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ON THE L^2-ORTHOGONALITY OF STEKLOV EIGENFUNCTIONS

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Abstract. This article analyzes the interior L^2-orthogonality of the Steklov eigenfunctions on rectangles $\Omega_{1\alpha}$. It is shown that most Steklov eigenfunctions are, indeed, pairwise orthogonal in $L^2(\Omega_{1\alpha})$, and pairs that are not orthogonal are *nearly orthogonal.* Explicit formulae for exact inner products in $L^2(\Omega_{1\alpha})$ of the eigenfunctions are found, and to elucidate the intricate formulae obtained, accompanying numerics are provided. Then envelopes that bound the calculated inner products are constructed that simplify the convoluted formulae. This leads to a straightforward description of the nearly orthogonal Steklov eigenfunctions. A consequence of the calculations is a tabulation of the mean value of Steklov eigenfunctions over $\Omega_{1\alpha}$.

1. Introduction

This article describes the exact, or near, orthogonality in $L^2(\Omega_{1\alpha})$ of the sequence of Steklov eigenfunctions in the case $\Omega_{1\alpha}$ is a rectangle in $\mathbb{R}^2$. This complements the well-known result of L^2-orthogonality of the sequence of Dirichlet eigenfunctions, as well as of the Neumann and Robin eigensystems, on more general domains. The question on L^2-orthogonality follows from [6], where Auchmuty and the second author analyzed the tensor product of pairs of these four systems.

Classes of Steklov eigenfunctions have been used, for instance, in the construction of bases of trace spaces of functions on the boundary (Auchmuty [1] and Kloucek et al. [14]), on the analysis of dewetting of thin films (Auchmuty and Klouček [5]), on the spectral representation of divergence-free vector fields (Auchmuty and Simpkins [7]), and on harmonic boundary value problems (as done by the first author in [4, 8, 9, 10, 11]). To the best of the authors knowledge, a complete description of the orthogonality in $L^2(\Omega)$, on more general regions Ω in $\mathbb{R}^N$ including rectangles in $\mathbb{R}^2$, of Steklov eigenfunctions has not been made because results and applications often employ orthogonality in $L^2(\partial\Omega)$ or (special) orthogonality in other Sobolev-Hilbert spaces.

After introducing notation in §2, and making precise in §3 the orthogonality issue studied here, collected in §4 is the explicit formulae for the harmonic Steklov eigendata on a rectangle $\Omega_{1\alpha}$ that is reprised from [3]. The main L^2-orthogonality results of this paper are given in §5.

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First, the exact L^2-norms on $\Omega_{1\alpha}$ and $\partial\Omega_{1\alpha}$ are recorded. Then explicit formulae for the inner product in $L^2(\Omega_{1\alpha})$ between Steklov eigenfunctions are given, and numerical work is shown to interpret the elaborate formulae. It is calcualted that the majority of the Steklov eigenfunctions are pairwise orthogonal in $L^2(\Omega_{1\alpha})$, and it is seen from the plots that those that are not orthogonal are nearly orthogonal at high frequencies. Calculation of the inner product of the constant Steklov eigenfunction and other Steklov eigenfunctions is re-interpreted as a calculation of the mean value of the Steklov eigenfunctions. To provide straightforward formulae that indicate near L^2-orthogonality of Steklov eigenfunctions, envelopes are constructed to bound the intricate formulae previously obtained. These envelope formulae are further simplified and this leads to the relation (5.16) that succinctly describes the *near orthogonality* in $L^2(\Omega_{1\alpha})$ of the sequence of Steklov eigenfunctions.

The findings in the present paper may be generalized to the case $\Omega_{1\alpha}$ is a cuboid in N-dimensions; see Girouard et al. [12] where the Steklov spectrum is carefully analyzed for cuboids. We expect that similar results hold, but that the formulae would be intense. Our work is an analysis of special functions.

2. Assumptions and notation

The analysis in this work will be over a retangle $\Omega_{1\alpha} := (-1,1)\times(-\alpha,\alpha)$ in $\mathbb{R}^2$, where α is a fixed constant in $(0,1]$ that is called the *aspect ratio* of the rectangle. Due to scaling and rotation properties of the Steklov problem, the analysis on $\Omega_{1\alpha}$ accounts for the Steklov analysis on any rectangle of $\mathbb{R}^2$. Denote by $\mathrm{d}\sigma$ the 1-dimensional Hausdorff measure, or arclength, so that the unit outward normal $\nu(z)$ is defined for σ a.e. $z \in \partial\Omega_{1\alpha}$. All functions in this work will take value in $[-\infty,\infty]$.

Let $L^p(\Omega_{1\alpha})$ and $L^p(\partial\Omega_{1\alpha})$ with $1 \le p \le \infty$, be the usual Lebesgue spaces with p-norm denoted by $\|u\|_{p,\Omega_{1\alpha}}$ or $\|u\|_{p,\partial\Omega_{1\alpha}}$ respectively. When $p=2$ these are real Hilbert spaces with inner products defined by

$$\langle u,v\rangle_{2,\Omega_{1\alpha}} := \int_{\Omega_{1\alpha}} uv\,dx\,dy \quad\text{and}\quad \langle u,v\rangle_{2,\partial\Omega_{1\alpha}} := \int_{\partial\Omega} uv\mathrm{d}\sigma.$$

Denote by $H^1(\Omega_{1\alpha})$ the usual real Sobolev space of functions on $\Omega_{1\alpha}$ that is a real Hilbert space under the standard H^1-inner product

$$[u,v]_{1,2,\Omega_{1\alpha}} = \int_{\Omega_{1\alpha}} [u\cdot v + \nabla u\cdot\nabla v]\,dx\,dy \tag{2.1}$$

where ∇u is the gradient of the function u; the associated norm is denoted by $\|u\|_{1,2,\Omega_{1\alpha}}$.

The trace γu of a continuous function u on $\overline{\Omega}_{1\alpha}$ to the boundary $\partial\Omega_{1\alpha}$ is its restriction to $\partial\Omega_{1\alpha}$. The boundary trace map on $H^1(\Omega_{1\alpha})$ is the linear extension of the map γ restricting Lipschitz continuous functions on $\overline{\Omega}_{1\alpha}$ to $\partial\Omega_{1\alpha}$. The region $\Omega_{1\alpha}$ is said to satisfy a *compact trace theorem* provided that the trace mapping $\gamma : H^1(\Omega_{1\alpha}) \to L^2(\partial\Omega_{1\alpha}, d\sigma)$ is compact. One inequality that implies the compact trace theorem for bounded regions in $\mathbb{R}^N$ with Lipschitz boundaries has been proved in [13, Theorem 1.5.1.10].

Instead of (2.1), one can use the ∂-inner product defined by

$$[u,v]_\partial := \int_{\Omega_{1\alpha}} \nabla u\cdot\nabla v\,dx\,dy + \frac{1}{|\partial\Omega_{1\alpha}|}\int_{\partial\Omega_{1\alpha}} uv\,\mathrm{d}\sigma. \tag{2.2}$$

Here, $|\partial\Omega_{1\alpha}| = 4(1+\alpha)$ is the length of the perimeter of the rectangle, and $d\sigma$ is integration with respect to arclength. The norm corresponding to $[\cdot,\cdot]_\partial$ is denoted by $\|u\|_\partial$. From Corollary 6.2 of [2], this norm is equivalent to the standard norm of $H^1(\Omega_{1\alpha})$.

A function $u \in H^1(\Omega_{1\alpha})$ is said to be *harmonic* on $\Omega_{1\alpha}$ if it satisfies

$$\int_{\Omega_{1\alpha}} \nabla u \cdot \nabla v \, dx\, dy = 0 \quad \text{for all } v \in C_c^1(\Omega_{1\alpha}) \tag{2.3}$$

where $C_c^1(\Omega_{1\alpha})$ is the set of all C^1-functions on $\Omega_{1\alpha}$ with compact support in $\Omega_{1\alpha}$. Denote by $\mathcal{H}(\Omega_{1\alpha})$ the space of all such harmonic functions on $\Omega_{1\alpha}$. The usual Sobolev space $H_0^1(\Omega_{1\alpha})$ is the closure of $C_c^1(\Omega_{1\alpha})$ in the $H^1(\Omega_{1\alpha})$-norm, and it is easy to see that $\mathcal{H}(\Omega_{1\alpha})$ is ∂-orthogonal to $H_0^1(\Omega_{1\alpha})$ so that $H^1(\Omega_{1\alpha})$ may be expressed as

$$H^1(\Omega_{1\alpha}) = H_0^1(\Omega_{1\alpha}) \oplus_\partial \mathcal{H}(\Omega_{1\alpha}) \tag{2.4}$$

where $\oplus_\partial$ represents a ∂-orthogonal decomposition as described in §5 of [2].

3. Steklov eigenfunctions and the L^2-orthogonality question

This article is about the harmonic Steklov eigenfunctions on $\Omega_{1\alpha}$, which are non-zero functions $s = s(x,y)$ in $H^1(\Omega_{1\alpha})$ satisfying, for some $\sigma \in \mathbb{R}$, the identity

$$\int_{\Omega_{1\alpha}} \nabla s \cdot \nabla v \, dx\, dy = \frac{\sigma}{|\partial\Omega_{1\alpha}|} \int_{\partial\Omega_{1\alpha}} sv \, d\sigma \quad \text{for all } v \in H^1(\Omega_{1\alpha}). \tag{3.1}$$

Equation (3.1) is the weak form of the boundary value problem

$$\Delta s = 0 \quad \text{in } \Omega_{1\alpha} \quad \text{and} \quad \frac{\partial u}{\partial n} = \frac{\sigma}{|\partial\Omega_{1\alpha}|} s \quad \text{on } \partial\Omega_{1\alpha}.$$

In a quite general bounded region Ω of $\mathbb{R}^N$ that includes rectangles, Auchmuty in [2] obtains a countable infinite sequence of Steklov eigenfunctions and proves, among other properties, that this sequence is orthogonal in $\mathcal{H}(\Omega)$ with respect to the ∂-inner product, and that the corresponding sequence of traces is orthogonal in $L^2(\partial\Omega, \mathrm{d}\sigma)$.

Determining the orthogonality in $L^2(\Omega_{1\alpha})$ of the sequence of Steklov eigenfunctions on the rectangle $\Omega_{1\alpha}$ is what this paper investigates and provides various results.

4. Steklov eigendata on rectangles $\Omega_{1\alpha}$ of $\mathbb{R}^2$

This section recapitulates the explicit Steklov spectral data for rectangles $\Omega_{1\alpha}$ that is contained in Auchmuty-Cho [3]. There the eigendata on $\Omega_{1\alpha}$ is organized into four classes according to symmetry. Here the Steklov eigenfunctions are denoted by u instead of s to indicate that they are unnormalized.

Class I Steklov eigenfunctions $u = u(x,y)$ are even in x and in y. The first such function is given by $u_{1,0}(x,y) := 1$, which corresponds to the zero eigenvalue $\sigma_{1,0} := 0$. Then there is a dichotomy for all other functions and values in this class given by

$$u_{1i}(x,y) := \cosh \beta_i x \cos \beta_i y \quad \text{corresponding to } \sigma_{1i} = \beta_i \tanh \beta_i,\ i \in \mathbb{N},$$
$$u_{1j}(x,y) := \cos \beta_j x \cosh \beta_j y \quad \text{corresponding to } \sigma_{1j} = \beta_j \tanh \alpha\beta_j,\ j \in \mathbb{N},$$

where β_i and β_j are the ascending, strictly positive zeros, respectively, of

$$\tan\alpha\beta + \tanh\beta = 0 \quad \text{and} \quad \tan\beta + \tanh\alpha\beta = 0. \tag{4.1}$$

These are called the *determining equations* in β for Class I Steklov eigendata.

Steklov eigenfunctions $u = u(x, y)$ in Class II are odd in x and in y. When $\alpha = 1$, the first such function is given by $u_{2,0}(x, y) = xy$, which corresponds to the eigenvalue $\sigma_{2,0} := 1$. All other eigendata in this class splits as

$$u_{2i}(x, y) := \sinh\beta_i x \sin\beta_i y \quad \text{corresponding to } \sigma_{2i} = \beta_i \coth\beta_i,\ i \in \mathbb{N},$$
$$u_{2j}(x, y) := \sin\beta_j x \sinh\beta_j y \quad \text{corresponding to } \sigma_{2j} = \beta_j \coth\alpha\beta_j,\ j \in \mathbb{N},$$

where β_i and β_j, in this case, are the ascending, strictly positive zeros, respectively, of

$$\tan\alpha\beta - \tanh\beta = 0 \quad \text{and} \quad \tan\beta - \tanh\alpha\beta = 0. \tag{4.2}$$

Now Class III functions are even in x and odd in y, and the class is separated as

$$u_{3i}(x, y) := \cosh\beta_i x \sin\beta_i y \quad \text{corresponding to } \sigma_{3i} = \beta_i \tan\beta_i,\ i \in \mathbb{N},$$
$$u_{3j}(x, y) := \cos\beta_j x \sinh\beta_j y \quad \text{corresponding to } \sigma_{3j} = \beta_j \tanh\alpha\beta_j,\ j \in \mathbb{N},$$

according to the respective determining equations

$$\tan\alpha\beta - \coth\beta = 0 \quad \text{and} \quad \tan\beta + \coth\alpha\beta = 0. \tag{4.3}$$

Finally, Class IV functions are odd in x and even in y, and the the two subclasses are

$$u_{4i}(x, y) := \sinh\beta_i x \cos\beta_i y \quad \text{corresponding to } \sigma_{4i} = \beta_i \coth\beta_i,\ i \in \mathbb{N},$$
$$u_{4j}(x, y) := \sin\beta_j x \cosh\beta_j y \quad \text{corresponding to } \sigma_{4j} = \beta_j \coth\alpha\beta_j,\ j \in \mathbb{N},$$

according to the respective determining equations

$$\tan\alpha\beta + \coth\beta = 0 \quad \text{and} \quad \tan\beta - \coth\alpha\beta = 0. \tag{4.4}$$

5. Exact or near orthogonality of Steklov eigenfunctions in $L^2(\Omega_{1\alpha})$

This section analytically treats the main question of orthogonality in $L^2(\Omega_{1\alpha})$ of the harmonic Steklov eigenfunctions catalogued in §4.

5.1. **Interior and boundary L^2-norms of Steklov eigenfunctions on $\Omega_{1\alpha}$.**

To calculate explicitly the orthogonality in $L^2(\Omega_{1\alpha})$ between Steklov eigenfunctions, their L^2-norms on $\Omega_{1\alpha}$ and on $\partial\Omega_{1\alpha}$ were found and are listed in Table 1.

Here, the indices i, j for β_i, β_j are suppresed in the formulae to elucidate the form of these norms, the $u_{\ell i}, u_{\ell j}$ are the unnormalized Steklov eigenfunctions of §4, and the following functions have been used

$$\operatorname{sinc}\theta := \frac{\sin\theta}{\theta} \quad \text{and} \quad \operatorname{sinhc}\theta := \frac{\sinh\theta}{\theta} \tag{5.1}$$

TABLE 1. L^2-norms of (unnormalized) Steklov eigenfunctions u.

u	$\|u\|^2_{2,\Omega_{1\alpha}}$	$\|u\|^2_{2,\partial\Omega_{1\alpha}}$
$u_{1,0}$	4α	$4(1+\alpha)$
u_{1i}	$(1+\operatorname{sinhc} 2\beta)(\alpha+\alpha \operatorname{sinc} 2\alpha\beta)$	$2\cos^2(\alpha\beta)[1+\operatorname{sinhc} 2\beta]+2\alpha\cosh^2(\beta)[1+\operatorname{sinc} 2\alpha\beta]$
u_{1j}	$(1+\operatorname{sinc} 2\beta)(\alpha+\alpha \operatorname{sinhc} 2\alpha\beta)$	$2\cosh^2(\alpha\beta)[1+\operatorname{sinc} 2\beta]+2\alpha\cos^2(\beta)[1+\operatorname{sinhc} 2\alpha\beta]$
$u_{2,0}$	$4/9$	$8/3$
u_{2i}	$(-1+\operatorname{sinhc} 2\beta)(\alpha-\alpha \operatorname{sinc} 2\alpha\beta)$	$2\sin^2(\alpha\beta)[-1+\operatorname{sinhc} 2\beta]+2\alpha\sinh^2(\beta)[1-\operatorname{sinc} 2\alpha\beta]$
u_{2j}	$(1-\operatorname{sinc} 2\beta)(-\alpha+\alpha \operatorname{sinhc} 2\alpha\beta)$	$2\sinh^2(\alpha\beta)[1-\operatorname{sinc} 2\beta]+2\alpha\sin^2(\beta)[-1+\operatorname{sinhc} 2\alpha\beta]$
u_{3i}	$(1+\operatorname{sinhc} 2\beta)(\alpha-\alpha \operatorname{sinc} 2\alpha\beta)$	$2\sin^2(\alpha\beta)[1+\operatorname{sinhc} 2\beta]+2\alpha\cosh^2(\beta)[1-\operatorname{sinc} 2\alpha\beta]$
u_{3j}	$(1+\operatorname{sinc} 2\beta)(-\alpha+\alpha \operatorname{sinhc} 2\alpha\beta)$	$2\sinh^2(\alpha\beta)[1+\operatorname{sinc} 2\beta]+2\alpha\cos^2(\beta)[-1+\operatorname{sinhc} 2\alpha\beta]$
u_{4i}	$(-1+\operatorname{sinhc} 2\beta)(\alpha+\alpha \operatorname{sinc} 2\alpha\beta)$	$2\cos^2(\alpha\beta)[-1+\operatorname{sinhc} 2\beta]+2\alpha\sinh^2(\beta)[1+\operatorname{sinc} 2\alpha\beta]$
u_{4j}	$(1-\operatorname{sinc} 2\beta)(\alpha+\alpha \operatorname{sinhc} 2\alpha\beta)$	$2\cosh^2(\alpha\beta)[1-\operatorname{sinc} 2\beta]+2\alpha\sin^2(\beta)[1+\operatorname{sinhc} 2\alpha\beta]$

5.2. **Mean-Value of Steklov eigenfunctions on $\Omega_{1\alpha}$.** The inner products in $L^2(\Omega_{1\alpha})$ of the first Steklov eigenfunction $u_{1,0} \equiv 1$ with the other Steklov eigenfunctions u_{li}, u_{lj}, where $l = 1, 2, 3, 4$ and $i, j \in \mathbb{N}$, lead to the calculations

$$\langle \tilde{u}_{1,0}, \tilde{u} \rangle_{2,\Omega_{1\alpha}} = \begin{cases} \frac{2 \operatorname{sinhc} \beta_i \ \operatorname{sinc} \alpha\beta_i}{\sqrt{(1+\operatorname{sinhc} 2\beta_i)(1+\operatorname{sinc} 2\alpha\beta_i)}} & \text{if } \tilde{u} = \tilde{u}_{1i} \\ \frac{2 \operatorname{sinc} \beta_j \ \operatorname{sinhc} \alpha\beta_j}{\sqrt{(1+\operatorname{sinc} 2\beta_j)(1+\operatorname{sinhc} 2\alpha\beta_j)}} & \text{if } \tilde{u} = \tilde{u}_{1j} \\ 0 & \text{if } u = u_{2,0},\ u_{2i},\ u_{2j}, \\ & \quad u_{3i},\ u_{3j},\ u_{4i},\ u_{4j} \end{cases} \tag{5.2}$$

where $\tilde{u}$ indicates that u is normalized with respect to the standard norm of $L^2(\Omega_{1\alpha})$.

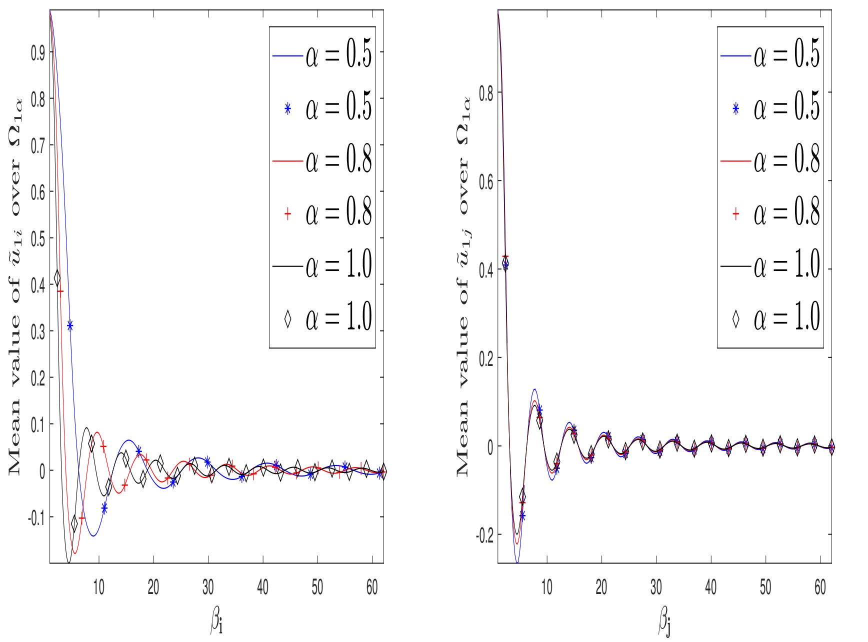
The graphs of $\beta_i \mapsto \langle \tilde{u}_{1,0}, \tilde{u}_{1i} \rangle_{2,\Omega_{1\alpha}}$ and $\beta_j \mapsto \langle \tilde{u}_{1,0}, \tilde{u}_{1j} \rangle_{2,\Omega_{1\alpha}}$ are in Figure 1, where discrete points on the graphs are at the roots β_i, β_j that determine u_{1i}, u_{1j}, respectively. As seen in Figure 1(a), the $L^2(\Omega_{1\alpha})$-angle between $u_{1,0}$ and u_{1i}, which are both Class I Steklov eigenfunctions, rapidly approaches $90°$ as $i \to \infty$. From Figure 1(b), the same is true of the angle between $u_{1,0}$ and u_{1j}. The third case in (5.2) shows that $u_{1,0}$ is orthogonal in $L^2(\Omega_{1\alpha})$ to Class II, III, and IV Steklov eigenfunctions.

This result can be interpreted as a result on the mean value of each L^2-normalized Steklov eigenfunction over the region $\Omega_{1\alpha}$ since

$$\langle \tilde{u}_{1,0}, \tilde{u} \rangle_{2,\Omega_{1\alpha}} = \frac{1}{4\alpha} \int_{\Omega_{1\alpha}} \tilde{u}\, dx \tag{5.3}$$

and 4α is the area of the rectangle $\Omega_{1\alpha}$. In this language, the Class II, III, IV Steklov eigenfunctions have mean value zero on $\Omega_{1\alpha}$, and the mean value on $\Omega_{1\alpha}$ of Class I Steklov eigenfunctions is nearly zero for high frequency u_{1i}, u_{1j}; by high frequency is meant that i, j are large values.

5.3. **L^2-orthogonality of Steklov eigenfunctions on $\Omega_{1\alpha}$.** The inner product in $L^2(\Omega_{1\alpha})$ of a Class I Steklov eigenfunction u_{1i}, with i fixed, and another Steklov eigenfunction, where the prime in u'_{1i} indicates a second Steklov eigenfunction of the form u_{1i}, is evaluated and leads to



(a) L^2-inner product between $\tilde{u}_{1,0}$ and $\tilde{u}_{1i}$; this equals the mean value of $\tilde{u}_{1i}$ over $\Omega_{1\alpha}$

(b) L^2-inner product between $\tilde{u}_{1,0}$ and $\tilde{u}_{1j}$; this equals the mean value of $\tilde{u}_{1j}$ over $\Omega_{1\alpha}$

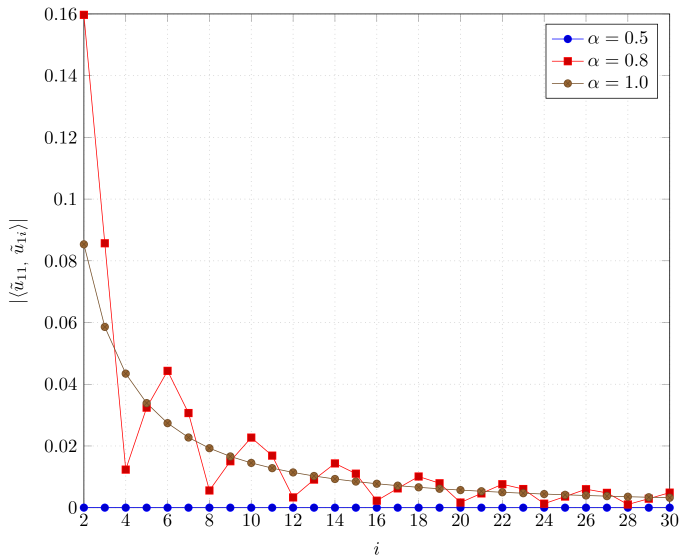
FIGURE 1. Near L^2-orthogonality between $\tilde{u}_{1,0}$ and Steklov eigenfunctions from Class I on $\Omega_{1\alpha}$ shown for $\alpha = 0.5$, 0.8, 1.0.

$$
\begin{aligned}
&\langle \tilde{u}_{1i}, \tilde{u} \rangle_{2,\Omega_{1\alpha}} \\
&= \begin{cases}
\frac{[\operatorname{sinhc}(\beta_i+\beta_i')+\operatorname{sinhc}(\beta_i-\beta_i')]\cdot[\operatorname{sinc}(\alpha(\beta_i+\beta_i'))+\operatorname{sinc}(\alpha(\beta_i-\beta_i'))]}{\sqrt{(1+\operatorname{sinhc}2\beta_i)(1+\operatorname{sinc}2\alpha\beta_i)(1+\operatorname{sinhc}2\beta_i')(1+\operatorname{sinc}2\alpha\beta_i')}} \\
\quad \text{if } \tilde{u} = \tilde{u}_{1i}' \\
\frac{4[\beta_i \sinh\beta_i \cos\beta_j + \beta_j \cosh\beta_i \sin\beta_j][\beta_i \sin(\alpha\beta_i)\cosh(\alpha\beta_j) + \beta_j \cos(\alpha\beta_i)\sinh(\alpha\beta_j)]}{\alpha\cdot(\beta_i^2+\beta_j^2)^2\cdot\sqrt{(1+\operatorname{sinhc}2\beta_i)(1+\operatorname{sinc}2\alpha\beta_i)(1+\operatorname{sinc}2\beta_j)(1+\operatorname{sinhc}2\alpha\beta_j)}} \\
\quad \text{if } \tilde{u} = \tilde{u}_{1j} \\
0 \quad \text{if } \tilde{u} = \tilde{u}_{2,0},\ \tilde{u}_{2i},\ \tilde{u}_{2j},\ \tilde{u}_{3i},\ \tilde{u}_{3j},\ \tilde{u}_{4i},\ \tilde{u}_{4j}.
\end{cases}
\end{aligned}
\tag{5.4}
$$

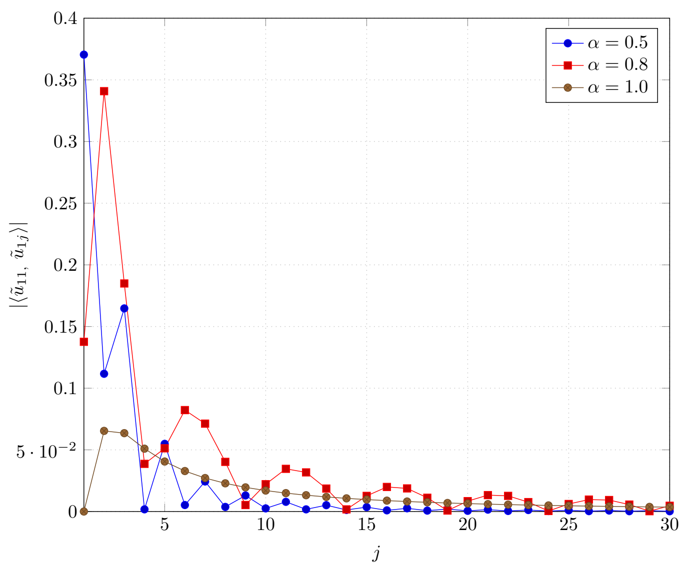
To facilitate the orthogonality discussion, the symbols $\perp$ and $\curlywedge$ will be used for the phrases *is orthogonal to* and *is nearly orthogonal to*, respectively. With this notation, the calculation in (5.4) shows that $u_{1i} \perp u_{\ell i}$ and $u_{1i} \perp u_{\ell j}$ in $L^2(\Omega_{1\alpha})$ for $\ell = 2, 3, 4$, and that $u_{1i} \curlywedge u_{1i}'$ and $u_{1i} \curlywedge u_{1j}$ in $L^2(\Omega_{1\alpha})$ for high frequency u_{1i}' and u_{1j}; see Figure 2.

These and the next computations for $\langle \tilde{u}_{\ell i}, \tilde{u} \rangle_{2,\Omega_{1\alpha}}$ are for fixed i; analogous formulae hold when i is replaced by j.

The $L^2(\Omega_{1\alpha})$-inner product of u_{2i}, a fixed Class II, Type 1 Steklov eigenfunction, with another Steklov eigenfunction is found and yields



(a) L^2-inner product between $\tilde{u}_{11}$ and $\tilde{u}_{1i}$ for $2 \le i \le 30$ on $\Omega_{1\alpha}$

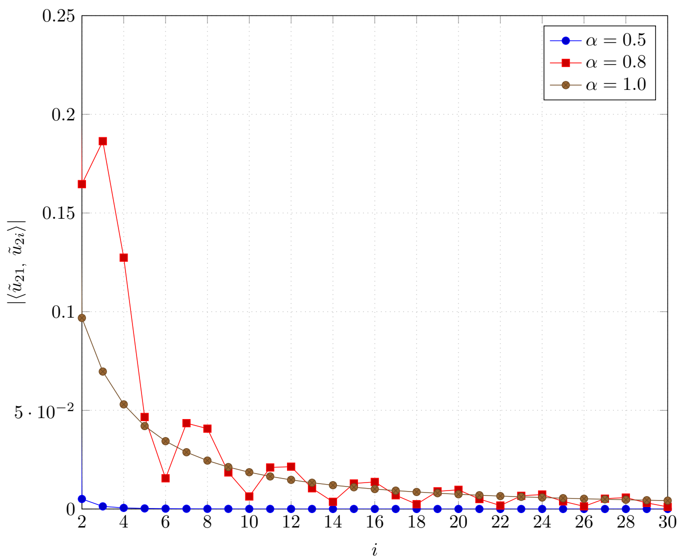


(b) L^2-inner product between $\tilde{u}_{11}$ and $\tilde{u}_{1j}$ for $1 \le j \le 30$ on $\Omega_{1\alpha}$

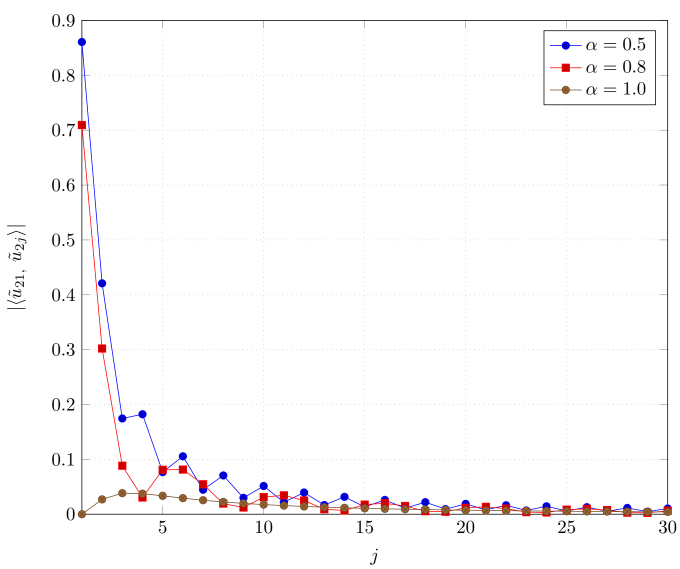
FIGURE 2. Near L^2-orthogonality of Class I Steklov eigenfunction on $\Omega_{1\alpha}$ shown for $\alpha = 0.5,\ 0.8,\ 1.0$.

$$\langle \tilde{u}_{2i}, \tilde{u}\rangle_{2,\Omega_{1\alpha}}$$
$$= \begin{cases} \frac{[\operatorname{sinhc}(\beta_i+\beta_i')-\operatorname{sinhc}(\beta_i-\beta_i')][\operatorname{sinc}(\alpha(\beta_i-\beta_i'))-\operatorname{sinc}(\alpha(\beta_i+\beta_i'))]}{\sqrt{(-1+\operatorname{sinhc}2\beta_i)(1-\operatorname{sinc}2\alpha\beta_i)(-1+\operatorname{sinhc}2\beta_i')(1-\operatorname{sinc}2\alpha\beta_i')}} \\ \quad \text{if } \tilde{u} = \tilde{u}_{2i}' \\ \frac{4[\beta_i\cosh\beta_i\sin\beta_j-\beta_j\sinh\beta_i\cos\beta_j][\beta_j\sin(\alpha\beta_i)\cosh(\alpha\beta_j)-\beta_i\cos(\alpha\beta_i)\sinh(\alpha\beta_j)]}{\alpha\cdot(\beta_i^2+\beta_j^2)^2\cdot\sqrt{(-1+\operatorname{sinhc}2\beta_i)(1-\operatorname{sinc}2\alpha\beta_i)(1-\operatorname{sinc}2\beta_j)(-1+\operatorname{sinhc}2\alpha\beta_j)}} \\ \quad \text{if } \tilde{u} = \tilde{u}_{2j} \\ 0 \quad \text{if } \tilde{u} = \tilde{u}_{3i},\ \tilde{u}_{3j},\ \tilde{u}_{4i},\ \tilde{u}_{4j} \end{cases} \tag{5.5}$$

This shows $u_{2i} \perp u_{\ell i}$ and $u_{2i} \perp u_{\ell j}$ in $L^2(\Omega_{1\alpha})$ for $\ell = 3, 4$, and that $u_{2i} \curlywedge u'_{2i}$ and $u_{2i} \curlywedge u_{2j}$ in $L^2(\Omega_{1\alpha})$ for high frequency u'_{2i} and u_{2j}; see Figure 3.



(a) L^2-inner product between $\tilde{u}_{21}$ and $\tilde{u}_{2i}$ for $2 \leq i \leq 30$ on $\Omega_{1\alpha}$



(b) L^2-inner product between $\tilde{u}_{21}$ and $\tilde{u}_{2j}$ for $1 \leq j \leq 30$ on $\Omega_{1\alpha}$

FIGURE 3. Near L^2-orthogonality of Class II Steklov eigenfunctions on $\Omega_{1\alpha}$ shown for $\alpha = 0.5,\ 0.8,\ 1.0$.

The inner product in $L^2(\Omega_{1\alpha})$ of u_{3i}, a fixed Class III, Type 1 Steklov eigenfunction, with another Steklov eigenfunction is computed and leads to

$$
\begin{aligned}
&\langle \tilde{u}_{3i}, \tilde{u}\rangle_{2,\Omega_{1\alpha}} \\
&= \begin{cases}
\frac{[\operatorname{sinhc}(\beta_i-\beta_i')+\operatorname{sinhc}(\beta_i+\beta_i')][\operatorname{sinc}(\alpha(\beta_i-\beta_i'))-\operatorname{sinc}(\alpha(\beta_i+\beta_i'))]}{\sqrt{(1+\operatorname{sinhc}2\beta_i)(1-\operatorname{sinc}2\alpha\beta_i)(1+\operatorname{sinhc}2\beta_i')(1-\operatorname{sinc}2\alpha\beta_i')}} \\
\quad \text{if } \tilde{u}=\tilde{u}_{3i}' \\
\frac{4[\beta_i\sinh\beta_i\cos\beta_j+\beta_j\cosh\beta_i\sin\beta_j][\beta_j\sin(\alpha\beta_i)\cosh(\alpha\beta_j)-\beta_i\cos(\alpha\beta_i)\sinh(\alpha\beta_j)]}{\alpha\cdot(\beta_i^2+\beta_j^2)^2\cdot\sqrt{(1+\operatorname{sinhc}2\beta_i)(1-\operatorname{sinc}2\alpha\beta_i)(1+\operatorname{sinc}2\beta_j)(-1+\operatorname{sinhc}2\alpha\beta_j)}} \\
\quad \text{if } \tilde{u}=\tilde{u}_{3j} \\
0 \quad \text{if } \tilde{u}=\tilde{u}_{4i},\ \tilde{u}_{4j}
\end{cases}
\end{aligned}
\tag{5.6}
$$

Thus, $u_{3i}\perp u_{4i}$ and $u_{3i}\perp u_{4j}$ in $L^2(\Omega_{1\alpha})$, and $u_{3i} \curlywedge u_{3i}'$ and $u_{3i}\curlywedge u_{3j}$ in $L^2(\Omega_{1\alpha})$ for high frequency u_{3i}' and u_{3j}; see Figure 4.

Lastly, the inner product in $L^2(\Omega_{1\alpha})$ of u_{4i}, a fixed Class IV, Type 1 Steklov eigenfunction, with another Class IV Steklov eigenfunction is evaluated and gives

$$
\begin{aligned}
&\langle \tilde{u}_{4i}, \tilde{u}\rangle_{2,\Omega_{1\alpha}} \\
&= \begin{cases}
\frac{[\operatorname{sinhc}(\beta_i+\beta_i')-\operatorname{sinhc}(\beta_i-\beta_i')]\cdot[\operatorname{sinc}(\alpha(\beta_i-\beta_i'))+\operatorname{sinc}(\alpha(\beta_i+\beta_i'))]}{\sqrt{(-1+\operatorname{sinhc}2\beta_i)(1+\operatorname{sinc}2\alpha\beta_i)(-1+\operatorname{sinhc}2\beta_i')(1+\operatorname{sinc}2\alpha\beta_i')}} \\
\quad \text{if } \tilde{u}=\tilde{u}_{4i}' \\
\frac{4[\beta_i\cosh\beta_i\sin\beta_j-\beta_j\sinh\beta_i\cos\beta_j][\beta_j\cos(\alpha\beta_i)\sinh(\alpha\beta_j)+\beta_i\sin(\alpha\beta_i)\cosh(\alpha\beta_j)]}{\alpha\cdot(\beta_i^2+\beta_j^2)^2\cdot\sqrt{(-1+\operatorname{sinhc}2\beta_i)(1+\operatorname{sinc}2\alpha\beta_i)(1-\operatorname{sinc}2\beta_j)(1+\operatorname{sinhc}2\alpha\beta_j)}} \\
\quad \text{if } \tilde{u}=\tilde{u}_{4j}
\end{cases}
\end{aligned}
\tag{5.7}
$$

This says $u_{4i}\curlywedge u_{4i}'$ and $u_{4i}\curlywedge u_{4j}$ in $L^2(\Omega_{1\alpha})$ for high frequency u_{4i}' and u_{4j}; see Figure 5.

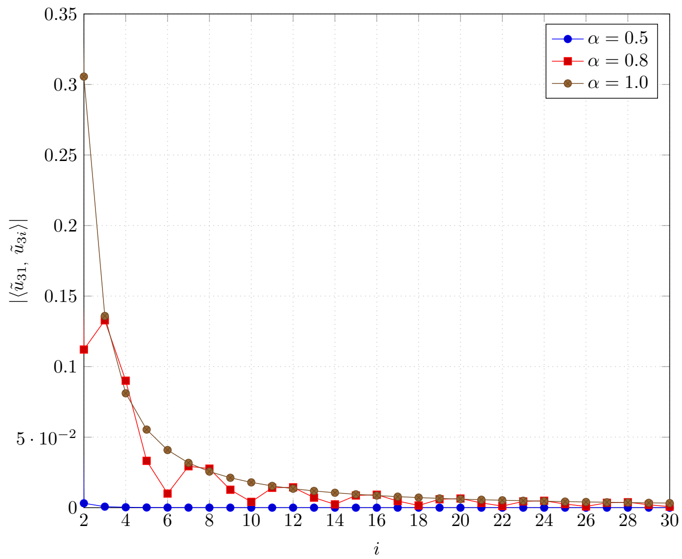
5.4. **Envelopes for the L^2-orthogonality.** Although exact formulae for inner products have been found, admittedly the expressions are formidable. The next result provides easier formulae for envelopes, or bounds, on these inner products, with the envelopes given in terms of the aspect ratio α of the rectangle $\Omega_{1\alpha}$ and the roots β_i, β_j that determine the Steklov data.

Theorem 5.1. *Let $\tilde{u}_{1,0}$ and $\tilde{u}_{\ell,i}$, with $\ell=1,2,3,4$, be the Steklov eigenfunctions normalized in $L^2(\Omega_{1\alpha})$ as described above.*

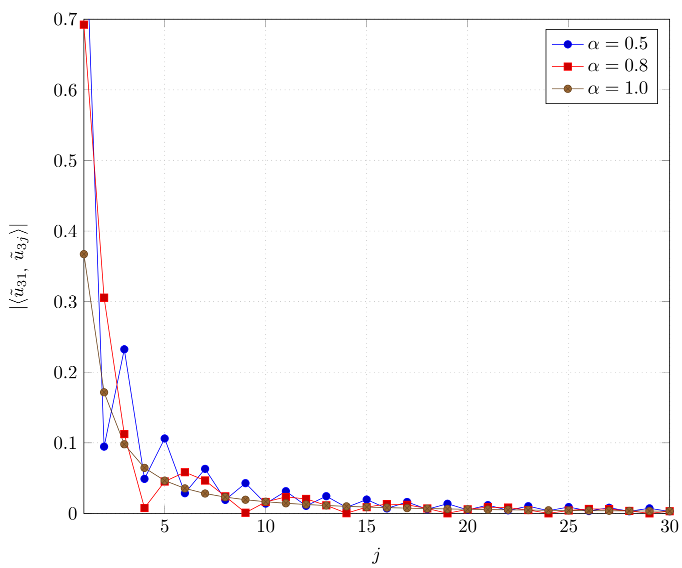
(1) *When the root β_i that determines $\tilde{u}_{1i}$ satisfies $\beta_i>\max\{\frac{e^{-2\beta_i}}{4},\frac{1}{2\alpha}\}$, we have*
$$
|\langle \tilde{u}_{1,0},\tilde{u}_{1i}\rangle_{2,\Omega_{1\alpha}}| \le \frac{2\sqrt{2}}{\beta_i\sqrt{\alpha(2\alpha\beta_i-1)}} \tag{5.8}
$$

(2) *When the root β_j that determines $\tilde{u}_{1j}$ satisfies $\beta_j>\max\{\frac{e^{-2\alpha\beta_j}}{4\alpha},\frac{1}{2}\}$, we have*
$$
|\langle \tilde{u}_{1,0},\tilde{u}_{1j}\rangle_{2,\Omega_{1\alpha}}| \le \frac{2\sqrt{2}}{\beta_j\sqrt{\alpha(2\beta_j-1)}} \tag{5.9}
$$

(3) *When the roots β_i,β_i' that determine $\tilde{u}_{1i}$, $\tilde{u}_{1i}'$ satisfy $\beta_i'>\frac{2}{\alpha}+\beta_i$ and*
$$
\beta_i>\max\{\frac{e^{-2\beta_i}}{4},\frac{1}{2\alpha}\} \quad \textit{and} \quad \beta_i'>\max\big\{\frac{e^{-2\beta_i'}}{4},\frac{1}{2\alpha}\big\}
$$



(a) L^2-inner product between $\tilde{u}_{31}$ and $\tilde{u}_{3i}$ for $2 \le i \le 30$ on $\Omega_{1\alpha}$

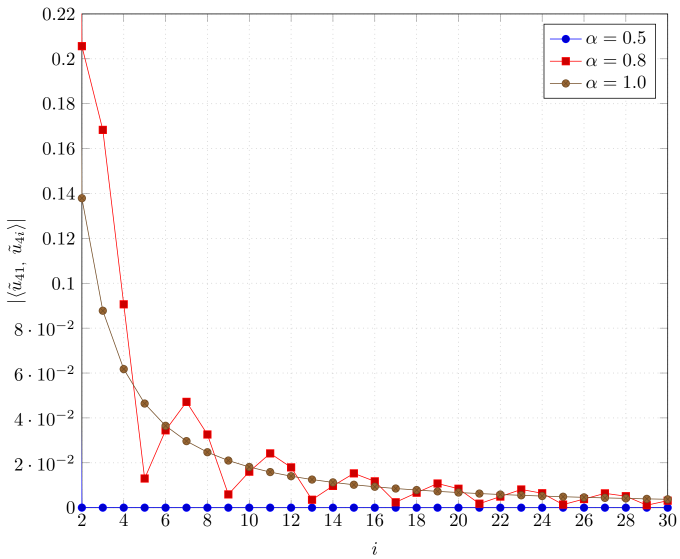


(b) L^2-inner product between $\tilde{u}_{31}$ and $\tilde{u}_{3j}$ for $1 \le j \le 30$ on $\Omega_{1\alpha}$

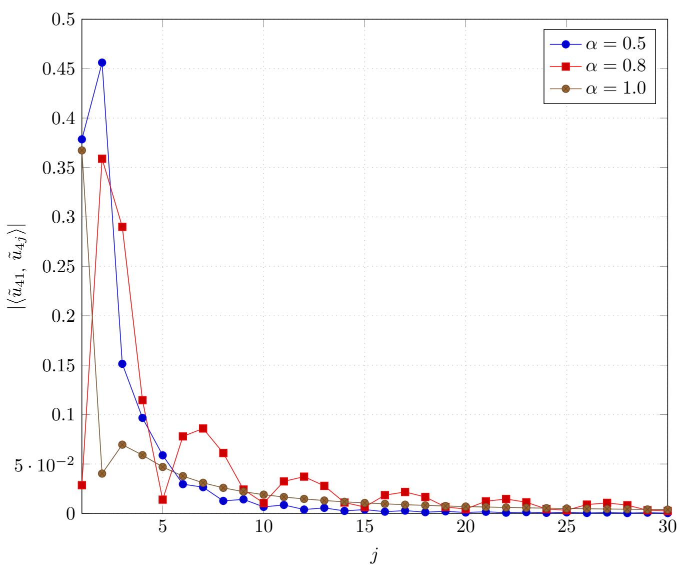
FIGURE 4. Near L^2-orthogonality of Class III Steklov eigenfunctions on $\Omega_{1\alpha}$ shown for $\alpha = 0.5,\ 0.8,\ 1.0$.

with $\alpha < 1$, we have

$$|\langle \tilde{u}_{1i}, \tilde{u}'_{1i}\rangle_{2,\Omega_{1\alpha}}| \le \frac{16\beta_i\beta'_i}{(\beta_i - \beta'_i)^2\sqrt{(2\alpha\beta_i - 1)(2\alpha\beta'_i - 1)}} \tag{5.10}$$



(a) L^2-inner product between $\tilde{u}_{41}$ and $\tilde{u}_{4i}$ for $2 \leq i \leq 30$ on $\Omega_{1\alpha}$



(b) L^2-inner product between $\tilde{u}_{41}$ and $\tilde{u}_{4j}$ for $1 \leq j \leq 30$ on $\Omega_{1\alpha}$

FIGURE 5. Near L^2-orthogonality of Class IV Steklov eigenfunctions on $\Omega_{1\alpha}$ shown for $\alpha = 0.5,\ 0.8,\ 1.0$.

(4) *When the roots β_i, β_j that determine $\tilde{u}_{1i}$, $\tilde{u}_{1j}$ satisfy*

$$\beta_i > \max\Big\{\frac{e^{-2\beta_i}}{4}, \frac{1}{2\alpha}\Big\} \quad \textit{and} \quad \beta_j > \max\Big\{\frac{e^{-2\alpha\beta_j}}{4\alpha}, \frac{1}{2}\Big\},$$

we have

$$|\langle \tilde{u}_{1i}, \tilde{u}_{1j}\rangle_{2,\Omega_{1\alpha}}| \le \frac{32(\beta_i+\beta_j)^2\beta_i\beta_j\ \cosh\beta_i\cosh\alpha\beta_j}{e^{\beta_i+\alpha\beta_j}\ (\beta_i^2+\beta_j^2)^2\sqrt{(2\alpha\beta_i-1)(2\beta_j-1)}} \tag{5.11}$$

Proof. Rewriting the formula (5.2) for normalized $\tilde{u}_{1,0}$ and $\tilde{u}_{1i}$ from Class I, we obtain

$$\langle \tilde{u}_{1,0}, \tilde{u}_{1i}\rangle_{2,\Omega_{1\alpha}} = \frac{4\sinh\beta_i\sin\alpha\beta_i}{\beta_i\sqrt{\alpha}\sqrt{(2\beta_i+\sinh 2\beta_i)(2\alpha\beta_i+\sin 2\alpha\beta_i)}} \le \frac{2\sqrt{2}}{\beta_i\sqrt{\alpha}\sqrt{2\alpha\beta_i-1}}$$

using $\sin\alpha\beta_i \le 1$ and $\sinh\beta_i \le \left(\frac{e_i^{\beta}}{2}\right)$ to obtain the majorizing numerator, and using the relation $\beta_i > \max\{\frac{e^{-2\beta_i}}{4}, \frac{1}{2\alpha}\}$ to obtain the smaller denominator at the end. Thus, the first assertion holds. An analogous majorization gives the envelope for $\langle \tilde{u}_{1,0}, \tilde{u}_{1j}\rangle_{2,\Omega_{1\alpha}}$.

Using that sine and cosine are bounded above by one, in the formula (5.4) the numerator of $\langle \tilde{u}_{1i}, \tilde{u}_{1j}\rangle_{2,\Omega_{1\alpha}}$ is majorized by

$$4[\beta_i\sinh\beta_i+\beta_j\cosh\beta_i][\beta_i\cosh(\alpha\beta_j)+\beta_j\sinh(\alpha\beta_j)].$$

For $\beta_i,\beta_j>0$, the relations $\sinh\beta_i<\cosh\beta_i$ and $\sinh\alpha\beta_j<\cosh\alpha\beta_j$ hold, which implies the numerator for $\langle \tilde{u}_{1i}, \tilde{u}_{1j}\rangle_{2,\Omega_{1\alpha}}$ is majorized by $4[\beta_i+\beta_j]^2\cosh\beta_i\cosh\alpha\beta_j$.

The denominator of $\langle \tilde{u}_{1i}, \tilde{u}_{1j}\rangle_{2,\Omega_{1\alpha}}$ can be rewritten as

$$\frac{(\beta_i^2+\beta_j^2)^2}{4\beta_i\beta_j}\sqrt{(2\beta_i+\sinh 2\beta_i)(2\alpha\beta_i+\sin 2\alpha\beta_i)(2\beta_j+\sin 2\beta_j)(2\alpha\beta_j+\sinh 2\alpha\beta_j)}.$$

When the roots β_i,β_j satisfy the prescribed inequalities, the factors under the square root are made smaller so that the denominator of $\langle \tilde{u}_{1i}, \tilde{u}_{1j}\rangle_{2,\Omega_{1\alpha}}$ is minorized by

$$\frac{(\beta_i^2+\beta_j^2)^2e^{\beta_i+\alpha\beta_j}}{8\,\beta_i\beta_j}\sqrt{(2\alpha\beta_i-1)(2\beta_j-1)}$$

These numerator and denominator results give the envelope for $|\langle \tilde{u}_{1i}, \tilde{u}_{1j}\rangle_{2,\Omega_{1\alpha}}|$ in the fourth assertion.

The third assertion that gives the envelope for $|\langle \tilde{u}_{1i}, \tilde{u}'_{1i}\rangle_{2,\Omega_{1\alpha}}|$ is a bit more tricky. From

$$\operatorname{sinhc}(\beta_i+\beta_i') \le \frac{\frac{1}{2}e^{\beta_i+\beta_i'}}{\beta_i+\beta_i'} \ \le\ \frac{\frac{1}{2}e^{\beta_i+\beta_i'}}{\beta_i'-\beta_i},$$

which holds since $\beta_i'>\beta_i$, and from

$$\operatorname{sinhc}(\beta_i-\beta_i') = \operatorname{sinhc}(\beta_i'-\beta_i) \le \frac{\frac{1}{2}e^{\beta_i'-\beta_i}}{\beta_i'-\beta_i} \ \le\ \frac{\frac{1}{2}e^{\beta_i'+\beta_i}}{\beta_i'-\beta_i}$$

it follows that

$$\operatorname{sinhc}(\beta_i+\beta_i')+\operatorname{sinhc}(\beta_i-\beta_i') \le \frac{e^{\beta_i+\beta_i'}}{\beta_i'-\beta_i}.$$

For the second factor in the numerator of $|\langle \tilde{u}_{1i}, \tilde{u}'_{1i}\rangle_{2,\Omega_{1\alpha}}|$, use $\operatorname{sinc}\theta \le \frac{1}{\theta}$ for $\theta>2$, to obtain

$$|\operatorname{sinc}(\alpha(\beta_i-\beta_i'))| = |\operatorname{sinc}(\alpha(\beta_i'-\beta_i))| \le \frac{1}{\alpha(\beta_i'-\beta_i)},$$

$$|\operatorname{sinc}(\alpha(\beta_i+\beta_i'))| \le \frac{1}{\alpha(\beta_i'+\beta_i)} \le \frac{1}{\alpha(\beta_i'-\beta_i)}$$

Thus
$$|\operatorname{sinc}(\alpha(\beta_i-\beta_i'))|+|\operatorname{sinc}(\alpha(\beta_i+\beta_i'))|\le \frac{2}{\alpha(\beta_i'-\beta_i)}.$$
The denominator for $|\langle \tilde u_{1i}, \tilde u_{1i}'\rangle_{2,\Omega_{1\alpha}}|$ can be rewritten as
$$\frac{1}{4\alpha\beta_i\beta_i'}\cdot\sqrt{(2\beta_i+\sinh 2\beta_i)(2\alpha\beta_i+\sin 2\alpha\beta_i)(2\beta_i'+\sinh 2\beta_i')(2\alpha\beta_i'+\sin 2\alpha\beta_i')}.$$
When β_i, β_i' satisfy the prescribed inequalities, this denominator is minorized by
$$\frac{e^{\beta_i+\beta_i'}}{8\alpha\beta_i\beta_i'}\sqrt{(2\alpha\beta_i-1)(2\alpha\beta_i'-1)}$$
Combining these numerator and denominator bounds and simplifying gives the third assertion. $\square$

Note that this theorem on envelopes provides exact inequalities for the inner products of Steklov eigenfunctions in Class I, and thus the envelope formulae are still somewhat intricate. However, a consequence of these bounding curves is the following succinct asymptotic estimates on the inner products.

Corollary 5.2. *Let $\tilde u_{1,0}$ and $\tilde u_{\ell,i}$, with $\ell = 1,2,3,4$, be the Steklov eigenfunctions normalized in $L^2(\Omega_{1\alpha})$ as described above.*

(1) *When the root β_i that determines $\tilde u_{1,i}$ satisfies $\beta_i > \max\{\frac{e^{-2\beta_i}}{4}, \frac{1}{2\alpha}\}$, we have*
$$|\langle \tilde u_{1,0}, \tilde u_{1i}\rangle_{2,\Omega_{1\alpha}}| \lessapprox \frac{2}{\alpha\beta_i^{3/2}}. \tag{5.12}$$

(2) *When the root β_j that determines $\tilde u_{1,j}$ satisfies $\beta_j > \max\left\{\frac{e^{-2\alpha\beta_j}}{4\alpha}, \frac{1}{2}\right\}$, we have*
$$|\langle \tilde u_{1,0}, \tilde u_{1j}\rangle_{2,\Omega_{1\alpha}}| \lessapprox \frac{2}{\sqrt{\alpha}\,\beta_j^{3/2}}. \tag{5.13}$$

(3) *When the roots β_i, β_i' that determine $\tilde u_{1i}$, $\tilde u_{1i}'$ satisfy $\beta_i' > \frac{2}{\alpha}+\beta_i$ and*
$$\beta_i > \max\big\{\frac{e^{-2\beta_i}}{4}, \frac{1}{2\alpha}\big\} \quad \textit{and} \quad \beta_i' > \max\big\{\frac{e^{-2\beta_i'}}{4}, \frac{1}{2\alpha}\big\}$$
with $\alpha < 1$, we have
$$|\langle \tilde u_{1i}, \tilde u_{1i}'\rangle_{2,\Omega_{1\alpha}}| \lessapprox \frac{8\sqrt{\beta_i\beta_i'}}{(\beta_i-\beta_i')^2\cdot\alpha}. \tag{5.14}$$

(4) *When the roots β_i, β_j that determine $\tilde u_{1i}$, $\tilde u_{1j}$ satisfy*
$$\beta_i > \max\big\{\frac{e^{-2\beta_i}}{4}, \frac{1}{2\alpha}\big\} \quad \textit{and} \quad \beta_j > \max\big\{\frac{e^{-2\alpha\beta_j}}{4\alpha}, \frac{1}{2}\big\},$$
we have
$$|\langle \tilde u_{1i}, \tilde u_{1j}\rangle_{2,\Omega_{1\alpha}}| \lessapprox \frac{4(\beta_i+\beta_j)^2\sqrt{\beta_i\beta_j}}{(\beta_i^2+\beta_j^2)^2\sqrt{\alpha}}. \tag{5.15}$$

Going a step further, note that for high frequency Steklov eigenfunctions $\tilde u_{1\ell}$, each of the four cases presented in the corollary simplify to the form
$$|\langle \tilde u_{1i}, \tilde u_{1\ell}\rangle_{2,\Omega_{1\alpha}}| \lessapprox C\cdot\beta_\ell^{-3/2} \tag{5.16}$$
for some constant C that depends on the aspect ratio α of $\Omega_{1\alpha}$ and the root β_i that determines the first factor $\tilde u_{1i}$. The relation (5.16) quantifies the statement

that Class I Steklov eigenfunctions are nearly (pairwise) orthogonal in $L^2(\Omega_{1\alpha})$, and from the above results, Class I is in fact exactly orthogonal in $L^2(\Omega_{1\alpha})$ to all other classes.

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