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NONHOMOGENEOUS SUBLINEAR FRACTIONAL SCHRÖDINGER EQUATIONS

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Abstract. We study the existence, uniqueness and multiplicity for the sublinear fractional problem

$$(-\Delta)^s u + V(x)u + a(x)|u|^p \operatorname{sgn}(u) = f \quad \text{in } \mathbb{R}^N,$$

where $s \in (0,1)$, $N > 2s$, $(-\Delta)^s$ is the fractional Laplacian, $p \in (0,1)$, $f \in L^2(\mathbb{R}^N) \cap L^{\frac{p+1}{p}}(\mathbb{R}^N)$, $V : \mathbb{R}^N \to \mathbb{R}$ and $a : \mathbb{R}^N \to \mathbb{R}$ are positive bounded functions.

1. Introduction

In this article we consider the nonlinear fractional Schrödinger equation

$$\begin{gathered} (-\Delta)^s u + V(x)u + a(x)|u|^p \operatorname{sgn}(u) = f \quad \text{in } \mathbb{R}^N \\ u \in H^s(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N) \end{gathered} \tag{1.1}$$

where $N > 2s$, $s \in (0,1)$, $p \in (0,1)$, $f \in L^2(\mathbb{R}^N) \cap L^{(p+1)/p}(\mathbb{R}^N)$, $V : \mathbb{R}^N \to \mathbb{R}$ and $a : \mathbb{R}^N \to \mathbb{R}$ are positive bounded functions. The nonlocal operator $(-\Delta)^s$ is the fractional Laplacian which is defined as

$$(-\Delta)^s u(x) = C_{N,s} \, \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy,$$

for any $u : \mathbb{R}^N \to \mathbb{R}$ sufficiently smooth. The symbol P.V. stands for the Cauchy principal value and $C_{N,s}$ is a dimensional constant depending on N and s; see [11].

In the previous decade a great attention has been devoted to the study of fractional and nonlocal operators of elliptic type since these operators arise in a quite natural way in many different contexts such as phase transition phenomena, minimal surface, game theory, continuum mechanics, crystal dislocation, optimization, water waves and so on. For more details the interested reader may consult [13, 22] and references therein.

A basic motivation for the study of problem (1.1) is related to the search of standing wave solutions of the type $\psi(x,t) = u(x)e^{-\imath ct}$ for the time dependent fractional Schrödinger equation

$$\imath \frac{\partial \psi}{\partial t} = (-\Delta)^s \psi + (V(x) + c)\psi - g(|\psi|) \tag{1.2}$$

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where $V : \mathbb{R}^N \to \mathbb{R}$ is an external potential and g is a suitable nonlinearity. The fractional Schrödinger equation (1.2) was introduced by Laskin [20] and it is a fundamental equation of fractional quantum mechanic in the study of particles on stochastic fields modeled by Levy processes.

Very recently, the study of problems of fractional Schrödinger equations has attracted the attention of many mathematicians. Indeed several existence and multiplicity results have been established, under different assumptions on the potential V and nonlinear term, by using suitable variational methods; see [3, 4, 5, 6, 7, 8, 14, 15, 16, 17]. In particular way, a special attention has been devoted to the study of fractional Schrödinger equations involving superlinear nonlinearities; see for instance [1, 2, 23]. On the contrary, to our knowledge, only few results deal with fractional problems with sublinear terms; see [18, 19, 25].

The aim of this article is to consider equation (1.1) under the following assumptions:

- (H1) $f \in L^2(\mathbb{R}^N) \cap L^{\frac{p+1}{p}}(\mathbb{R}^N)$, $f \geq 0$ ($f \not\equiv 0$);
- (H2) $V \in L^\infty(\mathbb{R}^N)$ and $\lim_{|x|\to\infty} V(x) = v_\infty \geq 0$;
- (H3) $a \in L^\infty(\mathbb{R}^N)$, $\lim_{|x|\to\infty} a(x) = a_\infty > 0$ and there exists $\alpha > 0$ such that $a(x) > \alpha$ a.e. in $\mathbb{R}^N$.

Our main result is the following theorem.

Theorem 1.1. *Assume that* (H1)–(H3) *hold. Then, there exists a positive constant c, such that for every $f > 0$ a.e. in $\mathbb{R}^N$, $\|f\|_{L^2(\mathbb{R}^N)} < c$, problem* (1.1) *admits a nonnegative solution $u_1 \in H^s(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)$ that converges to zero in $H^s(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)$ as $\|f\|_{L^2(\mathbb{R}^N)}$ tends to zero. Moreover:*

(i) *if*
$$\iint_{\mathbb{R}^{2N}} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)\psi^2 dx \geq 0 \tag{1.3}$$
for every $\psi \in C_c^\infty(\mathbb{R}^N)$, then the solution u_1 is unique;

(ii) *if*
$$\iint_{\mathbb{R}^{2N}} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)\psi^2 dx < 0 \tag{1.4}$$
for some $\psi \in C_c^\infty(\mathbb{R}^N)$, then there exists a second solution $u_2 \neq u_1$.

We note that when $s = 1$ Theorem 1.1 can be seen as the fractional analogue of [9, Theorem 1.1] in which the author studies existence, multiplicity and uniqueness of the corresponding nonhomogeneous elliptic equation
$$-\Delta u + V(x)u + a(x)|u|^p \operatorname{sgn}(u) = f \quad \text{in } \mathbb{R}^N.$$
We recall that in the classical setting, sublinear problems in the whole $\mathbb{R}^N$ in presence of a small perturbation have been widely investigated by many authors; see [9, 10, 12, 24]. In this paper, motivated by [9, 10], we continue the study started in [19] introducing the potentials $a(x)$ and $V(x)$. Borrowing some ideas from [9], we prove different existence and multiplicity results for (1.1). Clearly, due to the nonlocality of the fractional Laplacian $(-\Delta)^s$, a more careful analysis is needed to prove that the arguments developed in [9] also work in our setting.

The plan of this article is the following. In Section 2 we collect some useful preliminary results which we will use along the paper. In Section 3 we prove the existence of a first solution to (1.1) provided that f is sufficiently small in L^2 sense.

The last section is devoted to the proof of a second solution different from the previous one.

2. Preliminary results

In this section we briefly recall some properties of the fractional Sobolev spaces, and we introduce some notation that will be used.

For any $s \in (0,1)$, we define the homogeneous fractional Sobolev space

$$\mathcal{D}^{s,2}(\mathbb{R}^N) = \big\{ u \in L^{2_s^*}(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy < +\infty \big\}$$

which is the completion of $C_c^\infty(\mathbb{R}^N)$ under the norm given by

$$\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy.$$

The fractional Sobolev space $H^s(\mathbb{R}^N)$ can be described as

$$H^s(\mathbb{R}^N) = \big\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \big\}.$$

In this case the norm is defined as

$$\|u\|_{H^s(\mathbb{R}^N)} = \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} |u|^2 dx \Big)^{1/2}.$$

For the readers' convenience we recall the following embeddings.

Theorem 2.1 ([13])**.** *Let $s \in (0,1)$ and $N > 2s$. Then there exists a sharp constant $S_* = S(N,s) > 0$ such that for any $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$,*

$$\|u\|^2_{L^{2_s^*}(\mathbb{R}^N)} \le S_* \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy.$$

Moreover, $H^s(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [2, 2_s^*]$ and compactly in $L^q_{\rm loc}(\mathbb{R}^N)$ for any $q \in [1, 2_s^*)$.

Let $\mathbb{X} := H^s(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)$ equipped with the norm

$$\|u\|^2 := \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} |u|^{p+1} dx.$$

Since we are interested in weak solutions to (1.1), we look for critical points of the functional $\mathcal{I} : \mathbb{X} \to \mathbb{R}$ defined by

$$\begin{aligned}\mathcal{I}(u) = \frac{1}{2} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 dx + \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)|u|^{p+1} dx \\ - \int_{\mathbb{R}^N} f u\,dx.\end{aligned}$$

It is standard to check that $\mathcal{I}$ is well-defined in $\mathbb{X}$, $\mathcal{I} \in C^1(\mathbb{X}, \mathbb{R})$ and its differential is given by

$$\begin{aligned}\langle \mathcal{I}'(u), \varphi \rangle &= \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V(x) u \varphi dx \\ &\quad + \int_{\mathbb{R}^N} a(x)|u|^p \varphi dx - \int_{\mathbb{R}^N} f\varphi\,dx\end{aligned}$$

for any $u, \varphi \in \mathbb{X}$.

We begin by proving the following Lemma to obtain the existence of a local minimum for $\mathcal{I}$.

Lemma 2.2. *Assume that* (H1)–(H3) *hold. Then, there exist positive constants* κ, ρ *and* L *such that if* $\|f\|_{L^2(\mathbb{R}^N)} < L$ *then* $\mathcal{I}(u) \geq \kappa$ *whenever* $\|u\| = \rho$.

Proof. Let $u \in \mathbb{X}$. By using (H2), (H3), Hölder inequality and Young inequality we obtain

$$\begin{aligned}
\mathcal{I}(u) &\geq \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{\|V\|_{L^\infty(\mathbb{R}^N)}}{2}\int_{\mathbb{R}^N}|u|^2dx \\
&\quad + \frac{\alpha}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}dx - \Big(\int_{\mathbb{R}^N}|f|^2dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N}|u|^2dx\Big)^{1/2} \\
&\geq \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{\|V\|_{L^\infty(\mathbb{R}^N)}}{2}\int_{\mathbb{R}^N}|u|^2dx \\
&\quad + \frac{\alpha}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}dx - \frac{1}{2}\int_{\mathbb{R}^N}|f|^2dx - \frac{1}{2}\int_{\mathbb{R}^N}|u|^2dx \\
&= \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{1}{2}\left(\|V\|_{L^\infty(\mathbb{R}^N)}+1\right)\int_{\mathbb{R}^N}|u|^2dx \\
&\quad + \frac{\alpha}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}dx - \frac{1}{2}\int_{\mathbb{R}^N}|f|^2dx.
\end{aligned} \tag{2.1}$$

Now, by interpolation, there exists $r = \frac{(2_s^*-2)(p+1)}{2(2_s^*-(p+1))} \in (0,1)$ such that

$$\begin{aligned}
\int_{\mathbb{R}^N}|u|^2dx &\leq \Big(\int_{\mathbb{R}^N}|u|^{p+1}dx\Big)^{\frac{2r}{p+1}}\Big(\int_{\mathbb{R}^N}|u|^{2_s^*}dx\Big)^{\frac{2(1-r)}{2_s^*}} \\
&\leq S_*^{1-r}\Big(\int_{\mathbb{R}^N}|u|^{p+1}dx\Big)^{\frac{2r}{p+1}}\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1-r} \\
&\leq \frac{\alpha}{(p+1)(\|V\|_{L^\infty(\mathbb{R}^N)}+1)}\int_{\mathbb{R}^N}|u|^{p+1}dx \\
&\quad + C_1\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{\frac{(1-r)(p+1)}{p+1-2r}}
\end{aligned} \tag{2.2}$$

where we used Theorem 2.1, Hölder inequality and Young inequality. Therefore, putting together (2.1) and (2.2), we obtain

$$\begin{aligned}
\mathcal{I}(u) &\geq \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{1}{2}\frac{\alpha}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}dx \\
&\quad - C_1\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{\frac{(1-r)(p+1)}{p+1-2r}} + \frac{\alpha}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}dx \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^N}|f|^2dx \\
&= \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \frac{1}{2}\frac{\alpha}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}dx \\
&\quad - C_1\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\right)^{\frac{(1-r)(p+1)}{p+1-2r}} - \frac{1}{2}\int_{\mathbb{R}^N}|f|^2dx.
\end{aligned}$$

Now, since $\frac{2(1-r)(p+1)}{p+1-2r} > 2$, we can see that

$$\begin{aligned}
&\frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - C_1\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{\frac{(1-r)(p+1)}{p+1-2r}}\\
&\geq \frac{1}{4}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy
\end{aligned}$$

for $\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \leq \beta$, with β sufficiently small.

Hence, for $\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \leq \beta$ and $\|u\|_{L^{p+1}(\mathbb{R}^N)} \leq 1$, by using the fact that for any $x \geq 0$ and $y \in [0,1]$ it holds $(x+y)^2 \leq 2x^2 + 2y^{p+1}$, we obtain

$$\begin{aligned}
\mathcal{I}(u) &\geq \frac{1}{4}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \frac{\alpha}{2(p+1)}\int_{\mathbb{R}^N} |u|^{p+1}dx\\
&\quad - \frac{1}{2}\int_{\mathbb{R}^N} |f|^2 dx\\
&\geq \min\big\{\frac{1}{4}, \frac{\alpha}{2(p+1)}\big\}\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \int_{\mathbb{R}^N} |u|^{p+1}dx\Big)\\
&\quad - \frac{1}{2}\int_{\mathbb{R}^N} |f|^2 dx\\
&\geq C_2\|u\|^2 - \frac{1}{2}\|f\|^2_{L^2(\mathbb{R}^N)}.
\end{aligned} \tag{2.3}$$

Taking $\rho = \min\{\beta, 1\}$ and $\|f\|^2_{L^2(\mathbb{R}^N)} = C_2\rho^2$, then for $\|u\| = \rho$ we obtain $\mathcal{I}(u) \geq \kappa := \frac{1}{2}C_2\rho^2$. □

Let us define

$$m := \inf_{u\in B_\rho} \mathcal{I}(u), \tag{2.4}$$

where $B_\rho = \{u \in \mathbb{X} : \|u\| < \rho\}$ and ρ is defined in Lemma 2.2.

Lemma 2.3. *Under assumptions* (H1)–(H3) *it holds* $-\infty < m < 0$.

Proof. In view of (H1) there exists $\psi \in \mathbb{X}$ such that $\int_{\mathbb{R}^N} f\psi\,dx > 0$. Then we have

$$\begin{aligned}
\mathcal{I}(t\psi) = \frac{t^2}{2}&\iint_{\mathbb{R}^{2N}} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \frac{t^2}{2}\int_{\mathbb{R}^N} V(x)|\psi|^2 dx\\
&+ \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^N} a(x)|\psi|^{p+1}dx - \int_{\mathbb{R}^N} f\psi\,dx < 0
\end{aligned}$$

for t sufficiently small. As a consequence $m < 0$. It is clear that $m > -\infty$ in view of (2.3). □

3. Existence of a first solution

In this section we study of the existence and uniqueness of solutions to (1.1).

Theorem 3.1. *Assume that* (H1)–(H3) *hold. Then there exists* $u_1 \in \mathbb{X}$ *which is a nonnegative solution to* (1.1). *Moreover* u_1 *converges to zero in* $\mathbb{X}$ *as* $\|f\|_{L^2(\mathbb{R}^N)} \to 0$.

Proof. Let $\{u_n\} \subset \mathbb{X}$ be a minimizing sequence of (2.4). Since $\{u_n\}$ is bounded in $\mathbb{X}$, we may assume, up to a subsequence, that

$$\begin{gathered} u_n \rightharpoonup u_1 \quad \text{in } \mathbb{X}, \\ u_n \to u_1 \quad \text{in } L^q_{\rm loc}(\mathbb{R}^N), \forall q \in [1, 2^*_s), \\ u_n \to u_1 \quad \text{a.e. in } \mathbb{R}^N. \end{gathered}$$

Our aim is to prove that $u_n \to u_1$ in $\mathbb{X}$. Set $v_n = u_n - u_1$. Let us compute $\mathcal{I}(u_n)$:

$$\begin{aligned} \mathcal{I}(u_n) &= \mathcal{I}(v_n + u_1) \\ &= \frac{1}{2} \iint_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2s}}\, dx\, dy + \frac{1}{2} \iint_{\mathbb{R}^{2N}} \frac{|u_1(x) - u_1(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \\ &\quad + \iint_{\mathbb{R}^{2N}} \frac{(v_n(x) - v_n(y))(u_1(x) - u_1(y))}{|x-y|^{N+2s}}\, dx\, dy + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|v_n|^2 dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u_1|^2 dx + \int_{\mathbb{R}^N} V(x) v_n u_1 dx - \int_{\mathbb{R}^N} f v_n dx - \int_{\mathbb{R}^N} f u_1\, dx \\ &\quad + \frac{1}{p+1} \int_{\mathbb{R}^N} a(x)(|u_n|^{p+1} - (|u_1|^{p+1} + |v_n|^{p+1}))dx \\ &\quad + \frac{1}{p+1} \int_{\mathbb{R}^N} a(x)|u_1|^{p+1} dx + \frac{1}{p+1} \int_{\mathbb{R}^N} a(x)|v_n|^{p+1} dx \\ &= \mathcal{I}(u_1) + \frac{1}{2} \iint_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^2}{|x-y|^{N+2s}}\, dx\, dy + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|v_n|^2 dx \\ &\quad + \frac{1}{p+1} \int_{\mathbb{R}^N} a(x)|v_n|^{p+1} dx - \int_{\mathbb{R}^N} f v_n dx \\ &\quad + \iint_{\mathbb{R}^{2N}} \frac{(v_n(x) - v_n(y))(u_1(x) - u_1(y))}{|x-y|^{N+2s}}\, dx\, dy + \int_{\mathbb{R}^N} V(x) v_n u_1 dx \\ &\quad + \frac{1}{p+1} \int_{\mathbb{R}^N} a(x)(|u_n|^{p+1} - (|u_1|^{p+1} + |v_n|^{p+1}))dx. \end{aligned} \tag{3.1}$$

From the fact that $v_n \rightharpoonup 0$ in $\mathbb{X}$ we infer that

$$\begin{gathered} \iint_{\mathbb{R}^{2N}} \frac{(v_n(x) - v_n(y))(u_1(x) - u_1(y))}{|x-y|^{N+2s}}\, dx\, dy \to 0 \\ \int_{\mathbb{R}^N} f v_n dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} V(x) v_n u_1 dx \to 0. \end{gathered} \tag{3.2}$$

Moreover, taking into account that

$$-c|x|^p|y| \le |x+y|^{p+1} - (|x|^{p+1} + |y|^{p+1}) \le c|x|^p|y|$$

holds for any $x, y \in \mathbb{R}$ with $c > 0$ (independent of x and y), we obtain

$$\begin{aligned} &\Big| \frac{1}{p+1} \int_{\mathbb{R}^N} a(x)(|u_n|^{p+1} - (|u_1|^{p+1} + |v_n|^{p+1}))dx \Big| \\ &\le \frac{c}{p+1} \int_{\mathbb{R}^N} a(x)|v_n||u_1|^{p+1} dx \to 0. \end{aligned} \tag{3.3}$$

Let us note that

$$\begin{aligned} \rho^2 &> \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\ &= \iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \iint_{\mathbb{R}^{2N}} \frac{|u_1(x)-u_1(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\ &\quad + 2\iint_{\mathbb{R}^{2N}} \frac{(v_n(x)-v_n(y))(u_1(x)-u_1(y))}{|x-y|^{N+2s}}\,dx\,dy\,. \end{aligned} \tag{3.4}$$

Using (3.2) we can see that

$$\iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \le \rho^2$$

for n large enough. Then, taking the limit in (3.1) and by using (3.2) and (3.3), we obtain

$$\begin{aligned} m &= \mathcal{I}(u_1) + \lim_{n\to\infty}\Big\{\frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|v_n|^2dx \\ &\quad + \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)|v_n|^{p+1}dx\Big\} \\ &\ge \mathcal{I}(u_1) + \lim_{n\to\infty}\Big\{\frac{1}{4}\iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\ &\quad + \frac{\alpha}{2(p+1)}\int_{\mathbb{R}^N} a(x)|v_n|^{p+1}dx\Big\} \ge m \end{aligned} \tag{3.5}$$

where in the last inequality we have estimated as in Lemma 2.2. Then, (3.4) and (3.5) yield $v_n \to 0$ in $\mathbb{X}$, $0 > m = \mathcal{I}(u_1)$, $u_1 \in B_\rho$ and $\mathcal{I}'(u_1) = 0$. Thus u_1 is a weak solution to (1.1).

To prove that u_1 is a nonnegative solution to (1.1), note that $m \le \mathcal{I}(|u_n|) \le \mathcal{I}(u_n)$, thus $\mathcal{I}(|u_n|) \to m$, and we can choose $u_n \ge 0$. As before, up to a subsequence $u_n \to u_1$ a.e. in $\mathbb{R}^N$, $u_n \to u_1$ in $\mathbb{X}$ and u_1 is a nonnegative solution to (1.1).

Let $\{f_n\} \subset L^2(\mathbb{R}^N)$ be such that $\|f_n\|_{L^2(\mathbb{R}^N)} \to 0$ and let u_{f_n} be a solution to (1.1). Now we aim to prove that $u_{f_n} \to 0$ in $\mathbb{X}$.

Since u_{f_n} is a solution to (1.1), we obtain

$$\langle \mathcal{I}'(u_{f_n}), u_{f_n}\rangle = 0,$$

that is,

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{|u_{f_n}(x)-u_{f_n}(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)|u_{f_n}|^2dx + \int_{\mathbb{R}^N} a(x)|u_{f_n}|^{p+1}dx \\ &= \int_{\mathbb{R}^N} f_n u_{f_n}dx. \end{aligned} \tag{3.6}$$

Recalling that $\mathcal{I}(u_{f_n}) < 0$, by using (3.6) and Hölder inequality we deduce

$$\begin{aligned} 0 &> \mathcal{I}(u_{f_n}) \\ &= \frac{1}{2}\Big(-\int_{\mathbb{R}^N} a(x)|u_{f_n}|^{p+1}dx + \int_{\mathbb{R}^N} f_n u_{f_n}dx\Big) + \frac{1}{p+1}\int_{\mathbb{R}^N} a(x)|u_{f_n}|^{p+1}dx \\ &\quad - \int_{\mathbb{R}^N} f_n u_{f_n}dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1-p}{2(p+1)} \int_{\mathbb{R}^N} a(x)|u_{f_n}|^{p+1} dx - \frac{1}{2} \int_{\mathbb{R}^N} f_n u_{f_n} dx \\
&\geq \frac{1-p}{2(p+1)} \int_{\mathbb{R}^N} a(x)|u_{f_n}|^{p+1} dx - \frac{1}{2} \int_{\mathbb{R}^N} f_n u_{f_n} dx \\
&\geq \frac{(1-p)\alpha}{2(p+1)} \|u_{f_n}\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} - \frac{1}{2} \|f_n\|_{L^2(\mathbb{R}^N)} \|u_{f_n}\|_{L^2(\mathbb{R}^N)}.
\end{aligned}
$$

Observing that $p \in (0,1)$ and $\alpha > 0$, we obtain

$$
\begin{aligned}
0 &< \frac{(1-p)\alpha}{2(p+1)} \|u_{f_n}\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} \\
&\leq \frac{1}{2} \|f_n\|_{L^2(\mathbb{R}^N)} \|u_{f_n}\|_{L^2(\mathbb{R}^N)} \\
&\leq c \|f_n\|_{L^2(\mathbb{R}^N)} \to 0,
\end{aligned}
$$

thus $u_{f_n} \to 0$ in $L^{p+1}(\mathbb{R}^N)$.

Now, by using (3.6), Hölder inequality and Sobolev inequality, there exists $r \in (0,1)$ (defined in Lemma 2.2) such that

$$
\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u_{f_n}(x) - u_{f_n}(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&= -\int_{\mathbb{R}^N} V(x)|u_{f_n}|^2 dx - \int_{\mathbb{R}^N} a(x)|u_{f_n}|^{p+1} dx + \int_{\mathbb{R}^N} f_n u_{f_n} dx \\
&\leq \|V\|_{L^\infty(\mathbb{R}^N)} \|u_{f_n}\|^2_{L^2(\mathbb{R}^N)} + \|a\|_{L^\infty(\mathbb{R}^N)} \|u_{f_n}\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} + \|f_n\|_{L^2(\mathbb{R}^N)} \|u_{f_n}\|_{L^2(\mathbb{R}^N)} \\
&\leq \|V\|_{L^\infty(\mathbb{R}^N)} \Big(\|u_{f_n}\|^r_{L^{p+1}(\mathbb{R}^N)} \|u_{f_n}\|^{1-r}_{L^{2^*_s}(\mathbb{R}^N)} \Big) + \|a\|_{L^\infty(\mathbb{R}^N)} \|u_{f_n}\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} \\
&\quad + \|f_n\|_{L^2(\mathbb{R}^N)} \|u_{f_n}\|_{L^2(\mathbb{R}^N)} \\
&\leq C \|V\|_{L^\infty(\mathbb{R}^N)} \|u_{f_n}\|^r_{L^{p+1}(\mathbb{R}^N)} + \|a\|_{L^\infty(\mathbb{R}^N)} \|u_{f_n}\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} \\
&\quad + \|f_n\|_{L^2(\mathbb{R}^N)} \|u_{f_n}\|_{L^2(\mathbb{R}^N)} \to 0.
\end{aligned}
$$

Combining this with $u_{f_n} \to 0$ in $L^{p+1}(\mathbb{R}^N)$, we deduce the thesis. □

Now we recall the notion of supersolution and subsolution to (1.1):

Definition 3.2. We say that $\overline{u} \in \mathbb{X}$ is a weak supersolution to (1.1) if

$$
\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{(\overline{u}(x) - \overline{u}(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^N} V(x) \overline{u} \varphi \, dx \\
&+ \int_{\mathbb{R}^N} a(x) |\overline{u}|^p \operatorname{sgn}(\overline{u}) \varphi \, dx \\
&\leq \int_{\mathbb{R}^N} f \varphi \, dx
\end{aligned}
\tag{3.7}
$$

for any $\varphi \in \mathbb{X}$ such that $\varphi \geq 0$ a.e. in $\mathbb{R}^N$.

Definition 3.3. We say that $\underline{u} \in \mathbb{X}$ is a weak subsolution to (1.1) if

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{(\underline{u}(x)-\underline{u}(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)\underline{u}\varphi\,dx \\ &+ \int_{\mathbb{R}^N} a(x)|\underline{u}|^p \operatorname{sgn}(\underline{u})\varphi\,dx \\ &\le \int_{\mathbb{R}^N} f\varphi\,dx \end{aligned} \tag{3.8}$$

for any $\varphi \in \mathbb{X}$ such that $\varphi \ge 0$ a.e. in $\mathbb{R}^N$.

Theorem 3.4. *Assume that* (H1)–(H3) *and* (1.3) *hold. Let $\underline{u} \in \mathbb{X}$ be a subsolution to* (1.1) *and $\overline{u} \in \mathbb{X}$ be a supersolution to* (1.1)*. Then $\underline{u} \le \overline{u}$ a.e. in $\mathbb{R}^N$.*

Proof. From (3.7) and (3.8) it follows that

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{((\underline{u}-\overline{u})(x)-(\underline{u}-\overline{u})(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)(\underline{u}-\overline{u})\varphi\,dx \\ &+ \int_{\mathbb{R}^N} a(x)(|\underline{u}|^p \operatorname{sgn}(\underline{u}) - |\overline{u}|^p \operatorname{sgn}(\overline{u}))\varphi\,dx \le 0. \end{aligned} \tag{3.9}$$

Assume $\underline{u} \not\le \overline{u}$ and let $\varphi := (\underline{u}-\overline{u})^+$ in (3.9), then we have

$$\begin{aligned} 0 &\le \iint_{\mathbb{R}^{2N}} \frac{|((\underline{u}-\overline{u})(x)-(\underline{u}-\overline{u})(y))^+|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)|(\underline{u}-\overline{u})^+|^2 dx \\ &\le -\int_{\mathbb{R}^N} a(x)(|\underline{u}|^p \operatorname{sgn}(\underline{u}) - |\overline{u}|^p \operatorname{sgn}(\overline{u}))(\underline{u}-\overline{u})^+ < 0, \end{aligned}$$

and this gives a contradiction. Thus we have $\underline{u} \le \overline{u}$ a.e. in $\mathbb{R}^N$. □

At this point we are ready to prove that the problem (1.1) admits a unique weak solution.

Theorem 3.5. *Under assumptions* (H1)–(H3) *and* (1.3)*, problem* (1.1) *admits a unique solution.*

Proof. Let u_1 and u_2 be two solutions to (1.1). Then by Theorem 3.4 follows that $u_1 \le u_2$ and $u_2 \le u_1$, that is $u_1 = u_2$ a.e. in $\mathbb{R}^N$. □

4. Existence of a second solution

In this section we show the existence of a second solution to (1.1) under assumption (1.4).

Lemma 4.1. *Under assumption* (1.4)*, there exists $\varphi_0 \in \mathbb{X}\backslash B_\rho$ such that $\mathcal{I}(\varphi_0) < 0$.*

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ satisfying (1.4). Then, as $t \to +\infty$

$$\begin{aligned} \mathcal{I}(t\varphi) = &\frac{t^2}{2}\iint_{\mathbb{R}^{2N}} \frac{|\varphi(x)-\varphi(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \frac{t^2}{2}\int_{\mathbb{R}^N} V(x)|\varphi|^2 dx \\ &+ \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^N} a(x)|\varphi|^{p+1} dx - \int_{\mathbb{R}^N} f\varphi\,dx \to -\infty. \end{aligned}$$

Thus, choosing t_0 sufficiently large such that $\|t_0\varphi\| > \rho$ and $\mathcal{I}(t_0\varphi) < 0$, we can take $\varphi_0 = t_0\varphi$ to complete the proof. □

Let us consider the problem

$$M=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{I}(\gamma(t)) \tag{4.1}$$

where

$$\Gamma=\{\gamma\in C([0,1],\mathbb{X}):\gamma(0)=0,\ \gamma(1)=\varphi_0\}$$

with φ_0 as in Lemma 4.1.

Lemma 4.2. *Under assumption* (H1)–(H3) *and* (1.4) *it results* $M>0$.

Proof. Let $\gamma\in\Gamma$, then $\gamma(0)=0\in B_\rho$ and $\gamma(1)=\varphi_0\in\mathbb{X}\setminus B_\rho$. Then, there exists $\tau\in(0,1)$ such that $\|\gamma(\tau)\|=\rho$, and applying Lemma 2.2 we have $\mathcal{I}(\gamma(\tau))\geq\kappa>0$, and thus $M>0$. ☐

By applying Ekeland's variational principle to (4.1) there exists $\{u_n\}\subset\mathbb{X}$ such that $\mathcal{I}(u_n)\to M$ and $\mathcal{I}'(u_n)\to 0$ in $\mathbb{X}'$. In this case $\{u_n\}$ is called a (PS) sequence of the functional $\mathcal{I}$ at level M.

Theorem 4.3. *Under assumptions* (H1)–(H3) *and* (1.4), $\{u_n\}$ *is bounded in* $\mathbb{X}$.

Proof. Firstly note that it is possible to find a positive constant b such that, for n large,

$$|\langle\mathcal{I}'(u_n),u_n\rangle|\leq\|u_n\|\quad\text{and}\quad|\mathcal{I}(u_n)|<b.$$

By using Hölder inequality and Young inequality we can infer

$$\begin{aligned}
&b+\frac{1}{2}\|u_n\|\\
&\geq\mathcal{I}(u_n)-\frac{1}{2}\langle\mathcal{I}'(u_n),u_n\rangle\\
&\geq\alpha\Big(\frac{1}{p+1}-\frac{1}{2}\Big)\int_{\mathbb{R}^N}|u_n|^{p+1}dx-\frac{1}{2}\Big(\int_{\mathbb{R}^N}|f|^{\frac{p+1}{p}}dx\Big)^{\frac{p}{p+1}}\Big(\int_{\mathbb{R}^N}|u_n|^{p+1}dx\Big)^{\frac{1}{p+1}}\\
&\geq\frac{\alpha}{2}\Big(\frac{1}{p+1}-\frac{1}{2}\Big)\int_{\mathbb{R}^N}|u_n|^{p+1}dx-\frac{1}{2}\int_{\mathbb{R}^N}|f|^{\frac{p+1}{p}}dx;
\end{aligned}$$

therefore

$$\begin{aligned}
&\frac{\alpha}{2}\Big(\frac{1}{p+1}-\frac{1}{2}\Big)\int_{\mathbb{R}^N}|u_n|^{p+1}dx-\frac{1}{2}\Big(\int_{\mathbb{R}^N}|u_n|^{p+1}dx\Big)^{\frac{1}{p+1}}\\
&\leq C+\frac{1}{2}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy.
\end{aligned}$$

Let $\beta\in(1,p+1)$ and assume by contradiction that $\{u_n\}$ is not bounded in $L^{p+1}(\mathbb{R}^N)$. Then, for n large it results

$$\Big(\int_{\mathbb{R}^N}|u_n|^{p+1}dx\Big)^{\frac{\beta}{p+1}}<C+\frac{1}{2}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy. \tag{4.2}$$

By using (4.2), Hölder and Sobolev inequality, there exists $r\in(0,1)$ such that

$$\begin{aligned}
\int_{\mathbb{R}^N}|u_n|^2dx&\leq\Big(\int_{\mathbb{R}^N}|u_n|^{p+1}dx\Big)^{\frac{2(1-r)}{p+1}}\Big(\int_{\mathbb{R}^N}|u_n|^{2_s^*}dx\Big)^{\frac{2r}{2_s^*}}\\
&\leq S_*^r\Big(\int_{\mathbb{R}^N}|u_n|^{p+1}dx\Big)^{\frac{2(1-r)}{p+1}}\Big(\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^r\\
&\leq C_1\Big(\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{\frac{2(1-r)}{\beta}+r}
\end{aligned}$$

$$+ C_2\Big(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^r.$$

Now,

$$\begin{aligned}
b &> \mathcal{I}(u_n)\\
&\ge \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{\|V\|_{L^\infty(\mathbb{R}^N)}}{2}\int_{\mathbb{R}^N}|u_n|^2dx\\
&\quad + \frac{\alpha}{p+1}\int_{\mathbb{R}^N}|u_n|^{p+1}dx - \Big(\int_{\mathbb{R}^N}|f|^2dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N}|u_n|^2dx\Big)^{1/2}\\
&\ge \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - \frac{\|V\|_{L^\infty(\mathbb{R}^N)}+1}{2}\int_{\mathbb{R}^N}|u_n|^2dx\\
&\quad + \frac{\alpha}{p+1}\int_{\mathbb{R}^N}|u_n|^{p+1}dx - \frac{1}{2}\int_{\mathbb{R}^N}|f|^2dx\\
&\ge \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \frac{\alpha}{p+1}\int_{\mathbb{R}^N}|u_n|^{p+1}dx - \frac{1}{2}\int_{\mathbb{R}^N}|f|^2dx\\
&\quad - C_3\Big(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{\frac{2(1-r)}{\beta}+r}\\
&\quad - C_4\Big(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{r}\\
&\ge \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \frac{\alpha}{p+1}\int_{\mathbb{R}^N}|u_n|^{p+1}dx - \frac{1}{2}\int_{\mathbb{R}^N}|f|^2dx\\
&\quad - C_3\Big(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{\frac{2(1-r)}{\beta}+r}\\
&\quad - C_4\Big(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{r}.
\end{aligned}$$

Note that since $r \in (0,1)$, $\beta \in (1, p+1)$ and $p \in (0,1)$ we can infer that

$$0 < \frac{2(1-r)}{\beta} + r < 2,$$

from which it follows that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$, in contrast with (4.2). Thus, $\{u_n\}$ is bounded in $L^{p+1}(\mathbb{R}^N)$. From this we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^N}|u_n|^2dx &\le \Big(\int_{\mathbb{R}^N}|u_n|^{p+1}dx\Big)^{\frac{2(1-r)}{p+1}}\Big(\int_{\mathbb{R}^N}|u_n|^{2_s^*}dx\Big)^{\frac{2r}{2_s^*}}\\
&\le S_*^r\Big(\int_{\mathbb{R}^N}|u_n|^{p+1}dx\Big)^{\frac{2(1-r)}{p+1}}\Big(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{r}\\
&\le C\Big(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{r}.
\end{aligned} \tag{4.3}$$

Therefore, by (4.3) we obtain

$$\begin{aligned}
b &> \mathcal{I}(u_n)\\
&\ge \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - C\int_{\mathbb{R}^N}|u_n|^2dx + \frac{\alpha}{p+1}\int_{\mathbb{R}^N}|u_n|^{p+1}dx
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\int_{\mathbb{R}^N} |f|^2 dx \\
&\geq \frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy - C\Big(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^r \\
&- \frac{1}{2}\int_{\mathbb{R}^N} |f|^2 dx.
\end{aligned}
$$

This implies that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. □

Theorem 4.4. *Under assumptions* (H1)–(H3) *and* (1.4)*, problem* (1.1) *admits a second solution* $u_2 \in \mathbb{X}$.

Proof. By Theorem 4.3 $\{u_n\} \subset \mathbb{X}$ is bounded, thus

$$
\begin{aligned}
&u_n \rightharpoonup u_2 \quad \text{in } \mathbb{X}, \\
&u_n \to u_2 \quad \text{in } L^q_{\rm loc}(\mathbb{R}^N), \quad \forall q \in [1, 2_s^*), \\
&u_n \to u_2 \quad \text{a. e. in } \mathbb{R}^N.
\end{aligned} \tag{4.4}
$$

Let $\varphi \in C_c^\infty(\mathbb{R}^N)$, then we can infer that

$$
\begin{aligned}
&\lim_{n\to\infty}\Big\{\iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)u_n\varphi\,dx\Big\} \\
&= \iint_{\mathbb{R}^{2N}} \frac{(u_2(x)-u_2(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} V(x)u_2\varphi\,dx.
\end{aligned}
$$

From $u_n \to u_2$ in $L^{p+1}(\operatorname{supp}\varphi)$, it follows that there exist a subsequence still denoted by $\{u_n\}$ and a function $w \in L^{p+1}(\operatorname{supp}\varphi)$ such that $|u_n| \leq |w|$ and

$$
\begin{gathered}
|a||u_n|^p|\varphi| \leq \|a\|_{L^\infty(\mathbb{R}^N)}|w|^p|\varphi| \in L^1(\mathbb{R}^N), \\
a|u_n|^p \operatorname{sgn}(u_n)\varphi \to a|u_2|^p \operatorname{sgn}(u_2)\varphi \quad \text{a.e. in } \mathbb{R}^N.
\end{gathered}
$$

By using the Dominated convergence theorem we obtain

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N} a(x)|u_n|^p \operatorname{sgn}(u_n)\varphi\,dx = \int_{\mathbb{R}^N} a(x)|u_2|^p \operatorname{sgn}(u_2)\varphi\,dx.
$$

Therefore, for every $\varphi \in C_c^\infty(\mathbb{R}^N)$

$$
\langle \mathcal{I}'(u_n), \varphi\rangle \to \langle \mathcal{I}'(u_2), \varphi\rangle. \tag{4.5}
$$

Since $\{u_n\}$ is a (PS) sequence for $\mathcal{I}$ on $\mathbb{X}$, we have $\langle \mathcal{I}'(u_n), \varphi\rangle \to 0$, that combined with (4.5) gives $\langle \mathcal{I}'(u_2), \varphi\rangle$=0, hence u_2 is a weak solution to (1.1).

Now we prove that $u_2 \neq u_1$, where u_1 is the first solution to (1.1). Since $u_n \rightharpoonup u_2$ in $\mathbb{X}$, then up to a subsequence $\|u_2\| \leq \lim_{n\to\infty}\|u_n\|$. We distinguish two cases:

Case 1: compactness. We show that $u_n \to u_2$ in $\mathbb{X}$.
By Theorem (4.3) $\{u_n\}$ is bounded in $\mathbb{X}$, so up to a subsequence we can say that

$$
\begin{aligned}
&\lim_{n\to\infty}\Big[\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} |u_n|^{p+1}dx\Big] \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_2(x)-u_2(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} |u_2|^{p+1}dx
\end{aligned} \tag{4.6}
$$

from which it follows that

$$
\limsup_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^{p+1}dx = \iint_{\mathbb{R}^{2N}} \frac{|u_2(x)-u_2(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} |u_2|^{p+1}dx
$$

$$
\begin{aligned}
&- \limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\
&\le \iint_{\mathbb{R}^{2N}} \frac{|u_2(x)-u_2(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \int_{\mathbb{R}^N} |u_2|^{p+1}dx \\
&- \liminf_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy.
\end{aligned}
$$

We also know that

$$
\iint_{\mathbb{R}^{2N}} \frac{|u_2(x)-u_2(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \le \liminf_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy,
$$

$$
\int_{\mathbb{R}^N} |u_2|^{p+1}dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p+1}dx;
$$

therefore

$$
\int_{\mathbb{R}^N} |u_2|^{p+1}dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p+1}dx \le \limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p+1}dx \le \int_{\mathbb{R}^N} |u_2|^{p+1}dx
$$

so we deduce that

$$
u_n \to u_2 \quad \text{in } L^{p+1}(\mathbb{R}^N). \tag{4.7}
$$

Putting together (4.6) and (4.7) we obtain

$$
\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy = \iint_{\mathbb{R}^{2N}} \frac{|u_2(x)-u_2(y)|^2}{|x-y|^{N+2s}}\,dx\,dy.
$$

Now,

$$
\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|(u_n-u_2)(x)-(u_n-u_2)(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\
&= \iint_{\mathbb{R}^{2N}} \frac{|[u_n(x)-u_n(y)]-[u_2(x)-u_2(y)]|^2}{|x-y|^{N+2s}}\,dx\,dy \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \iint_{\mathbb{R}^{2N}} \frac{|u_2(x)-u_2(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \\
&\quad - 2\iint_{\mathbb{R}^{2N}} \frac{[u_n(x)-u_n(y)][u_2(x)-u_2(y)]}{|x-y|^{N+2s}}\,dx\,dy
\end{aligned} \tag{4.8}
$$

and by using (4.4) we obtain

$$
\begin{aligned}
&\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{[u_n(x)-u_n(y)][u_2(x)-u_2(y)]}{|x-y|^{N+2s}}\,dx\,dy \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_2(x)-u_2(y)|^2}{|x-y|^{N+2s}}\,dx\,dy.
\end{aligned} \tag{4.9}
$$

Combining (4.8) and (4.9) we have

$$
\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|(u_n-u_2)(x)-(u_n-u_2)(y)|^2}{|x-y|^{N+2s}}\,dx\,dy = 0. \tag{4.10}
$$

By (4.7) and (4.10) follows that $u_n \to u_2$ in $\mathbb{X}$.

Case 2: dichotomy. Assume that $\|u_2\| < \lim_{n\to\infty} \|u_n\|$. Let $v_n(x) = u_n(x) - u_2(x)$ be such that $v_n \rightharpoonup 0$ in $\mathbb{X}$.

Step 1: We show that there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$
v_n(\cdot + y_n) \rightharpoonup v_1 \neq 0 \quad \text{in } \mathbb{X}. \tag{4.11}
$$

Assume by contradiction that for any $R > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |v_n|^{p+1} dx = 0.$$

By using a variant of the Lions compactness principle (see [15, 23]) we can infer

$$v_n \to 0 \quad \text{in } L^q(\mathbb{R}^N) \quad \text{for all } q \in [p+1, 2_s^*). \tag{4.12}$$

Taking into account that $u_n(x) = v_n(x) + u_2(x)$, we obtain

$$\begin{aligned}
&\langle \mathcal{I}'(u_n), u_n\rangle \\
&= \iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}}\, dx\, dy + \iint_{\mathbb{R}^{2N}} \frac{|u_2(x)-u_2(y)|^2}{|x-y|^{N+2s}}\, dx\, dy \\
&\quad + 2\iint_{\mathbb{R}^{2N}} \frac{(v_n(x)-v_n(y))(u_2(x)-u_2(y))}{|x-y|^{N+2s}}\, dx\, dy + \int_{\mathbb{R}^N} V(x)|v_n|^2 dx \\
&\quad + \int_{\mathbb{R}^N} V(x)|u_2|^2 dx + 2\int_{\mathbb{R}^N} V(x) v_n u_2 dx + \int_{\mathbb{R}^N} a(x)(|u_n|^{p+1} - |u_2|^{p+1})\, dx \\
&\quad + \int_{\mathbb{R}^N} a(x)|u_2|^{p+1} dx - \int_{\mathbb{R}^N} f v_n dx - \int_{\mathbb{R}^N} f_2 dx \\
&= \langle \mathcal{I}'(u_2), u_2\rangle + \iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}}\, dx\, dy + \int_{\mathbb{R}^N} V(x)|v_n|^2 dx \\
&\quad + \int_{\mathbb{R}^N} a(x)\big(|u_n|^{p+1} - (|u_2|^{p+1} + |v_n|^{p+1})\big)\, dx + \int_{\mathbb{R}^N} a(x)|v_n|^{p+1} dx \\
&\quad - \int_{\mathbb{R}^N} f v_n dx + 2\iint_{\mathbb{R}^{2N}} \frac{(v_n(x)-v_n(y))(u_2(x)-u_2(y))}{|x-y|^{N+2s}}\, dx\, dy \\
&\quad + 2\int_{\mathbb{R}^N} V(x) v_n u_2 dx.
\end{aligned}$$

Putting together this and (4.12) we infer

$$\begin{aligned}
0 &= \lim_{n\to\infty} \langle \mathcal{I}'(u_n), u_n\rangle \\
&= \langle \mathcal{I}'(u_2), u_2\rangle + \lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^2}{|x-y|^{N+2s}}\, dx\, dy + \int_{\mathbb{R}^N} a(x)|v_n|^{p+1} dx.
\end{aligned}$$

Hence $v_n \to 0$ in $\mathbb{X}$, and this gives a contradiction.

Step 2: Now we prove that $\{y_n\}$ is not a bounded sequence. Assume by contradiction that $\{y_n\}$ is bounded. Then, up to a subsequence $y_n \to y$. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$. From the facts that $y_n \to y$ and $v_n \rightharpoonup 0$ in $\mathbb{X}$ it follows that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \varphi(x-y_n) v_n(x)\, dx = 0. \tag{4.13}$$

By using (4.11) we can infer

$$\begin{aligned}
\lim_{n\to\infty} \int_{\mathbb{R}^N} \varphi(x-y_n) v_n(x)\, dx &= \lim_{n\to\infty} \int_{\mathbb{R}^N} \varphi(x) v_n(x+y_n)\, dx \\
&= \int_{\mathbb{R}^N} \varphi(x) v_1(x)\, dx.
\end{aligned} \tag{4.14}$$

Putting together (4.13) and (4.14) we deduce that

$$\int_{\mathbb{R}^N} \varphi(x) v_1(x)\, dx = 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N),$$

that implies that $v_1(x) = 0$ a.e. in $\mathbb{R}^N$, and this is a contradiction because of (4.11). Thus $\{y_n\}$ is not bounded.

Step 3: v_1 is a solution to (4.21). Since $\{y_n\}$ is not bounded,

$$u_n(x + y_n) \rightharpoonup v_1 \quad \text{in } \mathbb{X}. \tag{4.15}$$

Now, let $\varphi \in C_c^\infty(\mathbb{R}^N)$. Since $\{u_n\}$ is a (PS) sequence for $\mathcal{I}$, we have $\langle \mathcal{I}'(u_n), \varphi(\cdot - y_n)\rangle \to 0$. On the other hand we have

$$\begin{aligned}
&\langle \mathcal{I}'(u_n), \varphi(\cdot - y_n)\rangle \\
&= \iint_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\varphi(x - y_n) - \varphi(y - y_n))}{|x - y|^{N+2s}}\, dx\, dy \\
&\quad + \int_{\mathbb{R}^N} V(x)u_n(x)\varphi(x - y_n)\, dx + \int_{\mathbb{R}^N} a(x)|u_n(x)|^p \operatorname{sgn}(u_n(x))\varphi(x - y_n)\, dx \\
&\quad - \int_{\mathbb{R}^N} f(x)\varphi(x - y_n)\, dx \\
&= \iint_{\mathbb{R}^{2N}} \frac{(u_n(x + y_n) - u_n(y + y_n))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}}\, dx\, dy \\
&\quad + \int_{\mathbb{R}^N} V(x + y_n)u_n(x + y_n)\varphi(x)\, dx \\
&\quad + \int_{\mathbb{R}^N} a(x + y_n)|u_n(x + y_n)|^p \operatorname{sgn}(u_n(x + y_n))\varphi(x)\, dx \\
&\quad - \int_{\mathbb{R}^N} f(x)\varphi(x - y_n)\, dx.
\end{aligned} \tag{4.16}$$

Since $|y_n| \to +\infty$, $f \in L^2(\mathbb{R}^N)$ and $\varphi \in C_c^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} f(x)\varphi(x - y_n)\, dx \to 0. \tag{4.17}$$

Moreover, by (4.15) we have

$$\begin{aligned}
&\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{(u_n(x + y_n) - u_n(y + y_n))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}}\, dx\, dy \\
&= \iint_{\mathbb{R}^{2N}} \frac{(v_1(x) - v_1(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}}\, dx\, dy.
\end{aligned} \tag{4.18}$$

Since $u_n(\cdot + y_n) \to v_1$ in $L^2(\operatorname{supp}\varphi)$, there exists a subsequence denoted again by $u_n(\cdot + y_n)$ and a function $h \in L^2(\operatorname{supp}\varphi)$ such that $|u_n(\cdot + y_n)| \le |h|$. Then, by (H2) it follows that

$$V(x + y_n)u_n(x + y_n)\varphi \to v_\infty v_1\varphi \quad \text{a.e. in } \mathbb{R}^N$$
$$|V(x + y_n)u_n(x + y_n)\varphi| \le \|V\|_{L^\infty(\mathbb{R}^N)}|h||\varphi| \in L^1(\mathbb{R}^N).$$

Thus, by the Dominated Convergence Theorem we obtain

$$\int_{\mathbb{R}^N} V(x + y_n)u_n(x + y_n)\varphi(x)\, dx \to v_\infty \int_{\mathbb{R}^N} v_1\varphi\, dx. \tag{4.19}$$

Similarly, since $u_n(\cdot + y_n) \to v_1$ in $L^{p+1}(\operatorname{supp}\varphi)$, there exist a subsequence denoted again by $u_n(\cdot+y_n)$ and a function $\tilde{h} \in L^{p+1}(\operatorname{supp}\varphi)$ such that $|u_n(\cdot+y_n)| \le$

$|\tilde{h}|$. Then, by (H3) we infer that

$$a(x+y_n)|u_n(x+y_n)|^p \operatorname{sgn}(u_n(x+y_n))\varphi \to a_\infty |v_1|^p \operatorname{sgn}(v_1)\varphi \quad \text{a.e. in } \mathbb{R}^N,$$
$$|a(x+y_n)|u_n(x+y_n)|^p\varphi| \le \|a\|_{L^\infty(\mathbb{R}^N)} |\tilde{h}|^p|\varphi| \in \mathrm{L}^1(\mathbb{R}^\mathrm{N}).$$

Thus, by the Dominated Convergence Theorem we obtain

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{R}^N} a(x+y_n)|u_n(x+y_n)|^p \operatorname{sgn}(u_n(x+y_n))\varphi(x)\,dx \\ &= a_\infty \int_{\mathbb{R}^N} |v_1|^p \operatorname{sgn}(v_1)\varphi\,dx. \end{aligned} \tag{4.20}$$

Putting together (4.16), (4.17), (4.18), (4.19) and (4.20) we obtain

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{(v_1(x)-v_1(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy + v_\infty \int_{\mathbb{R}^N} v_1\varphi\,dx \\ &+ a_\infty \int_{\mathbb{R}^N} |v_1|^p (v_1)\varphi\,dx = 0, \end{aligned}$$

that is v_1 is a weak solution to

$$\begin{gathered} (-\Delta)^s u + v_\infty u + a_\infty |u|^p \operatorname{sgn}(u) = 0 \quad \text{in } \mathbb{R}^N \\ u \in H^s(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N) \end{gathered} \tag{4.21}$$

But this problem only possesses the trivial solution, thus $v_1 = 0$ and this is an absurd in view of (4.11).

From Steps 1, 2 and 3 we conclude that the dichotomy does not occur. Then $\mathcal{I}(u_n) \to M = \mathcal{I}(u_2) > 0 (> \mathcal{I}(u_1))$ and $u_2 \neq u_1$. □

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