}}\big(\sup_{s\in(0,T)} \|w_1\|_{\vec{J}}\big). \end{aligned} \tag{4.2}$$

The first inequality is justified by Lemma 2.5 and the fact that uw_1 is integrable, and the second by Hölder's inequality. To justify the third inequality note that

$$\|u\|_{T,p,\vec{I}} \le T\|u\|_{C((0,T);L_{\vec{I}}(\mathbb{R}^n))} < \infty.$$

Since

$$\|e^{-\bar{A}t}e^{t\Delta v_0}\|_{\vec{J}} \le C\|G_t\|_{L^\infty(\mathbb{R}^n)}\|\nabla v_0\|_{\vec{J}} \le C\|\nabla v_0\|_{\vec{J}},$$

it follows that $\|w_1\|_{C((0,T),L_{\vec{I}}(\mathbb{R}^n))} < \infty$.

Applying the results found in [6], [11] and using (4.2) to estimate T_2 yields:

$$\begin{aligned}\|G(t;u,w_1)\|_{T,p,\vec{I}} &\le \Big\| \int_0^t (-\Delta)^{1/2} e^{(t-s)\Delta}(uw_1)ds\Big\|_{T,p,\vec{I}}\\ &\le \Big\| \int_0^t (-\Delta) e^{(t-s)\Delta}(-\Delta)^{-1/2}(uw_1)ds\Big\|_{T,p,\vec{I}}\\ &\le C\|(-\Delta)^{-1/2}(uw_1)\|_{T,p,\vec{I}}\\ &\le C\|u\|_{L^p((0,T);L_{\vec{I}})}\big(\sup_{s\in(0,T)}\|w_1\|_{\vec{J}}\big)\end{aligned} \tag{4.3}$$

Now applying the same process to T_4 it follows that $w_2(t)\in C((0,T);L_{\vec{I}}(\mathbb{R}^n))$, and $\|\nabla v\|_{T,p,\vec{J}}<\infty$, and so

$$\|(-\Delta)^{-1/2}w_2\nabla v\|_{T,p,\vec{I}} \le C\|\nabla v\|_{T,p,\vec{J}}\Big(\sup_{s\in(0,T)}\|w_2\|_{\vec{I}}\Big).$$

Hence,

$$\|G(t;w_2,\nabla v)\|_{T,p,\vec{I}} \le C\|\nabla v\|_{T,p,\vec{J}}\Big(\sup_{s\in(0,T)}\|w_2\|_{\vec{I}}\Big). \tag{4.4}$$

Turning our attention to T_1, Lemma 2.1 yields

$$\sup_{s\in(0,\infty)} s^{1/2}\big\|e^{s\Delta}\nabla v_0\big\|_{L^\infty(\mathbb{R}^n)} \le C\sup_{s\in(0,\infty)} s^{1/2}s^{-1/2}\|\nabla v_0\|_{\vec{J}} = C\|\nabla v_0\|_{\vec{J}}.$$

Using Minkowski's inequality, and the Hölder and Young inequalities for Lorentz spaces we find that

$$\begin{aligned}&\|G(t;u,e^{-\bar{A}t}e^{t\Delta}\nabla v_0)\|_{T,p,\vec{I}}\\ &\le \Big\|\int_0^t \|\nabla e^{(t-s)\Delta}(ue^{-\bar{A}s}e^{s\Delta}\nabla v_0)\|_{\vec{I}}ds\Big\|_{L^p}\\ &\le C\Big\|\int_0^t s^{-1/2}(t-s)^{-1/2}\|s^{1/2}e^{s\Delta}\nabla v_0\|_{L^\infty}\|u\|_{\vec{I}}ds\Big\|_{L^p}\\ &\le C\|\nabla v_0\|_{\vec{J}}\Big\|\int_0^t s^{-1/2}(t-s)^{-1/2}\|u\|_{\vec{I}}ds\Big\|_{L^p}\\ &\le C\|s^{-1/2}\|u\|_{\vec{I}}\|_{\mathcal{L}^{\frac{2p}{p+2},p}}\\ &\le C\|\|u\|_{\vec{I}}\|_{\mathcal{L}^{p,p}}\\ &= C\|u\|_{L^p((0,T);L_{\vec{I}}(\mathbb{R}^n))}.\end{aligned} \tag{4.5}$$

To estimate T_3 observe that because $1=\sum_{k=1}^n\big(\frac{1}{I_k}-\frac{1}{J_k}\big)$, Lemma 2.1 gives

$$\sup_{s\in(0,\infty)} s^{1/2}\|e^{s\Delta}u_0\|_{\vec{J}} \le C\|u_0\|_{\vec{I}}.$$

Estimating as before yields

$$\begin{aligned}\|G(t;e^{t\Delta}u_0,\nabla v)\|_{T,p,\vec{I}} &\le C\Big\|\int \|e^{s\Delta}u_0\|_{\vec{J}}\|\nabla v\|_{\vec{J}}(t-s)^{-1/2}s^{1/2}s^{-1/2}ds\Big\|_{L^p}\\ &\le C\|u_0\|_{\vec{I}}\|\nabla v\|_{\vec{J}}\\ &\le C\|u_0\|_{\vec{I}}\|u\|_{L_{\vec{I}}}\end{aligned} \tag{4.6}$$

where the integral representation of v in terms of u and the Sobolev embedding theorem to obtain the last inequality. Altogether, (4.5), (4.3), (4.6) and (4.4) give

$$\|u\|_{T,p,\vec{I}} \leq C(n,p) f(T) \|u\|_{T,p,\vec{I}}.$$

Observing that $f(T) \to 0$ as $T \to 0$ it follows that for $T_1 \leq T_{\max}$ small enough $u = 0$ and so $v = 0$, where T_{max} is the maximal existence time. Let T_1 be the maximal value for which $u = 0$. By a linear shift the calculation above applies equally well when the time interval $(T_1, T_1 + \delta)$ with $T_1 + \delta \leq T_{\max}$, which is a contradiction to the definition of T_1. Hence $T_1 = T_{\max}$, proving the theorem. □

5. Appendix

We provide the proof of Lemma 3.1 via induction and the Monotone Convergence theorem.

Proof. Limiting values $(a, b) = \lim_{n\to\infty}(a_n, b_n)$ must satisfy

$$a = a_1 + Cab$$
$$b = b_1 + CKa.$$

Thus, a solves the quadratic equation

$$0 = a_1 + (b_1 C - 1)a + CK^2 a^2,$$

and so

$$a = \frac{-(b_1 C - 1) \pm \sqrt{(b_1 C - 1)^2 - 4a_1 KC^2}}{2KC^2}.$$

Provided that $-(b_1 C - 1) > 0$ and $(b_1 C - 1)^2 - 4a_1 KC^2 > 0$, it is clear that both values of a are positive. Choose the smaller value of a, setting

$$a = \frac{-(b_1 C - 1) - \sqrt{(b_1 C - 1)^2 - 4a_1 KC^2}}{2KC^2}.$$

Then

$$b = b_1 + KC\Big(\frac{-(b_1 C - 1) - \sqrt{(b_1 C - 1)^2 - 4a_1 KC^2}}{2KC^2}\Big)$$

Note that both $\{a_n\}_n$ and $\{b_n\}_n$ are monotonically increasing if $a_1, b_1 > 0$, giving $(a, b) = \lim_{n\to\infty}(a_n, b_n)$ if $a_n \leq a$ and $b_n \leq b$ for all $n \in \mathbb{N}$. Observe that if $a_k \leq a$ and $b_k \leq b$ for all $k \leq n$ then

$$a_{n+1} = a_1 + Ca_n b_n \leq a_1 + Cab = a,$$
$$b_{n+1} = b_1 + KCa_n \leq b_1 + KCa = b.$$

By mathematical induction it suffices to show that $a_1 \leq a$ and $b_1 \leq b$. The inequality for b_1 is trivially true, so all that remains is to show that $a_1 \leq a$. A straightforward calculation shows that this inequality holds if $0 \leq a_1^2(2KC^2)^2$, which is clearly true. Observe that $-(b_1 C - 1) > 0$ iff $b_1 < \frac{1}{C}$ and $(b_1 C - 1)^2 - 4a_1 KC^2 > 0$ if and only if $a_1 < \frac{(b_1 C-1)^2}{4KC^2}$. □

References

[1] G. Arumugam, J. Tyagi; *Keller-Segel Chemotaxis Models: A Review* Acta Applied Math, (2021) 171:6.

[2] O. V. Besov, V. P. Il'In, S. M. Nikol'skii; *Integral Representations of Functions and Imbedding Theorems, Volume I.* V. H. Winston & Sons, 1978.

[3] Biler et al.; *The 8π-problem for radially symmetric solutions of a chemotaxis model in the plane*, Mathematical Methods in the Applied Sciences, 2006 29, 1563–1583.

[4] A. Blanchet, J. Dolbeault, B. Perthame; *Two-dimensional Keller-Segel model: Optimal critical mass and qualitative properties of the solutions*, Electronic Journal of Differential Equations, 2006 (2006) No. 44, 1–33.

[5] L. Corrias, B. Perthame, H. Zaag; *Global solutions of some chemotaxis and angiogenesis systems in high space dimensions*, Milan Journal of Mathematics, 72 (2004), 611–616.

[6] H. Dong, D. Kim; *On lp-estimates for elliptic and parabolic equations with ap weights*, Transmissions of the American Mathematical Society, Volume 307, Number 7, (2018), 5081–5130.

[7] E.F. Keller, L.A. Segel; *Initiation of Slime Mold Aggregation Viewed as an Instability* Journal of Theoretical Biology, (1970) 26, 399–415.

[8] H. Kozono, Y. Sugiyama; *Keller-Segel system of parabolic-parabolic type with initial data in weak $L^{n/2}(\mathbb{R}^n)$ and its application to self-similar solutions*, Indiana University Mathematics Journal, 57 (2008) 1467–1500.

[9] H. Kozono, Y. Sugiyama, T. Wachi; *Existence and uniqueness theorem on mild solutions to the Keller-Segel system in the scaling invariant space*, Journal of Differential Equations 252 (2012), 1213–1228.

[10] H. Kozono, M. Yamazaki; *Uniqueness criterion of weak solutions to the stationary Navier-Stokes equations in exterior domains*, Nonlinear Analysis, 38 (1999), 959–970.

[11] N. V. Krylov; *Rubio de Francia extrapolation theorem and related topics in the theory of elliptic and parabolic equations. A survey*, Algebra i Analiz, 2020, Volume 32, Issue 3, 5–38.

[12] T. Phan; *Well-Posedness for the Navier-Stokes equations in critical mixed-norm Lebesgue spaces*, Journal of Evolution Equations (2019), 1–24.

[13] T. Phan, T. Robertson; *On Masuda uniqueness theorem for Leray–Hopf weak solutions in mixed-norm spaces*, European Journal of Mechanics - B/Fluids 90 (2021), 18–28.

Timothy Robertson
Department of Mathematics, University of Tennessee, Knoxville, USA
Email address: trober41@vols.utk.edu