Two nonlinear days in Urbino 2017,
Electronic Journal of Differential Equations, Conference 25 (2018), pp. 77–85.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

DIFFERENTIABILITY VERSUS APPROXIMATE DIFFERENTIABILITY

LUIGI D'ONOFRIO

In memory of Anna Aloe

Abstract. One of the main tools in geometric function theory is the fact that the *area formula* is true for Lipschitz mapping; if f is differentiable a.e. (in the classic sense) then f can be exhausted up to a set of zero measure; the restriction of f, set by set, is Lipschitz [6, Theorem 3.18]. The aim of this survey is to clarify the regularity assumptions for a map to be differentiable a.e., and to give some auxiliary results when it is not, using the notion of approximate differentiability.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$, and let $f : \Omega \to \Omega' = f(\Omega)$ be a homeomorphism ($f \in \mathrm{Hom}(\Omega, \Omega')$ in short). We recall two classical notions:

- f satisfies the Lusin condition (also called condition (N), and denoted as $f \in (N)$) if
$$E \subset \Omega \text{ with } |E| = 0 \text{ implies } |f(E)| = 0 \tag{1.1}$$
where $|\cdot|$ denotes the Lebesgue measure.
- f is a *Sobolev homeomorphism* if f belongs to $W^{1,1}_{\mathrm{loc}}$. For $n \geq 2$, a mapping f is a *bi-Sobolev map* if f and f^{-1} are Sobolev homeomorphisms.

For a homeomorphism $f : \Omega \xrightarrow{\text{onto}} \Omega'$, condition (1.1) holds if and only if f maps measurable sets to measurable sets. Moreover, if f is differentiable at every point x of the Borel set $B \subset \Omega$ and $J_f(x)$ is the Jacobian determinant of f at x, then the *weak area formula* holds on B; that is,
$$\int_B \eta(f(z))|J_f(z)|\mathrm{d}z \leq \int_{f(B)} \eta(w)\mathrm{d}w \tag{1.2}$$
for any nonnegative Borel-measurable function η on $\mathbb{R}^n$.

Note that for $n \geq 3$, there is a homeomorphism of class $W^{1,n-1}_{\mathrm{loc}}((-1,1)^n, \mathbb{R}^n)$ such that both f and f^{-1} are nowhere differentiable [3, Example 5.2]. For this reason the aim of this survey (based mainly on results contained in [4, 5]) is to understand when dealing with mappings of $W^{1,p}$ with $p < n-1$, the notion of

2010 *Mathematics Subject Classification.* 46E35.
Key words and phrases. Sobolev homeomorphism; Lusin condition; approximate differentiability.

differentiability (that fails in this setting) can be replaced by the notion of approximate differentiability on the change of variable formula.

However, the condition (N) plays a fundamental role for these mappings. Indeed for such f, condition (N) is equivalent to the *area formula*

$$\int_B \eta(f(z))|J_f(z)|\mathrm{d}z = \int_{f(B)} \eta(w)\mathrm{d}w. \tag{1.3}$$

If the homeomorphism f satisfies the natural assumption $f \in W^{1,n}_{\rm loc}(\Omega, \mathbb{R}^n)$, then f satisfies the condition (N). This is due to Reshetnjak [29], and is a sharp result in the scale of $W^{1,p}(\Omega, \mathbb{R}^n)$-homeomorphisms thanks to an example of Ponomarev [27, 28] of a $W^{1,p}$-homeomorphisms $f : [0,1]^n \to [0,1]^n$, $p < n$ violating condition (N).

2. Lusin condition and differentiability almost everwhere

If $f \in \mathrm{Hom}(\Omega, \Omega')$ we decompose Ω as

$$\Omega = \mathcal{R}_f \cup \mathcal{Z}_f \cup \mathcal{E}_f \,,$$

where

$$\mathcal{R}_f = \{z \in \Omega : f \text{ is differentiable at z and } J_f(z) \neq 0\}, \tag{2.1}$$

$$\mathcal{Z}_f = \{z \in \Omega : f \text{ is differentiable at z and } J_f(z) = 0\}, \tag{2.2}$$

$$\mathcal{E}_f = \{z \in \Omega : f \text{ is not differentiable at z } \} \tag{2.3}$$

Differentiability is understood in the classical sense. Since f is continuous, these are Borel sets. Clearly we have

$$f(\mathcal{R}_f) = \mathcal{R}_{f^{-1}}. \tag{2.4}$$

Let us recall the *weak area formula* from Federer [6, Theorem 3.1.8]. Let $B \subset \Omega$ be a Borel measurable set and assume that $f : \Omega \xrightarrow{\text{onto}} \Omega'$ is a homeomorphism such that f is differentiable at every point of B, then for any $\eta : \mathbb{R}^n \to [0, +\infty[$ Borel measurable function we have

$$\int_B \eta(f(z))|J_f(z)|\mathrm{d}z \leq \int_{f(B)} \eta(w)\mathrm{d}w \tag{2.5}$$

This follows from the area formula (1.3) which is valid for Lipschitz mappings and from the fact that the set of differentiability can be exhausted up to a set of zero measure by sets the restriction to which of f is Lipschitz [[6] Theorem 3.1.8]. Hence, for an a.e. differentiable homeomorphism on Ω we can decompose Ω into pairwise disjoint sets

$$\Omega = Z \cup \cup_{k=1}^{\infty} \Omega_k \tag{2.6}$$

such that $|Z| = 0$ and f_{Ω_k} is Lipschitz.

We note the following consequence of (2.5). If $B' \subset f(\Omega)$ is a Borel subset with $|B'| = 0$, then $J_f(x) = 0$ for a.e. $x \in f^{-1}(B')$. Indeed

$$\int_{f^{-1}(B')} |J_f(z)|\mathrm{d}z \leq \int_{B'} \mathrm{d}w = |B'| = 0.$$

For example, if f^{-1} is differentiable a.e. on $f(\Omega)$, then $J_f(x) = 0$ for a.e. $x \in f^{-1}(\mathcal{E}_{f^{-1}})$ where

$$\mathcal{E}_{f^{-1}} = \{z \in \Omega : f^{-1} \text{ is not differentiable at z}\}.$$

We say that *the area formula* holds for f on B if (2.5) is valid as an equality; that is,

$$\int_B \eta(f(z))|J_f(z)|\mathrm{d}z = \int_{f(B)} \eta(w)\mathrm{d}w \tag{2.7}$$

for all $\eta : \mathbb{R}^n \to [0, +\infty[$ Borel measurable function.

For a Sobolev homeomorphisms $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ (that is if the coordinate functions of f belong to the Sobolev space $W^{1,1}(\Omega)$ of L^1-functions $u : \Omega \to R$ whose gradient $|\nabla u|$ belongs to $L^1(\Omega)$) it is well known that there exists a set $\tilde{\Omega}$ of full measure such that the area formula holds for f on $\tilde{\Omega}$. Also, the area formula holds on each set on which the Lusin condition (N) is satisfied (this follows from the area formula for Lipschitz mappings, from the a.e. approximate differentiability of f (see [6, Theorem 3.1.4]) and the already mentioned general property of a.e. differentiable functions [6, Theorem 3.1.8], namely that Ω can be exhausted up to a set of measure zero by sets the restriction to which of f is Lipschitz continuous).

So, if we choose the Borel set $B = \mathcal{Z}_f$ as defined in (2.2) then by (1.3), we deduce

$$|f(\mathcal{Z}_f)| = 0$$

which is a weak version of the classical Sard lemma.

Let $f : \Omega \xrightarrow{\text{onto}} \Omega'$ be a homeomorphism. Then f maps every Borel set $B \subset \Omega$ onto a Borel set. Note that here we need to restrict ourselves to Borel sets B only since the homeomorphic image of a measurable set need not remain measurable.

In fact if f is a Cantor type homeomorphism $f : [0, 1] \to [0, 2]$ such that a zero set N_0, $|N_0| = 0$ is mapped to a positive set $P_0' = f(N_0)$, $|P_0'| > 0$ and E' is a non measurable set contained in P_0' (recall that every set of Lebesgue positive measure contains a non measurable subset) then $f^{-1}(E')$ is contained in the null set N_0 hence it is measurable.

The question of the differentiability in the classical sense of a homeomorphisms has a rather simple positive answer in the case $n = 2$ thanks to a classical Theorem by Gehring-Lehto (see [10, 19, 22]).

Theorem 2.1. *Let Ω and Ω' be bounded domains in the plane and suppose that $f \in Hom(\Omega, \Omega')$ has finite partial derivatives a.e. in Ω, then f is differentiable a.e. in Ω.*

As a consequence, if $f = (u, v) : \Omega \subset \mathbb{R}^2 \xrightarrow{\text{onto}} \Omega' \subset \mathbb{R}^2$ is a Sobolev-homeomorphism, then f and f^{-1} are differentiable a.e. ([16]). The fairly well-known Theorem of Gehring-Lehto is one of the few facts from real analysis that carry geometric information up from the infinitesimal level. Its proof uses properties of the plane, in fact the Theorem at this stage of generality (f a BV-homeomorphism or f a $W^{1,1}$-homeomorphism) is false in higher dimension.

In the general case $n \geq 2$, the minimal integrability conditions on the partial derivatives of a Sobolev homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ needed to guarantee a.e. differentiability have been found by J. Onninen [26], generalizing a classical result of Stein [32].

It turns out that if $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ and $|Df| \in L^{n-1,1}(\Omega)$ (where the Lorentz space $L^{p,1}(\Omega)$, $1 \leq p < \infty$ is defined as the class of all measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L^{p,1}(\Omega)} = p \int_0^\infty |\{z \in \Omega : |u(z)| > t\}|^{1/p}\mathrm{d}t$$

is finite) then the homeomorphisms f and f^{-1} are differentiable a.e.

This is sharp after an example of a $W^{1,n-1}$-homeomorphism f ($n \geq 3$) which is bi-Sobolev (that is, $f^{-1} \in W^{1,1}$) and both f, f^{-1} are nowhere differentiable [3]. Let us prove the following useful result which generalizes [19, Lemma 3.4].

Proposition 2.2 ([4])**.** *If f is a Sobolev homeomorphism such that $J_f \geq 0$, then f^{-1} satisfies condition (N), if and only if, $J_f(z) > 0$ for a.e. $z \in \Omega$.*

Proof. Suppose first that $f^{-1} \in (N)$ and denote by $\tilde{\Omega}$ a subset of Ω of full measure such that the area formula (2.7) with $B = \tilde{\Omega}$ holds true. Hence,

$$|f(\{z \in \tilde{\Omega} : J_f(z) = 0\})| = 0$$

and by condition (N), for f^{-1} we have

$$|\{z \in \Omega : J_f(z) = 0\}| = |\{z \in \tilde{\Omega} : J_f(z) = 0\} \cup (\Omega \setminus \tilde{\Omega})| = 0.$$

Conversely, suppose $J_f(z) > 0$ a.e. and let us prove that $f^{-1} \in (N)$. Assuming by contradiction that there exists $|N_0'| = 0$, $N_0' \subset \Omega'$ with $|f^{-1}(N_0')| > 0$, then we have

$$\int_{f^{-1}(N_0')} J_f \leq |f(f^{-1}(N_0'))| = |N_0'| = 0.$$

Hence $J_f = 0$ on the positive set $f^{-1}(N_0') \subset \Omega$ and this is a contradiction. □

Let us prove a simple characterization of condition (N) for a function f.

Proposition 2.3 ([4])**.** *If $f : \Omega \xrightarrow{\text{onto}} \Omega'$ is a Sobolev homemorphism, $J_f \geq 0$ and*

$$\int_B J_f = |f(B)|$$

for any Borel set $B \subset \Omega$, then $f \in (N)$ on every Borel set $B \subset \Omega$.

Proof. By contradiction, assume that there exists a subset $E \subset B : |E| = 0$ and $|f(E)| > 0$. Then

$$\int_B J_f = \int_{B \setminus E} J_f \leq |f(B \setminus E)| = |f(B)| - |f(E)| < |f(B)|$$

which is a contradiction. □

An interesting application of condition (N) is the following result on the inverse of an a.e. differentiable homeomorphism, which in the plane and has an interesting counterpart.

Proposition 2.4 ([4])**.** *Let $f \in Hom(\Omega, \Omega')$ be differentiable a.e.. If f satisfies condition* (N)*, then the inverse f^{-1} is differentiable a.e..*

Proof. We notice that the area formula (1.3) holds on each set on which f satisfies condition (N); in particular it holds on $\mathcal{R}_f \cup \mathcal{Z}_f$, that is the set where f is differentiable:

$$\int_{\mathcal{R}_f \cup \mathcal{Z}_f} \eta(f(z))|J_f(z)|\mathrm{d}z = \int_{f(\mathcal{R}_f \cup \mathcal{Z}_f)} \eta(w)\mathrm{d}w. \tag{2.8}$$

In particular, we have the following version of Sard Lemma,

$$|f(\mathcal{Z}_f)| = 0. \tag{2.9}$$

Since f is differentiable a.e., $\mathcal{E}_f$ has measure zero and by condition (N), $f(\mathcal{E}_f)$ has measure zero. We note that f^{-1} is differentiable in $f(\mathcal{R}_f)$ which is a subset of full measure of $f(\Omega)$; indeed,

$$f(\Omega) \setminus f(\mathcal{R}_f) = f(\mathcal{Z}_f) \cup f(\mathcal{E}_f)$$

has measure zero by (2.9) and condition (N). □

By $A \triangle B$, we denote the set $(A \cup B) \setminus (A \cap B)$. By $A = B$ a.e. we mean $|A \triangle B| = 0$.

Proposition 2.5. *Let $f \in \operatorname{Hom}(\Omega, \Omega')$ and assume that f and f^{-1} are differentiable a.e. and both satisfy condition* (N)*, then f essentially maps $\mathcal{E}_f$ to $\mathcal{Z}_{f^{-1}}$ and f^{-1} maps $\mathcal{E}_{f^{-1}}$ to $\mathcal{Z}_f$ in the sense that*

$$|f(\mathcal{E}_f) \triangle \mathcal{Z}_{f^{-1}}| = |f(\mathcal{E}_{f^{-1}}) \triangle \mathcal{Z}_f| = 0.$$

3. Differentiability versus approximate differentiability

As we have already observed the notion of differentiability can not guaranteed for homeomorphism that are in $W^{1,n-1}$. To avoid this problem the notion of approximate differentiability comes to the play, so we would like to know if some results presented in the previous section are still valid. For example, let us consider the following issue: let Ω and Ω' be domains in $\mathbb{R}^n$, if $f : \Omega \xrightarrow{\text{onto}} \Omega'$ is a homeomorphism approximately differentiable at $x_0 \in \Omega$ with nonzero Jacobian, $J_f(x_0) \neq 0$, is it true that f^{-1} is approximately differentiable at $y_0 = f(x_0)$ and that

$$J_{f^{-1}}(y_0) = \frac{1}{J_f(x_0)}? \tag{3.1}$$

This is true for $n = 1$, because a homeomorphism $h : [a, b] \subset \mathbb{R} \xrightarrow{\text{onto}} [a', b'] \subset \mathbb{R}$ is strictly monotone, hence approximate differentiability is equivalent to ordinary differentiability [15].

Recall that f is approximately differentiable at $x_0 \in \Omega$ with approximate gradient $Df(x_0)$, if there is a set $A \subset \Omega$ of *density* one at x_0, i.e.

$$\lim_{r\to 0} \frac{|A \cap \mathbb{B}_r(x_0)|}{|\mathbb{B}_r(x_0)|} = 1\,, \tag{3.2}$$

where $\mathbb{B}_r(x)$ denotes the closed ball of center x and radius r, such that

$$\lim_{y\to x_0,\, y\in A} \frac{|f(y) - f(x_0) - Df(x_0)(y - x_0)|}{|y - x_0|} = 0. \tag{3.3}$$

Lebesgue's density Theorem guarantees that almost every point of a measurable set $A \subset \Omega$ is a point of density one for A [30, p. 129]. In view of the notion of approximate differentiability, we decompose the set Ω in a different way than in Section 2:

$$\Omega = \mathscr{R}_f \cup \mathscr{Z}_f \cup \mathscr{E}_f\,,$$

where

$$\begin{gathered}\mathscr{R}_f = \{x \in \Omega : f \text{ is approximately differentiable at } x \text{ and } J_f(x) \neq 0\},\\ \mathscr{Z}_f = \{x \in \Omega : f \text{ is approximately differentiable at } x \text{ and } J_f(x) = 0\},\\ \mathscr{E}_f = \{x \in \Omega : f \text{ is not approximately differentiable at } x\}\end{gathered}$$

The following version of chain rule was established in [7, 14, 15].

Theorem 3.1. *Let $f : \Omega \subset \mathbb{R}^n \xrightarrow{\text{onto}} \Omega' \subset \mathbb{R}^n$ be a bi-Sobolev map. Then f and f^{-1} are approximately differentiable a.e. and there exists a Borel set $B \subset \mathscr{R}_f$ with $|\mathscr{R}_f \setminus B| = 0$ such that $f(B) \subset \mathscr{R}_{f^{-1}}$ where*

$$\mathscr{R}_{f^{-1}} = \{y \in \Omega' : f^{-1} \text{ is approximately differentiable at } y \text{ and } J_{f^{-1}}(y) \neq 0\} \quad (3.4)$$

with $|\mathscr{R}_{f^{-1}} \setminus f(B)| = 0$. Also we have

$$Df^{-1}(y) = \big(Df(f^{-1}(y))\big)^{-1} \quad \forall y \in f(B).$$

The proof in [15] relies on the fact that if $g : \Omega \to \mathbb{R}^n$ is a Lipschitz map and $J_g(x_0) \neq 0$ for a point $x_0 \in \Omega$, then

$$A \text{ being a set of density 1 at } x_0 \text{ implies } g(A) \text{ is a set of density 1 at } g(x_0). \quad (3.5)$$

So we are naturally induced to consider the following problem: how density points of measurable sets are transformed under Sobolev or bi-Sobolev maps?

Actually a result of Buczolich [2] guarantees the preservation of density points of Ω under any bi-Lipschitz map $f : \Omega \subset \mathbb{R}^n \xrightarrow{\text{onto}} \Omega' \subset \mathbb{R}^n$.

Note that if $f : \Omega \xrightarrow{\text{onto}} \Omega'$ preserves density points, then f satisfies the Lusin $(\mathcal{N})$-property.

Let us emphasize that for almost every $x \in \Omega$ the density of E at x is one if and only if the density of $f(E)$ at $f(x)$ is one. Here we want to prove that, under suitable assumptions on f, for *all* $x \in \Omega$ we have preservations of density points.

3.1. **Density points for $n = 1$.** We shall consider here Q-quasiminimizers, $Q \geq 1$, of the one-dimensional Dirichlet integral

$$u \to \int |u'|^p dx \quad p > 1 \quad (3.6)$$

whose definition goes back to Giaquinta-Giusti [12, 11, 20]). Let (a, b) be an open interval in $\mathbb{R}$ and $h \in W^{1,p}_{\text{loc}}((a, b))$; then h is a Q-quasiminimizer of (3.6) if for all $[c, d] \subset (a, b)$

$$\int_c^d |h'|^p dx \leq Q \int_c^d |k'|^p dx \quad (3.7)$$

whenever $k \in h + W^{1,p}_0((c, d))$. We say that h is a quasiminimizer of a Dirichlet integral if there exists $p > 1$ such that h is a quasiminimizer of (3.6).

Note that if $h : (a, b) \to (a', b')$ satisfies the bi-Lipschitz condition, i.e. there exists $L > 1$ such that

$$\frac{1}{L} \leq |h'(x)| \leq L \quad (3.8)$$

for a.e. $x \in (a, b)$, then for any $p > 1$ it satisfies (3.7) with $Q = L^2$. Indeed, by (3.8) we have

$$\frac{1}{d-c}\int_c^d (h')^2 \leq \frac{L}{d-c}\int_c^d h' \leq \Big(L \fint_c^d h'\Big) L \fint_c^d h'.$$

Hence

$$\fint_c^d (h')^2 \leq L^2 \Big(\fint_c^d h'\Big)^2.$$

In [4] the authors proved that quasiminimizers (which are far away from being bi-Lipschitz mappings) are $W^{1,r}$-biSobolev maps (that is $f, f^{-1} \in W^{1,r}$ for a certain

$r > 1$) and preserve density points. We emphasize that this can be interpreted also as a regularity result for one dimensional Q-quasiminima.

Theorem 3.2 ([4]). *Let $h \in W^{1,p}_{\rm loc}((a,b))$, $p > 1$, be a non-constant Q-quasiminimizer of* (3.6). *Then there exists $r > 1$ such that h is a $W^{1,r}$-biSobolev map which preserves density points together with its inverse.*

3.2. **Density points for $n = 2$.** Now, let us consider a special class of bi-Sobolev maps: the quasiconformal mappings. We will say that $f : \Omega \xrightarrow{\text{onto}} \Omega'$ is a K-quasiconformal map, $K \geq 1$, if f is a bi-Sobolev map such that

$$|Df(x)|^2 \leq K J_f(x) \tag{3.9}$$

for a.e. $x \in \Omega$. Since f and f^{-1} satisfy the $(\mathcal{N})$-condition of Lusin, i.e. they map sets of measure zero onto sets of measure zero (see [1], [15]), then for any $E \subset \Omega$ measurable, $f(E)$ is measurable; hence for a.e. $x \in \Omega$, x is a density point for E if and only if $f(x)$ is a density point for $f(E)$. We remark that this holds for *all* $x \in \Omega$, according to a result by Gehring-Kelly [9].

Theorem 3.3 ([9]). *If $f : \Omega \xrightarrow{\text{onto}} \Omega'$ is a K-quasiconformal map, then f and f^{-1} preserve density points.*

This is a consequence of the following invariant form of the area distortion Theorem: there is a constant $C(K)$ such that for any K-quasiconformal map f, we have

$$\frac{1}{C(K)}\Big(\frac{|E|}{|Q|}\Big)^K \leq \frac{|f(E)|}{|f(Q)|} \leq C(K)\Big(\frac{|E|}{|Q|}\Big)^{1/K} \tag{3.10}$$

for any cube $Q \subset \Omega$ and for any subset $E \subset Q$ [1, Theorem 13.1.5)].

3.3. **Density points for $n \geq 2$.** We say that a homeomorphism $f : \Omega \subset \mathbb{R}^n \xrightarrow{\text{onto}} \Omega' \subset \mathbb{R}^n$ satisfies the *uniform area inequality* if there exist constants C, α with $0 < \alpha \leq 1 \leq C$ such that

$$\frac{|f(E)|}{|f(B)|} \leq C\Big(\frac{|E|}{|B|}\Big)^\alpha \tag{3.11}$$

for any ball $B \subset \Omega$ and for any measurable set $E \subset B$.

Theorem 3.4. *Let $f : \Omega \xrightarrow{\text{onto}} \Omega'$ be a continuous mapping satisfying the uniform area inequality* (3.11). *If f is differentiable at x_0 with $J_f(x_0) \neq 0$ and A is a set of density one at x_0, then the density of $f(A)$ at $f(x_0)$ is one.*

Our aim is to give the chain rule formula (3.1) assuming only that f is an a.e. approximately differentiable homeomorphism not necessarily of Sobolev class and with no assumptions on f^{-1}. In this sense the following result is a generalized version of the inverse function Theorem.

Theorem 3.5 ([5]). *Let $f : \Omega \subset \mathbb{R}^n \xrightarrow{\text{onto}} \Omega' \subset \mathbb{R}^n$ be a homeomorphism. If f is approximately differentiable a.e. then there exists a Borel set $B \subset \mathscr{R}_f$ with $|B| = |\mathscr{R}_f|$ such that $f(B) \subset \mathscr{R}_{f^{-1}}$ with $|f(B)| = |\mathscr{R}_{f^{-1}}|$ and*

$$Df^{-1}(f(x))Df(x) = Id \quad J_{f^{-1}}(f(x))J_f(x) = 1 \quad \textit{for all } x \in B.$$

Moreover, if $|B| > 0$ then $|f(B)| > 0$.

As a consequence, we have the following corollary.

Corollary 3.6 ([5])**.** *Let* $f : \Omega \subset \mathbb{R}^n \overset{\text{onto}}{\longrightarrow} \Omega' \subset \mathbb{R}^n$ *be a homeomorphism. If* f *is approximately differentiable a.e. then*

$$|\mathscr{R}_{f^{-1}}| = |f(\mathscr{R}_f)|. \tag{3.12}$$

Corollary 3.7 ([5])**.** *Let* $f : \Omega \subset \mathbb{R}^n \overset{\text{onto}}{\longrightarrow} \Omega' \subset \mathbb{R}^n$ *be a homeomorphism. If* f *is approximately differentiable almost everywhere and* f *satisfies the Lusin* $(\mathcal{N})$ *condition, then* f^{-1} *is approximately differentiable almost everywhere.*

In general, it is not true that $f(\mathscr{R}_f) \subset \mathscr{R}_{f^{-1}}$, as it happens when f is differentiable in the classical sense. D'Onofrio-Sbordone-Schiattarella [5] provided the following example.

Example 3.8. Let $n \geq 2$. There is a bi-Sobolev homeomorphism $f : \mathbb{B}_1(0) \overset{\text{onto}}{\longrightarrow} \mathbb{B}_1(0)$ with $f \in W^{1,n}(\mathbb{B}_1(0), \mathbb{R}^n)$ such that f is not differentiable at 0, f is approximately differentiable at 0, $J_f(0) \neq 0$ and f^{-1} is not approximately differentiable at $f(0) = 0$.

We emphasize that the map in this example satisfies Lusin condition as it belongs to $W^{1,n}$, and it does not preserve density points.

Acknowledgments. The author is member of the *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* (GNAMPA) of the *Istituto Nazionale di Alta Matematica* (INdAM). The manuscript was realized within the auspices of the INdAM - GNAMPA Project and of *Sostegno alla Ricerca* projest of University of Napoli "Parthenope".

References

[1] K. Astala, T. Iwaniec, G. Martin; *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton Mathematical Series, 2011.

[2] Z. Buczolich; *Density points and bi-Lipschitz functions in* $\mathbb{R}^m$, Proc. Amer. Math. Soc. **116** (1992), no. 1, 53–59.

[3] M. Csörney, S. Hencl, J. Maly; *Homeomorphism in the Sobolev space* $W^{1,n-1}$, J. Reine Angew. Math. **644**, (2010), 221–235.

[4] L. D'Onofrio, C. Sbordone, R. Schiattarella; *The Grand Sobolev nomeomorphisms and their measurability properties*, Adv. Nonlinear Stud. **12** (2012), 767–782.

[5] L. D'Onofrio, C. Sbordone, R. Schiattarella; *On the approximate differentiability of inverse maps*, J. Fixed Point Theory Appl. **15** (2014), 773–799.

[6] H. Federer; *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer- Verlag, New York, 1969 (second edition 1996).

[7] N. Fusco, G. Moscariello, C. Sbordone; *The limit of* $W^{1,1}$*-homeomorphisms with finite distortion*, Calc. Var. **33** (2008), 377–390.

[8] F. W. Gehring; *The* L^p *-integrability of the partial derivatives of quasiconformal mappings*, Acta Math. **130**, (1973), 265–277.

[9] F. W. Gehring, J. C. Kelly; *Quasi-conformal mappings and Lebesgue density* Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973), pp. 171–179. Ann. of Math. Studies, No. **79**, Princeton Univ. Press, Princeton, NJ, 1974.

[10] F. W. Gehring, O. Lehto; *On the total differentiability of functions of a complex variable*, Ann. Acad. Sci. Fenn. Ser. A1. Math. **272**, (1959) 3–8.

[11] M. Giaquinta, E. Giusti; *Quasiminima* Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 2, 79–107.

[12] M. Giaquinta, E. Giusti; *On the regularity of the minima of variational integrals*, Acta Math. **148** (1982), 31–46.

[13] M. Giaquinta, G. Modica, J. Souček; *Area and the area formula*. Proceedings of the Second International Conference on Partial Differential Equations (Italian) (Milan, 1992). Rend. Sem. Mat. Fis. Milano **62** (1992), 53–87.

[14] S. Hencl; *Sharpness of the assumptions for the regularity of a homeomorphism,* Michigan Math. J. **59** (2010), no. 3, 667–678.

[15] S. Hencl, P. Koskela; *Lectures on mappings of finite distortion*, Lecture Notes in Mathematics 2096, Springer, (2014).

[16] S. Hencl, P. Koskela, J. Onninen; *Homeomorphisms of bounded variation*, Arch. Ration. Mech. Anal. 186 (2007), no. 3, 351–360.

[17] S. Hencl, G. Moscariello, A. Passarelli di Napoli, C. Sbordone; *Bisobolev mappings and elliptic equation in the plane*, J. Math. Anal. Appl. **355** (2009), 22–32.

[18] T. Iwaniec, G. Martin; *Geometric function theory and non-linear analysis*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2001.

[19] O. Lehto, K. Virtanen; *Quasiconformal Mappings in the Plane*. Springer-Verlag, New York (1973).

[20] O. Martio, C. Sbordone; *Quasiminimizers in one dimension: integrability of the derivative, inverse function and obstacle problems*, Ann. Mat. Pura Appl., (4) **186** (2007), no. 4, 579–590.

[21] O. Martio, W. Ziemer; *Lusin's condition (N) and mappings with nonnegative Jacobians.* Michigan Math. J., **39** (1992), no. 3, 495–508.

[22] D. Menchoff; *Sur les differencelles totales des fonctions univa-lentes*, Math. Ann. **105**, (1931) 75-85.

[23] G. Moscariello, A. Passarelli di Napoli; *The Regularity of the Inverses of Sobolev Homeomorphisms with Finite Distortion*, J. Geom. Anal. **24** (2014), no. 1, 571–594.

[24] G. Moscariello, A. Passarelli di Napoli, C. Sbordone; *Planar ACL-homeomorphisms: critical points of their components.* Commun. Pure Appl. Anal. **9** (2010), no. 5, 1391–1397.

[25] J. Niewiarowski; *Density-preserving homeomorphisms*, Fund. Math. **106** (1980), no. 2, 77–87.

[26] J. Onninen; *Differentiability of monotone Sobolev functions.* Real Anal. Exchange, **26** (2000/01), no. 2, 761–772.

[27] S. Ponomarev; *Examples of homeomorphisms in the class $ACTL^p$ which do not satisfy the absolute continuity condition of Banach*, Dokl. Akad. Nauk USSR, **201** (1971), 1053-1054.

[28] S. Ponomarev; *Property N of homeomorphisms of the class W^{1p}* , Sibirsk. Math. Zh., (Russian) **28** (1987), 140–148.

[29] Yu. G. Reshetnyak; *Some geometrical properties of functions and mappings with generalized derivatives*, Sibirsk. Math. Zh. (Russian), **7** (1966), 886-919.

[30] S. Saks; *Theory of the integral*, Dover Publications, Inc., New York (1964).

[31] C. Sbordone, R. Schiattarella; *Critical points for Sobolev homeomorphisms* Rendiconti Lincei Matematica ed Applicazioni **22**, 2011, n.2, 207–222

[32] E. M. Stein; *Editor's note: The differentiability of functions in $\mathbb{R}^n$*, Annals of Mathematics, **113**, (1981), 383–385.

[33] J. Väisälä; *Lectures on n-dimensional quasiconformal mappings.* Lecture Notes in Mathematics, Vol. 229. Springer-Verlag, Berlin-New York, 1971.

Luigi D'Onofrio

University of Napoli "Parthenope ", Italy

E-mail address: donofrio@uniparthenope.it