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NONLOCAL FRACTIONAL PROBLEMS AND ∇-THEOREMS

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In memory of Anna Aloe

ABSTRACT. We prove the multiplicity result in [12] under more general assumptions. More precisely, we prove the existence of three nontrivial solutions for a nonlocal problem when a parameter approaches one of the eigenvalues of the leading operator, without assuming the Ambrosetti-Rabinowitz condition.

1. Introduction

In this article we prove the existence of three nontrivial solutions for a class of nonlocal problems when a parameter approaches one of the eigenvalues of the leading operator and when the nonlinear terms has superlinear and subcritical behaviour. The result is in the spirit of [12], but here the result is proved under more general assumptions.

Going into details, we consider a class of problems near resonance whose prototype is

$$\begin{gathered} (-\Delta)^s u = \lambda u + f(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \end{gathered} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz continuous boundary, $\lambda \in \mathbb{R}$, f is a Carathéodory function which is superlinear and subcritical in the sense of the fractional Sobolev exponent. Moreover, $s \in (0,1)$ and $(-\Delta)^s$ is the fractional Laplace operator, which (up to normalization factors) may be defined as

$$-(-\Delta)^s u = \int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u}{|y|^{n+2s}}\, dy\,, \quad x \in \mathbb{R}^n. \tag{1.2}$$

Actually, we shall consider more general nonlocal operators, in place of $(-\Delta)^s$, and thus we will focus on problems of the form

$$\begin{gathered} -\mathcal{L}_K u = \lambda u + f(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega, \end{gathered} \tag{1.3}$$

where the nonlocal operator $\mathcal{L}_K$ is defined as

$$\mathcal{L}_K u(x) = \int_{\mathbb{R}^n} \Big(u(x+y)+u(x-y)-2u(x)\Big) K(y)\, dy\,, \quad x \in \mathbb{R}^n, \tag{1.4}$$

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and $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ is such that

$$K(-x)=K(x)\quad\text{for any } x\in\mathbb{R}^n\setminus\{0\}, \tag{1.5}$$

$$mK\in L^1(\mathbb{R}^n),\quad\text{where } m=\min\{|x|^2,1\}, \tag{1.6}$$

$$\text{there exists } \theta>0 \text{ such that } K\geq\theta|x|^{-(n+2s)} \text{ for every } x\in\mathbb{R}^n\setminus\{0\}. \tag{1.7}$$

We notice that, similarly to [4, Lemma 3.5], an equivalent formulation for $\mathcal{L}_K$ is given, as usual up to some positive constant, by

$$\begin{aligned}\mathcal{L}_K u(x)&=\mathrm{P.V.}\int_{\mathbb{R}^n}\big(u(x)-u(y)\big)K(x-y)\,dy\\&=\lim_{\varepsilon\to0}\int_{|x|\geq\varepsilon}\big(u(x)-u(y)\big)K(x-y)\,dy\end{aligned}\tag{1.8}$$

for every $x\in\mathbb{R}^n$, *P.V.* standing for the "Cauchy principal value".

Before stating our result, we recall that the "boundary condition" $u=0$ in $\mathbb{R}^n\setminus\Omega$ leads to settle the problem in a particular functional setting, namely, in view of (1.8), a weak solution of (1.3) is a function $u\in X_0$ such that

$$\int_{\mathbb{R}^n\times\mathbb{R}^n}(u(x)-u(y))(\varphi-\varphi(y))K(x-y)\,dxdy=\lambda\int_\Omega u\varphi\,dx+\int_\Omega f(x,u)\varphi dx \tag{1.9}$$

for every $\varphi\in X_0$. Here X_0 is defined as follows: first, X is the linear space

$$\begin{aligned}X=\Big\{&u\in\mathcal{M}(\mathbb{R}^n): u_{|\Omega}\in L^2(\Omega)\text{ and the map}\\&(x,y)\mapsto(g(x)-g(y))\sqrt{K(x-y)}\in L^2\big(\mathbb{R}^n\times\mathbb{R}^n\setminus(\mathcal{C}\Omega\times\mathcal{C}\Omega),dxdy\big)\Big\},\end{aligned}$$

where $\mathcal{C}\Omega:=\mathbb{R}^n\setminus\Omega$. Finally,

$$X_0=\{g\in X: g=0\text{ a.e. in }\mathbb{R}^n\setminus\Omega\}.$$

We recall that

$$\langle u,v\rangle_{X_0}:=\int_{\mathbb{R}^n\times\mathbb{R}^n}\big(u(x)-u(y)\big)\big(\varphi-\varphi(y)\big)K(x-y)\,dx\,dy$$

makes X_0 a Hilbert space, see [26, Lemma 7].

Moreover, we also need to recall that $-\mathcal{L}_K$ admits a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ of eigenvalues having finite multiplicity and with the property that

$$\begin{gathered}0<\lambda_1<\lambda_2\leq\dots\leq\lambda_k\leq\lambda_{k+1}\leq\dots,\\\lambda_k\to+\infty\quad\text{as } k\to+\infty.\end{gathered}\tag{1.10}$$

In addition, if e_k is the eigenfunction corresponding to λ_k normalized in $L^2(\Omega)$, then $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of X_0, see [25, 27].

Finally, we say that eigenvalue λ_k, $k\geq2$, has multiplicity $m\in\mathbb{N}$ if

$$\lambda_{k-1}<\lambda_k=\dots=\lambda_{k+m-1}<\lambda_{k+m},$$

and in such a case the set of all eigenfunctions associated to λ_k coincides with $\mathrm{span}\{e_k,\dots,e_{k+m-1}\}$.

In this article, for any $k\in\mathbb{N}$ we set

$$\begin{gathered}H_k=\mathrm{span}\{e_1,\dots,e_k\},\\H_k^\perp=\big\{u\in X_0:\langle u,e_j\rangle_{X_0}=0\text{ for any } j=1,\dots,k\big\},\end{gathered}$$

so that H_k has precisely dimension k.

In this way, the variational characterization of the eigenvalues (see [27]) gives

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(y)|^2 K(x-y)\,dx\,dy \geq \lambda_{k+1}\int_\Omega |u|^2\,dx \text{ for all } u\in H_k^\perp. \tag{1.11}$$

On the other hand, by the orthogonality properties of the eigenvalues, a standard Fourier decomposition gives

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(y)|^2 K(x-y)\,dx\,dy \leq \lambda_k\int_\Omega |u|^2\,dx \text{ for all } u\in H_k. \tag{1.12}$$

The aim of this paper is to exploit a critical point theorem *of mixed type*, one of the so-called ∇*-theorems*, introduced by Marino and Saccon [10] (see also [9, 11, 17]), which permit to provide multiplicity results in a very elegant way. These theorems have been successfully employed in several contexts, see, for instance, [8, 18, 19, 20, 22, 23, 28, 29, 30]. In particular, one theorem of this type was used in [12] for showing a multiplicity result for a problem like (1.3), assuming that f satisfies a growth condition of the Ambrosetti-Rabinowitz type. Here we want to obtain the same result in a more general setting. Indeed, we assume:

(A1) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying the following conditions: there exist $a_1,a_2>0$ and $q\in(2,2^*)$, $2^*=2n/(n-2s)$ such that

$$|f(x,t)|\leq a_1+a_2|t|^{q-1} \quad \text{a.e. } x\in\Omega, t\in\mathbb{R}\,; \tag{1.13}$$

$$\lim_{|t|\to 0}\frac{f(x,t)}{|t|}=0 \quad \text{uniformly in } x\in\Omega\,; \tag{1.14}$$

$$f(x,t)t-2F(x,t)>0 \quad \text{for a.e. } x\in\Omega \text{ and all } t\in\mathbb{R},\ t\neq 0, \tag{1.15}$$

$$\lim_{|t|\to\infty}\frac{F(x,t)}{t^2}=+\infty \quad \text{uniformly in } x\in\Omega, \tag{1.16}$$

there exist positive constants $p>\max\left\{\frac{2n}{n+2s}(q-1),q-1\right\}$, $a_3>0$ and $R>0$ such that

$$f(x,t)t-2F(x,t)\geq a_3|t|^p \quad \text{for a.e. } x\in\Omega \text{ and every } |t|\geq R; \tag{1.17}$$

$$F(x,t)\geq 0 \text{ for a.e. } x\in\Omega \text{ and all } t\in\mathbb{R}. \tag{1.18}$$

Here

$$F(x,t):=\int_0^t f(x,\tau)d\tau \quad \text{for a.e. } x\in\Omega \text{ and all } t\in\mathbb{R}. \tag{1.19}$$

As an example for f we can take $f(x,t)=a(x)|t|^{q-2}t$, with $a\in L^\infty(\Omega)$, $\inf_\Omega a>0$ (see [21]) and $q\in(2,2^*)$.

Remark 1.1. The common Ambrosetti-Rabinowitz condition, i.e. there exists $\mu>2$ and $R\geq 0$ such that

$(AR)\qquad 0<F(x,t)\leq f(x,t)t,$

for a.e. $x\in\Omega$ and all $|t|>R$, is not sufficient to ensure that $F(x,\cdot)$ can be estimated from below by a superquadratic power, while it would be if (AR) holds for every $(x,t)\in\bar\Omega\times\mathbb{R}$ (see [21]). For this reason, it seems natural, in this general context, to assume *a priori* some kind of control from below, as we do in (1.17), though we *do not* require $p > 2$. Indeed,

$$\frac{2n}{n+2s}(q-1) \geq 2$$

if and only if

$$q \geq 4\frac{n+s}{n+2s} > 2^*,$$

which is not an admissible occurrence.

On the other hand, by (1.13) and (1.19) it is clear that $p \leq q$.

Very close assumptions on f were assumed in [6] for studying a fourth order problem in bounded domains through the same approach via $\nabla-$theorems. Inspired by [6], our main result reads as follows.

Theorem 1.2. *Let $s \in (0,1)$, $n > 2s$ and Ω be an open bounded subset of $\mathbb{R}^n$ with continuous boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ satisfy* (1.5)–(1.7) *and let f satisfy* (A1). *Then, for every eigenvalue λ_k of $-\mathcal{L}_K$, $k \geq 2$, there exists a left neighborhood $\mathcal{O}_k$ of λ_k such that problem* (1.3) *admits at least three nontrivial weak solutions for all $\lambda \in \mathcal{O}_k$.*

Corollary 1.3. *Under the assumptions of Theorem 1.2, for every $k \geq 2$ there exists a left neighborhood O_k of the $k-th$ eigenvalue λ_k of $(-\Delta)^s$, such that, if $\lambda \in O_k$, then* (1.1) *admits at least three nontrivial weak solutions.*

This article is organized in the following way: in Section 2 we recall some notions and notations which will be used throughout the paper. In Section 3 we prove that the energy functional associated to the problem enjoys some good geometric structures. In Section 4 we prove the ∇-condition, the main ingredient of the critical point tool that we shall use, which is Theorem 5.1. Finally, in Section 5 we prove the main multiplicity result of this paper, i.e. Theorem 1.2, by coupling the result of the ∇-theorem due to Marino and Saccon in [10] with a classical Linking theorem (see [24, Theorem 5.3]), obtaining the existence of three nontrivial solutions for problem (1.3). We remark that while the existence of two nontrivial solutions near resonance is free due to bifurcation theory, the fine estimates on the critical value provided by Theorem 5.1 permit to compare the critical value obtained with a Linking Theorem and find a third nontrivial solution, being the energy of the last solution higher than that of the former two ones.

A last comment on the notation: we will use several times the symbol c or C to denote absolute constants, which, however, may be different from previous ones denoted in the same way.

2. Preliminaries

First of all, we need some notation. In the sequel we endow the space X_0 with the norm defined as (see [26, Lemma 6])

$$\|g\| = \Big(\int_{\mathbb{R}^n\times\mathbb{R}^n} |g(x)-g(y)|^2 K(x-y)dx\,dy\Big)^{1/2}, \tag{2.1}$$

which is obviously related to the so-called *Gagliardo norm*

$$\|g\|_{H^s(\Omega)} = \|g\|_{L^2(\Omega)} + \Big(\int_{\Omega\times\Omega} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2} \tag{2.2}$$

of the usual fractional Sobolev space $H^s(\Omega)$. For further details on the fractional Sobolev spaces we refer to [1, 4, 14] and to the references therein. We only recall the following embeddings, which will be repeatedly used and for whose proofs we refer to [26]:

$$\begin{aligned} X_0 &\hookrightarrow L^\nu(\Omega) \quad \text{for every } \nu \in [1, 2^*], \\ X_0 &\hookrightarrow\hookrightarrow L^\nu(\Omega) \quad \text{for every } \nu \in [1, 2^*). \end{aligned} \tag{2.3}$$

Problem (1.9) has a variational structure: indeed, it is the Euler-Lagrange equation of the functional $\mathcal{J}_\lambda : X_0 \to \mathbb{R}$ defined as

$$\mathcal{J}_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \frac{\lambda}{2}\int_\Omega u^2\,dx - \int_\Omega F(x,u)\,dx.$$

Note that when functional $\mathcal{J}_\lambda$ is Fréchet differentiable at $u \in X_0$, we have that for any $\varphi \in X_0$

$$\begin{aligned} \langle \mathcal{J}_\lambda'(u), \varphi\rangle = &\int_{\mathbb{R}^n\times\mathbb{R}^n} \big(u(x)-u(y)\big)\big(\varphi-\varphi(y)\big)K(x-y)\,dx\,dy \\ &- \lambda\int_\Omega u\varphi\,dx - \int_\Omega f(x,u)\varphi\,dx, \end{aligned}$$

where we have denoted by $\langle\cdot,\cdot\rangle$ the duality between X_0' and X_0. Thus, critical points of $\mathcal{J}_\lambda$ are solutions to problem (1.9). We remark that (A1) ensures that $\mathcal{J}_\lambda$ is actually of class C^1, and so we can find solutions to (1.9) by looking for critical points to $\mathcal{J}_\lambda$. This is what we shall do using the ∇-theorem in the form of Theorem 5.1 (see Section 5) and the classical Linking Theorem.

We conclude this section recalling that problems of the form (1.3) have been widely investigated in latest years, under different assumptions on λ and f. The literature in this context is huge, and we only refer to some recent papers and the references therein, quoting, in addition to the already cited ones, [2, 3, 5, 7, 13, 15].

3. Geometry of the ∇-theorem

In this section we show that if k and m in $\mathbb{N}$ are such that

$$\lambda_{k-1} < \lambda < \lambda_k = \dots = \lambda_{k+m-1} < \lambda_{k+m}, \tag{3.1}$$

then $\mathcal{J}_\lambda$ satisfies the geometric setting of Theorem 5.1 with

$$X_1 := H_{k-1}, X_2 := \operatorname{span}\{e_k,\dots,e_{k+m-1}\} \quad X_3 := H_{k+m-1}^\perp\,.$$

Proposition 3.1. *Let k and m in $\mathbb{N}$ be such that* (3.1) *holds and let f satisfy* (A1). *Then, there exist ρ, R, with $R > \rho > 0$, such that*

$$\sup_{\{u\in X_1,\|u\|\le R\}\cup\{u\in X_1\oplus X_2:\|u\|=R\}} \mathcal{J}_\lambda(u) < \inf_{\{u\in X_2\oplus X_3:\|u\|=\rho\}} \mathcal{J}_\lambda(u)\,.$$

Proof. Take $u \in X_1$. Then by (1.18) it is straightforward to see that

$$\mathcal{J}_\lambda(u) \le \frac{\lambda_{k-1}-\lambda}{2}\int_\Omega u^2 dx \le 0, \tag{3.2}$$

since $\lambda_{k-1} < \lambda$. Moreover, by (1.16) there exists $M > 0$ such that

$$F(x,t) \ge (\lambda_k - \lambda)t^2 - M$$

for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$. Thus, if $u \in X_1 \oplus X_2$, we obtain

$$\mathcal{J}_\lambda(u) \le \frac{\lambda_k-\lambda}{2}\int_\Omega u^2 dx - (\lambda_k-\lambda)\int_\Omega u^2 dx + M|\Omega| = -\frac{\lambda_k-\lambda}{2}\int_\Omega u^2 dx + M|\Omega|,$$

and, being all norms equivalent in $X_1 \oplus X_2$, we obtain that

$$\lim_{u \in X_1 \oplus X_2,\, \|u\| \to \infty} \mathcal{J}_\lambda(u) = -\infty. \tag{3.3}$$

Now, by (1.13) and (1.14) we obtain that, fixed $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$F(x,t) < \frac{\varepsilon}{2} t^2 + M_\varepsilon |t|^q \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

Then, if $u \in X_2 \oplus X_3$, by (1.11) and (2.3) we obtain

$$\mathcal{J}_\lambda(u) \geq \frac{1}{2}\Big(1 - \frac{\lambda + \varepsilon}{\lambda_k}\Big)\|u\|^2 - M_\varepsilon \int_\Omega |u|^q dx \geq \frac{1}{2}\Big(1 - \frac{\lambda + \varepsilon}{\lambda_k}\Big)\|u\|^2 - \tilde{M}_\varepsilon \|u\|^q$$

for some $\tilde{M}_\varepsilon > 0$. Choosing $\varepsilon < \lambda_k - \lambda$, we can find $\rho > 0$ so small that

$$\inf_{\{u \in X_2 \oplus X_3 : \|u\| = \rho\}} \mathcal{J}_\lambda(u) > 0. \tag{3.4}$$

By (3.2), (3.3) and (3.4), the claim follows. $\square$

4. ∇-condition

To prove the $\nabla-$condition, we denote by $P_C : X_0 \to C$ the orthogonal projection of X_0 onto a closed subspace C, and we recall the following concept.

Definition 4.1. Let C be a closed subspace of X_0 and $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$. We say that $\mathcal{J}_\lambda$ verifies $(\nabla)(\mathcal{J}_\lambda, C, a, b)$ if there exists $\gamma > 0$ such that

$$\inf\big\{\|P_C \nabla \mathcal{J}_\lambda(u)\| : a \leq \mathcal{J}_\lambda(u) \leq b, \operatorname{dist}(u, C) \leq \gamma\big\} > 0\,.$$

Roughly speaking, the condition $(\nabla)(\mathcal{J}_\lambda, C, a, b)$ requires that $\mathcal{J}_\lambda$ has no critical points $u \in C$ such that $a \leq \mathcal{J}_\lambda(u) \leq b$, with some uniformity. The main purpose of this section is to prove the following result.

Proposition 4.2. *Let k and m in $\mathbb{N}$ be such that* (3.1) *holds and let f satisfy* (A1)*. Then, for any $\sigma > 0$ with $\sigma < \min\{\lambda_{k+m} - \lambda_k, \lambda_k - \lambda_{k-1}\}$ there exists $\varepsilon_\sigma > 0$ such that for any $\lambda \in [\lambda_{k-1} + \sigma, \lambda_{k+m} - \sigma]$ and for any $\varepsilon', \varepsilon'' \in (0, \varepsilon_\sigma)$, with $\varepsilon' < \varepsilon''$, functional $\mathcal{J}_\lambda$ satisfies $(\nabla)(\mathcal{J}_\lambda, H_{k-1} \oplus H_{k+m-1}^\perp, \varepsilon', \varepsilon'')$.*

Of course, in our case $C = H_{k-1} \oplus H_{k+m-1}^\perp$, and without mentioning any longer, we assume (3.1) and (A1). We start by proving the following result.

Lemma 4.3. *For any σ such that $0 < \delta < \min\{\lambda_{k+m} - \lambda_k, \lambda_k - \lambda_{k-1}\}$ there exists $\varepsilon_\sigma > 0$ such that for any $\lambda \in [\lambda_{k-1} + \sigma, \lambda_{k+m} - \sigma]$ the unique critical point u of $\mathcal{J}_\lambda$ constrained on $H_{k-1} \oplus H_{k+m-1}^\perp$ with $\mathcal{J}_\lambda(u) \in [-\varepsilon_\sigma, \varepsilon_\sigma]$, is the trivial one.*

Proof. We argue by contradiction and we suppose that there exists $\bar{\sigma} > 0$, a sequence $\{\mu_j\}_{j \in \mathbb{N}}$ in $\mathbb{R}$ with

$$\mu_j \in [\lambda_{k-1} + \bar{\sigma}, \lambda_{k+m} - \bar{\sigma}] \tag{4.1}$$

and a sequence $\{u_j\}_{j \in \mathbb{N}} \subset H_{k-1} \oplus H_{k+m-1}^\perp \setminus \{0\}$ such that

$$\langle \mathcal{J}'_{\mu_j}(u_j), \varphi \rangle = 0 \quad \text{for any } \varphi \in H_{k-1} \oplus H_{k+m-1}^\perp \text{ and any } j \in \mathbb{N}, \tag{4.2}$$

$$\begin{aligned} \mathcal{J}_{\mu_j}(u_j) = \frac{1}{2} &\int_{\mathbb{R}^n \times \mathbb{R}^n} |u_j - u_j(y)|^2 K(x-y)\, dx\, dy \\ &- \frac{\mu_j}{2} \int_\Omega |u_j|^2\, dx - \int_\Omega F(x, u_j)\, dx \to 0 \quad \text{as } j \to +\infty\,. \end{aligned} \tag{4.3}$$

By (1.17) and (1.13) we obtain the existence of $a_4 > 0$ such that

$$f(x,t)t - 2F(x,t) \geq a_3|t|^p - a_4 \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}. \tag{4.4}$$

Taking $\varphi = u_j$ in (4.2) and using (4.4), we obtain that for any $j \in \mathbb{N}$,

$$\begin{aligned} 2\mathcal{J}_{\mu_j}(u_j) - \langle \mathcal{J}'_{\mu_j}(u_j), u_j\rangle &= \int_\Omega (f(x,u_j)u_j - F(x,u_j))\,dx \\ &\geq a_3\int_\Omega |u_j|^p dx - a_5 \end{aligned}$$

for some positive constant a_5. Hence, by (4.2) and (4.3), we immediately get that

$$(u_j)_{j\in\mathbb{N}} \quad \text{is bounded in } L^p(\Omega). \tag{4.5}$$

Now, let $v_j \in H_{k-1}$ and $w_j \in H_{k+m-1}^{\perp}$ be such that $u_j = v_j + w_j$ for any $j \in \mathbb{N}$. Choosing $\varphi = v_j - w_j$ in (4.2) and taking into account the orthogonality properties of v_j and w_j, we have that for any $j \in \mathbb{N}$

$$\begin{aligned} 0 &= \langle \mathcal{J}'_{\mu_j}(u_j), v_j - w_j\rangle \\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} |v_j(x) - v_j(y)|^2 K(x-y)\,dx\,dy \\ &\quad - \int_{\mathbb{R}^n\times\mathbb{R}^n} |w_j(x) - w_j(y)|^2 K(x-y)\,dx\,dy \\ &\quad - \mu_j\int_\Omega |v_j|^2\,dx + \mu_j\int_\Omega |w_j|^2\,dx - \int_\Omega f(x,u_j)(v_j - w_j)\,dx\,. \end{aligned} \tag{4.6}$$

By (1.12) and (1.11), equation (4.6) implies that for any $j \in \mathbb{N}$,

$$\begin{aligned} \int_\Omega f(x,u_j)(v_j - w_j)\,dx &= \int_{\mathbb{R}^n\times\mathbb{R}^n} |v_j(x) - v_j(y)|^2 K(x-y)\,dx\,dy \\ &\quad - \int_{\mathbb{R}^n\times\mathbb{R}^n} |w_j(x) - w_j(y)|^2 K(x-y)\,dx\,dy \\ &\quad - \mu_j\int_\Omega |v_j|^2\,dx + \mu_j\int_\Omega |w_j|^2\,dx \\ &\leq \frac{\lambda_{k-1} - \mu_j}{\lambda_{k-1}}\|v_j\|^2 + \frac{\mu_j - \lambda_{k+m}}{\lambda_{k+m}}\|w_j\|^2 \\ &\leq -\frac{\bar{\sigma}}{\lambda_{k-1}}\|v_j\|^2 - \frac{\bar{\sigma}}{\lambda_{k+m}}\|w_j\|^2 \\ &\leq -\frac{\bar{\sigma}}{\lambda_{k+m}}\|u_j\|^2. \end{aligned} \tag{4.7}$$

Hence, there exists $C > 0$ such that

$$\|u_j\|^2 \leq C\int_\Omega f(x,u_j)(v_j - w_j)\,dx \quad \text{for all } j \in \mathbb{N}. \tag{4.8}$$

Now, since

$$\frac{2n}{n+2s}(q-1) \leq p < 2^*,$$

we immediately get that

$$1 < \frac{p}{p-q+1} \leq 2^*,$$

and so, by (2.3), there exists $C > 0$ such that
$$\|u\|_{\frac{p}{p-q+1}} \le C\|u\| \quad \text{for every } u \in X_0. \tag{4.9}$$
As a consequence, by the Hölder inequality and (4.5), we obtain
$$\int_\Omega |u_j|^{q-1}|v_j - w_j|\,dx \le \|u_j\|_p^{q-1}\|v_j - w_j\|_{p-q+1} \le c\|v_j - w_j\|_{L^{2^*}(\Omega)} \tag{4.10}$$
for some $c > 0$. Moreover, by (1.13) we obtain
$$\Big|\int_\Omega f(x,u_j)(v_j - w_j)\,dx\Big| \le a_1\int_\Omega |v_j - w_j|\,dx + a_2\int_\Omega |u_j|^{q-1}|v_j - w_j|\,dx. \tag{4.11}$$
Hence, by (4.11), (2.3), (4.8) and (4.10), we obtain the existence of two constants C_1, $C_2 > 0$ such that for all $j \in \mathbb{N}$
$$\|u_j\|^2 \le C_1\|v_j - w_j\| + C_2\|v_j - w_j\| = C_3\|v_j + w_j\| = C_3\|u_j\|,$$
and so $(u_j)_{j\in\mathbb{N}}$ is bounded in X_0. Then, we can assume that there exists $u_\infty \in H_{k-1} \oplus H^\perp_{k+m-1}$ such that, by (4.2),
$$\begin{aligned} &\int_{\mathbb{R}^n\times\mathbb{R}^n} \big(u_j(x) - u_j(y)\big)\big(\varphi(x) - \varphi(y)\big)K(x-y)\,dx\,dy \\ &\to \int_{\mathbb{R}^n\times\mathbb{R}^n} \big(u_\infty(x) - u_\infty(y)\big)\big(\varphi - \varphi(y)\big)K(x-y)\,dx\,dy \quad \text{for any } \varphi \in X_0\,, \end{aligned} \tag{4.12}$$
$$\begin{aligned} &u_j \to u_\infty \quad \text{in } L^q(\mathbb{R}^n) \\ &u_j \to u_\infty \quad \text{a.e. in } \mathbb{R}^n \end{aligned} \tag{4.13}$$
as $j \to +\infty$. Now, taking $\varphi = u_j$ in (4.2) and using (4.4), we obtain that for any $j \in \mathbb{N}$
$$0 = 2\mathcal{J}_{\mu_j}(u_j) - \langle \mathcal{J}'_{\mu_j}(u_j), u_j\rangle = \int_\Omega \big(f(x,u_j)u_j - F(x,u_j)\big)dx.$$
Passing to the limit in the equation above, by (1.13) and (4.13), we obtain
$$0 = \int_\Omega (f(x,u_\infty)u_\infty - F(x,u_\infty))\,dx,$$
and so (1.15) implies $u_\infty \equiv 0$.

From (4.8) we also obtain
$$\begin{aligned} \|u_j\|^2 &\le C\Big(\int_\Omega |f(x,u_j)|^{\frac{q}{q-1}}\,dx\Big)\|v_j - w_j\|_{L^q(\Omega)} \\ &\le \tilde{C}\Big(\int_\Omega |f(x,u_j)|^{\frac{q}{q-1}}\,dx\Big)\|v_j - w_j\| \\ &= \tilde{C}\Big(\int_\Omega |f(x,u_j)|^{\frac{q}{q-1}}\,dx\Big)\|u_j\| \end{aligned}$$
for some $\tilde{C} > 0$ and all $j \in \mathbb{N}$. Since $u_j \ne 0$, we obtain
$$\|u_j\| \le C\Big(\int_\Omega |f(x,u_j)|^{\frac{q}{q-1}}\,dx\Big) \tag{4.14}$$
for some $C > 0$ and all $j \in \mathbb{N}$. Now, if $u_j \to 0$ in X_0, from (4.14) and (1.14) we would get
$$1 \le \lim_{j\to\infty} C\frac{\big(\int_\Omega |f(x,u_j)|^{\frac{q}{q-1}}\,dx\big)}{\|u_j\|} = 0,$$

which is absurd. Hence, we can assume that there is $A > 0$ such that $\|u_j\| \geq C$ for all $j \in \mathbb{N}$. Hence, (4.14) and the fact that $u_j \to 0$ in $L^q(\Omega)$ would give

$$A \leq \lim_{j\to\infty} C\Big(\int_\Omega |f(x,u_j)|^{\frac{q}{q-1}}\,dx\Big) = 0,$$

again a contradiction. The proof is complete. □

Before going on, we recall that, as showed in [12], we have

$$\nabla\mathcal{J}_\lambda(u) = u - \mathcal{L}_K^{-1}(\lambda u + f(x,u)) \tag{4.15}$$

for all $u \in X_0$, where

$$\mathcal{L}_K^{-1} : L^\nu(\Omega) \to X_0 \text{ is a compact operator for all } \nu \in [1, 2^*). \tag{4.16}$$

Moreover,

$$\langle u, \mathcal{L}_K^{-1} v\rangle_{X_0} = \int_\Omega uv\,dx \tag{4.17}$$

for every $u, v \in X_0$. The second lemma we need in order to prove the ∇-condition is the following one.

Lemma 4.4. *Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in X_0 such that*

$$\{\mathcal{J}_\lambda(u_j)\}_{j\in\mathbb{N}} \text{ is bounded in } \mathbb{R}, \tag{4.18}$$

$$P_{\mathrm{span}\{e_k,\dots,e_{k+m-1}\}} u_j \to 0 \quad \text{in } X_0, \tag{4.19}$$

$$P_{H_{k-1}\oplus H_{k+m-1}^\perp} \nabla\mathcal{J}_\lambda(u_j) \to 0 \quad \text{in } X_0 \text{ as } j \to +\infty. \tag{4.20}$$

Then, $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0.

Proof. Assume by contradiction that $\{u_j\}_{j\in\mathbb{N}}$ is unbounded in X_0; hence, we can assume that

$$\|u_j\| \to +\infty \tag{4.21}$$

as $j \to +\infty$ and that there exists $u_\infty \in X_0$ such that

$$\begin{gathered} \frac{u_j}{\|u_j\|} \rightharpoonup u_\infty \quad \text{in } X_0 \\ \frac{u_j}{\|u_j\|} \to u_\infty \quad \text{in } L^\nu(\Omega) \text{ for any } \nu \in [1,2^*) \end{gathered} \tag{4.22}$$

as $j \to +\infty$.

Now, for shortness, set $P_{\mathrm{span}\{e_k,\dots,e_{k+m-1}\}} =: P$, $P_{H_{k-1}\oplus H_{k+m-1}^\perp} =: Q$, and write

$$u_j = Pu_j + Qu_j\,,$$

where $Pu_j \to 0$ as $j \to \infty$ by (4.19).

First, by (1.13) and Hölder's inequality, since $p > q-1$, we have that there exists $c_1 > 0$ such that for a.e. $x \in \Omega$ and all $j \in \mathbb{N}$

$$|f(x,u_j)Pu_j| \leq c_1\|Pu_j\|_\infty\big(1 + \|u_j\|_p^{q-1}\big).$$

Recalling (4.15), we have

$$\begin{aligned} \langle Q\nabla\mathcal{J}_\lambda(u_j), u_j\rangle_{X_0} &= \langle \nabla\mathcal{J}_\lambda(u_j), u_j\rangle_{X_0} - \langle P\nabla\mathcal{J}_\lambda(u_j), u_j\rangle_{X_0} \\ &= \|u_j\|^2 - \lambda\int_\Omega |u_j|^2\,dx - \int_\Omega f(x,u_j)u_j\,dx \\ &\quad - \langle P\big(u_j - \mathcal{L}_K^{-1}(\lambda u_j + f(x,u_j))\big), u_j\rangle_{X_0}. \end{aligned} \tag{4.23}$$

Since $\langle Pu, v\rangle_{X_0} = \langle u, Pv\rangle_{X_0}$ for any $u, v \in X_0$, by (4.17) (4.23) reads

$$\begin{aligned}\langle Q\nabla\mathcal{J}_\lambda(u_j), u_j\rangle_{X_0} &= 2\mathcal{J}_\lambda(u_j) + 2\int_\Omega F(x,u_j)dx - \int_\Omega f(x,u_j)u_j dx \\ &\quad - \|Pu_j\|^2 + \lambda\int_\Omega |Pu_j|^2\,dx + \int_\Omega f(x,u_j)Pu_j dx.\end{aligned} \tag{4.24}$$

As a consequence, by (1.17) there exists $c_2 > 0$ such that

$$\begin{aligned}2\mathcal{J}_\lambda(u_j) - \langle Q\nabla\mathcal{J}_\lambda(u_j), u_j\rangle_{X_0} &= \int_\Omega \left(f(x,u_j)u_j dx - 2F(x,u_j)\right)dx \\ &\quad + \|Pu_j\|^2 - \lambda\int_\Omega |Pu_j|^2\,dx - \int_\Omega f(x,u_j)Pu_j\,dx \\ &\geq a_3\int_\Omega |u_j|^p dx - c_2 + \|Pu_j\|^2 - \lambda\int_\Omega |Pu_j|^2\,dx \\ &\quad - c_1\|Pu_j\|_\infty\big(1 + \|u_j\|_p^{q-1}\big).\end{aligned}$$

Recalling (4.18)-(4.20) and that $p > q-1$, we easily obtain

$$\lim_{j\to\infty} \frac{\|u_j\|_p^{q-1}}{\|u_j\|} = 0, \tag{4.25}$$

since X_2 has finite dimension, and so all norms are equivalent. As a consequence of (4.25) we also get

$$u_\infty = 0. \tag{4.26}$$

Now, by (4.18), (4.21) and (4.26) we obtain

$$\frac{\mathcal{J}_\lambda(u_j)}{\|u_j\|^2} = \frac{1}{2} - \frac{\lambda}{2}\frac{\int_\Omega |u_j|^2\,dx}{\|u_j\|^2} - \frac{\int_\Omega F(x,u_j)\,dx}{\|u_j\|^2} \to 0,$$

which implies that

$$\frac{\int_\Omega F(x,u_j)\,dx}{\|u_j\|^2} \to \frac{1}{2} \tag{4.27}$$

as $j \to +\infty$. But, by (1.13), proceeding as for (4.9),

$$\Big|\int_\Omega F(x,u_j)\,dx\Big| \leq a_1\int_\Omega |u_j|\,dx + \frac{a_2}{q}\int_\Omega |u_j|^q dx \leq \tilde{a_1}\|u_j\| + \tilde{a_2}\|u_j\|_p^{q-1}\|u_j\|,$$

and by (4.25) we obtain a contradiction with (4.27). □

As a consequence of Lemmas 4.3 and 4.4, we are able to prove Proposition 4.2.

Proof of Proposition 4.2. Assume by contradiction that there exists $\sigma > 0$ such that for every $\varepsilon_0 > 0$ there exist $\bar\lambda \in [\lambda_{k-1} + \sigma, \lambda_{k+m} - \sigma]$ and $\varepsilon' < \varepsilon''$ in $(0, \varepsilon_0)$ such that

$$(\nabla)(\mathcal{J}_{\bar\lambda}, H_{k-1} \oplus H_{k+m-1}^\perp, \varepsilon', \varepsilon'') \quad \text{does not hold.} \tag{4.28}$$

Take $\varepsilon > 0$ associated to σ according to Lemma 4.3.

By (4.28) we can find a sequence $\{u_j\}_{j\in\mathbb{N}}$ in X_0 such that

$$\begin{gathered}\mathcal{J}_{\bar\lambda}(u_j) \in [\varepsilon', \varepsilon''] \quad \text{for all } j \in \mathbb{N}, \\ \operatorname{dist}(u_j, H_{k-1} \oplus H_{k+m-1}^\perp) \to 0 \\ P_{H_{k-1}\oplus H_{k+m-1}^\perp}\nabla\mathcal{J}_{\bar\lambda}(u_j) \to 0 \quad \text{in } X_0\end{gathered} \tag{4.29}$$

as $j \to +\infty$.

By Lemma 4.4 we obtain that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0, and so there exists $u_\infty \in X_0$ such that, up to a subsequence,

$$\begin{gathered} u_j \rightharpoonup u_\infty \quad \text{in } X_0 \\ u_j \to u_\infty \quad \text{in } L^\nu(\Omega) \text{ for any } \nu \in [1, 2^*) \\ u_j \to u_\infty \quad \text{a.e. in } \Omega \end{gathered} \tag{4.30}$$

as $j \to +\infty$.

Now, by (4.15) we have

$$\begin{aligned} P_{H_{k-1}\oplus H_{k+m-1}^\perp}\nabla \mathcal{J}_{\bar\lambda}(u_j) = u_j &- P_{\mathrm{span}\{e_k,\dots,\,e_{k+m-1}\}}u_j \\ &- P_{H_{k-1}\oplus H_{k+m-1}^\perp}\mathcal{L}_K^{-1}(\bar\lambda u_j + f(x,u_j)). \end{aligned} \tag{4.31}$$

Hence, recalling that $\mathcal{L}_K^{-1} : L^{q'}(\Omega) \to X_0$ is a compact operator, see (4.16), and that $f(x,u_j) \to f(x,u_\infty)$ in $L^{q'}(\Omega)$ by Krasnoselskii's Theorem, see [16, Theorem 2.75], we obtain that

$$P_{H_{k-1}\oplus H_{k+m-1}^\perp}\mathcal{L}_K^{-1}(\bar\lambda u_j + f(x,u_j)) \to P_{H_{k-1}\oplus H_{k+m-1}^\perp}\mathcal{L}_K^{-1}(\bar\lambda u_\infty + f(x,u_\infty))$$

as $j \to +\infty$ and so, taking into account (4.29), (4.30) and (4.31), we deduce that

$$u_j \to P_{H_{k-1}\oplus H_{k+m-1}^\perp}\mathcal{L}_K^{-1}(\bar\lambda u_\infty + f(x,u_\infty)) = u_\infty \quad \text{in } X_0 \tag{4.32}$$

as $j \to +\infty$.

Moreover, again by (4.29), we obtain that u_∞ is a critical point of $\mathcal{J}_{\bar\lambda}$ constrained on $H_{k-1} \oplus H_{k+m-1}^\perp$. Hence, Lemma 4.3 yields that $u_\infty \equiv 0$. However, $0 < \varepsilon' \le \mathcal{J}_{\bar\lambda}(u_j)$ for every $j \in \mathbb{N}$, so that, by continuity of $\mathcal{J}_{\bar\lambda}$, we find $\mathcal{J}_{\bar\lambda}(u_\infty) > 0$, which is absurd. □

5. Proof of Theorem1.2

The proof of Theorem 1.2 relies on the combination of Theorem 5.1 below with a classical Linking Theorem, see [24, Theorem 5.3].

Theorem 5.1 ([10, Theorem 2.10])**.** *Let H be a Hilbert space and X_1, X_2, X_3 be three subspaces of H such that $H = X_1 \oplus X_2 \oplus X_3$ with $0 < \dim X_i < \infty$ for $i = 1, 2$. Let $\mathcal{I} : H \to \mathbb{R}$ be a $C^{1,1}$ functional. Let $\rho, \rho', \rho'', \rho_1$ be such that $0 < \rho_1$, $0 \le \rho' < \rho < \rho''$ and*

$$\Delta = \{u \in X_1 \oplus X_2 : \rho' \le \|P_2u\| \le \rho'', \|P_1u\| \le \rho_1\}, \quad T = \partial_{X_1\oplus X_2}\Delta\,,$$

where $P_i : H \to X_i$ is the orthogonal projection of H onto X_i, $i = 1, 2$, and

$$S_{23}(\rho) = \{u \in X_2 \oplus X_3 : \|u\| = \rho\}, \quad B_{23}(\rho) = \{u \in X_2 \oplus X_3 : \|u\| < \rho\}\,.$$

Assume that

$$a' = \sup \mathcal{I}(T) < \inf \mathcal{I}(S_{23}(\rho)) = a''\,.$$

Let a, b be such that $a' < a < a''$, $b > \sup \mathcal{I}(\Delta)$ and the assumption $(\nabla)(\mathcal{I}, X_1 \oplus X_3, a, b)$ holds; the Palais-Smale condition holds at any level $c \in [a, b]$. Then $\mathcal{I}$ has at least two critical points in $\mathcal{I}^{-1}([a, b])$.

If, furthermore,

$$-\infty < \inf \mathcal{I}(B_{23}(\rho)), \quad \textit{and} \quad a_1 < \inf \mathcal{I}(B_{23}(\rho)),$$

and the Palais-Smale condition holds at every $c \in [a_1, b]$, then $\mathcal{I}$ has another critical level between a_1 and a'.

Hence, let us start showing that $\mathcal{J}_\lambda$ satisfies the *Palais-Smale condition* at any level, i.e. for all $c \in \mathbb{R}$ every sequence $\{u_j\}_{j\in\mathbb{N}} \subset X_0$ such that

$$\mathcal{J}_\lambda(u_j) \to c, \tag{5.1}$$

$$\mathcal{J}_\lambda'(u_j) \to 0 \quad \text{in } X_0' \tag{5.2}$$

as $j \to +\infty$, admits a strongly convergent subsequence in X_0.

Proposition 5.2. *Let $\lambda > 0$ and let f satisfy* (A1). *Then, $\mathcal{J}_\lambda$ satisfies the Palais-Smale condition at any level $c \in \mathbb{R}$.*

Proof. Let $c \in \mathbb{R}$ and let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence satisfying (5.1) and (5.2). Assume by contradiction that $\{u_j\}_{j\in\mathbb{N}}$ is not bounded, and so assume that $\|u_j\| \to \infty$ as $j \to \infty$. We claim that

$$\frac{Pu_j}{\|u_j\|} \to 0 \quad \text{as } j \to \infty, \tag{5.3}$$

where P is the same projection of Lemma 4.4. Indeed, by (1.17) we can find $A, B > 0$ such that

$$f(x,t)t - 2F(x,t) \geq A|t| - B \quad \text{for a.e. } x \in \mathbb{R} \text{ and all } t \in \mathbb{R}.$$

Then

$$\begin{aligned} 2\mathcal{J}_\lambda(u_j) - \mathcal{J}_\lambda'(u_j)(u_j) &= \int_\Omega (f(x,u_j)u_j dx - 2F(x,u_j))\,dx \\ &\geq A\int_\Omega |u_j|\,dx - B|\Omega| \\ &\geq \|Pu_j\|_1 - A\|Qu_j\|_1 - B|\Omega|, \end{aligned} \tag{5.4}$$

where Q is as in Lemma 4.4.

Now, write $Qu_j = v_j + w_j$, where $v_j \in X_1$ and $w_j \in X_3$ for every $j \in \mathbb{N}$. As in (4.9), we obtain

$$\int_\Omega |u_j|^{q-1}|v_j| dx \leq C\|u_j\|_p^{q-1}\|v_j\|, \tag{5.5}$$

$$\int_\Omega |u_j|^{q-1}|w_j| dx \leq C\|u_j\|_p^{q-1}\|w_j\| \tag{5.6}$$

for some $C > 0$ and all $j \in \mathbb{N}$. Hence, by (5.2) and (1.17) there exists $c_2 > 0$ such that

$$\begin{aligned} 2\mathcal{J}_\lambda(u_j) - \langle \nabla\mathcal{J}_\lambda(u_j), u_j\rangle_{X_0} &= \int_\Omega (f(x,u_j)u_j dx - 2F(x,u_j))\,dx \\ &\geq a_3\int_\Omega |u_j|^p dx - c_2, \end{aligned}$$

so that

$$\lim_{j\to\infty} \frac{\int_\Omega |u_j|^p dx}{\|u_j\|} = 0, \tag{5.7}$$

and so also (4.25) holds again.

Now, by (5.2), (1.13), (1.12) and (5.5) we obtain

$$\begin{aligned} \|v_j\|o(1) &= \langle \nabla\mathcal{J}_\lambda(u_j), -v_j\rangle_{X_0} \\ &= -\|v_j\|^2 + \lambda\int_\Omega v_j^2 dx + \int_\Omega f(x,u_j)v_j\,dx \end{aligned}$$

$$
\begin{aligned}
&\geq \Big(-1+\frac{\lambda}{\lambda_{k-1}}\Big)\|v_j\|^2 - a_1\|v_j\|_1 - a_2\int_\Omega |u_j|^{q-1}|v_j|\,dx\\
&\geq \frac{\lambda-\lambda_{k-1}}{\lambda_{k-1}}\|v_j\|^2 - c\|v_j\| - d\|u_j\|_p^{q-1}\|v_j\|\\
&= \frac{\lambda-\lambda_{k-1}}{\lambda_{k-1}}\|v_j\|^2 - \|v_j\|\big(c+d\|u_j\|_p^{q-1}\|v_j\|\big)
\end{aligned}
$$

for some constants $c, d > 0$ and where $o(1) \to 0$ as $j \to \infty$. By (4.25), the previous inequality implies that

$$
\lim_{j\to\infty}\frac{\|v_j\|}{\|u_j\|} = 0. \tag{5.8}
$$

Similarly, by using (1.11), we find

$$
\begin{aligned}
\|w_j\|o(1) &= \langle \nabla\mathcal{J}_\lambda(u_j), w_j\rangle_{X_0}\\
&= \|w_j\|^2 - \lambda\int_\Omega w_j^2dx - \int_\Omega f(x,u_j)w_j\,dx\\
&\geq \Big(1-\frac{\lambda}{\lambda_{k+m}}\Big)\|v_j\|^2 - a_1\|w_j\|_1 - a_2\int_\Omega |u_j|^{q-1}|w_j|\,dx\\
&\geq \frac{\lambda_{k+m}-\lambda}{\lambda_{k-1}}\|w_j\|^2 - c\|w_j\| - d\|u_j\|_p^{q-1}\|w_j\|\\
&= \frac{\lambda_{k+m}-\lambda}{\lambda_{k+m}}\|w_j\|^2 - \|w_j\|(c+d\|u_j\|_p^{q-1}\|w_j\|),
\end{aligned}
$$

and by (4.25) we find that

$$
\lim_{j\to\infty}\frac{\|w_j\|}{\|u_j\|} = 0. \tag{5.9}
$$

By (5.8) and (5.9), recalling that $Qu_j = v_j + w_j$, we finally get

$$
\lim_{j\to\infty}\frac{\|Qu_j\|}{\|u_j\|} = 0. \tag{5.10}
$$

Since by (2.3) there exists $c > 0$ such that

$$
\|Qu_j\|_1 \leq c\|Qu_j\|,
$$

using (5.10) in (5.4), being X_2 finite-dimensional, (5.3) holds.

Now, proceeding as in the proof of Lemma 4.4 we finally find that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in X_0. By (1.13) it is standard to prove that $\{u_j\}_{j\in\mathbb{N}}$ is pre-compact, and so the Palais-Smale condition holds at every level. □

Lemma 5.3. *Assume* (3.1) *and* (A1). *Then*

$$
\lim_{\lambda\to\lambda_k}\sup_{u\in H_{k+m-1}}\mathcal{J}_\lambda(u) = 0.
$$

Proof. First of all, note that $\mathcal{J}_\lambda$ attains a maximum in H_{k+m-1} by (1.16).

Now, assume by contradiction that there exist $\{\mu_j\}_{j\in\mathbb{N}}$, such that

$$
\mu_j \to \lambda_k \tag{5.11}
$$

as $j \to +\infty$, $\{u_j\}_{j\in\mathbb{N}}$ in H_{k+m-1} and $\varepsilon > 0$ such that for any $j \in \mathbb{N}$

$$
\mathcal{J}_{\mu_j}(u_j) = \max_{u\in H_{k+m-1}}\mathcal{J}_{\mu_j}(u) \geq \varepsilon. \tag{5.12}
$$

If $\{u_j\}_{j\in\mathbb{N}}$ were bounded, we could assume that $u_j \to u_\infty$ in H_{k+m-1}. Then, by (5.11) we would get

$$\mathcal{J}_{\mu_j}(u_j) \to \mathcal{J}_{\lambda_k}(u_\infty)$$

as $j \to +\infty$. By (5.12), (1.12) and (1.18) we would find that

$$\begin{aligned}\varepsilon \le \mathcal{J}_{\lambda_k}(u_\infty) &= \frac{1}{2}\|u_\infty\|^2 - \frac{\lambda_k}{2}\int_\Omega |u_\infty|^2\,dx - \int_\Omega F(x,u_\infty)\,dx\\ &\le \frac{1}{2}(\lambda_{k+m-1}-\lambda_k)\int_\Omega |u_\infty|^2\,dx - \int_\Omega F(x,u_\infty)\,dx \le 0,\end{aligned}$$

which is absurd.

Otherwise, if $\{u_j\}_{j\in\mathbb{N}}$ were unbounded in X_0, we could assume that $\|u_j\| \to +\infty$ as $j \to +\infty$. Therefore, (5.12) and (1.16) would imply

$$0 < \varepsilon \le \mathcal{J}_{\mu_j}(u_j) = \frac{1}{2}\|u_j\|^2 - \frac{\mu_j}{2}\int_\Omega |u_j|^2\,dx - \int_\Omega F(x,u_j)\,dx. \tag{5.13}$$

Notice that (1.18) and Fatou's Lemma imply, since all norms are equivalent in H_{k+m-1}, that

$$\lim_{j\to\infty}\int_\Omega \frac{F(x,u_j)}{\|u_j\|^2}dx = +\infty,$$

and so from (5.13) we would get

$$0 < \varepsilon \le \mathcal{J}_{\mu_j}(u_j) = \|u_j\|^2\Big(\frac{1}{2} - \frac{\mu_j}{2}\int_\Omega \frac{|u_j|^2}{\|u_j\|^2}\,dx - \int_\Omega \frac{F(x,u_j)}{\|u_j\|^2}dx\Big) = -\infty$$

another contradiction, and so the lemma holds. □

Applying Theorem 5.1 to $\mathcal{J}_\lambda$ we have a preliminary result.

Proposition 5.4. *Assume* (3.1) *and* (A1)*. Then, there exists a left neighborhood $\mathcal{O}_k$ of λ_k such that for all $\lambda \in \mathcal{O}_k$, problem* (1.3) *has two nontrivial solutions u_i such that*

$$0 < \mathcal{J}_\lambda(u_i) \le \sup_{u\in H_{k+m-1}} \mathcal{J}_\lambda(u)$$

for $i = 1, 2$.

Proof. To apply Theorem 5.1 to $\mathcal{J}_\lambda$, fix $\sigma > 0$ and find ε_σ as in Proposition 4.2. Then, for all $\lambda \in [\lambda_{k-1}+\sigma, \lambda_{k+m}-\sigma]$ and for every $\varepsilon', \varepsilon'' \in (0, \varepsilon_\sigma)$, functional $\mathcal{J}_\lambda$ satisfies $(\nabla)(\mathcal{J}_\lambda, H_{k-1}\oplus H_{k+m-1}^\perp, \varepsilon', \varepsilon'')$.

By Lemma 5.3 there exists $\sigma_1 \le \sigma$ such that, if $\lambda \in (\lambda_k - \sigma_1, \lambda_k)$, then

$$\sup_{u\in H_{k+m-1}} \mathcal{J}_\lambda(u) = \varepsilon''. \tag{5.14}$$

Moreover, since $\lambda < \lambda_k$, Proposition 3.1 holds and $\mathcal{J}_\lambda$ satisfies the Palais-Smale condition at any level by Proposition 5.2.

Then, we can apply Theorem 5.1 and find two critical points u_1, u_2 of $\mathcal{J}_\lambda$ with

$$\mathcal{J}_\lambda(u_i) \in [\varepsilon', \varepsilon''], \quad i = 1, 2, \tag{5.15}$$

i.e. u_1 and u_2 are nontrivial solutions of (1.3) such that

$$0 < \mathcal{J}_\lambda(u_i) \le \varepsilon'', \quad i = 1, 2.$$

□

We are now ready to conclude with the following result.

Proof of Theorem 1.2. Mimicking the proof of Proposition 3.1 we see that for every $u \in H_{k+m-1}^{\perp}$,

$$\mathcal{J}_\lambda(u) \geq \frac{1}{2}\Big(1 - \frac{\lambda+\varepsilon}{\lambda_{k+m}}\Big)\|u\|^2 - \tilde{M}_\varepsilon \|u\|^q,$$

so that, for ε small, there exists $\rho > 0$ such that

$$\inf_{u \in H_{k+m-1}^{\perp},\, \|u\|=\rho} \mathcal{J}_\lambda(u) \geq \frac{1}{2}\Big(1 - \frac{\lambda+\varepsilon}{\lambda_{k+m}}\Big)\rho^2 - \tilde{M}_\varepsilon \rho^q := \alpha_\rho > 0.$$

By Lemma 5.3, we can choose λ so close to λ_k that

$$\sup_{u \in H_{k+m-1}} \mathcal{J}_\lambda(u) < \alpha_\rho. \tag{5.16}$$

Hence, the classical Linking Theorem ensures the existence of a solution u_3 of problem (1.3) with

$$\mathcal{J}_\lambda(u_3) \geq \inf_{u \in H_{k+m-1}^{\perp},\, \|u\|=\varrho} \mathcal{J}_\lambda(u) \geq \alpha_\rho. \tag{5.17}$$

Choosing σ_1 such that in (5.14) $\varepsilon'' < \alpha_\rho$, we obtain

$$\mathcal{J}_\lambda(u_i) \leq \sup_{u \in H_{k+m-1}} \mathcal{J}_\lambda(u) < \mathcal{J}_\lambda(u_3)$$

and so $u_3 \neq u_i$, $i = 1, 2$. The proof of Theorem 1.2 is complete. □

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