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GLOBAL WEAK SOLUTIONS TO DEGENERATE COUPLED DIFFUSION-CONVECTION-DISPERSION PROCESSES AND HEAT TRANSPORT IN POROUS MEDIA

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Abstract. In this contribution we prove the existence of weak solutions to degenerate parabolic systems arising from the coupled moisture movement, transport of dissolved species and heat transfer through partially saturated porous materials. Physically motivated mixed Dirichlet-Neumann boundary conditions and initial conditions are considered. Existence of a global weak solution of the problem is proved by means of semidiscretization in time and by passing to the limit from discrete approximations. Degeneration occurs in the nonlinear transport coefficients which are not assumed to be bounded below and above by positive constants. Degeneracies in all transport coefficients are overcome by proving suitable a priori L^∞-estimates for the approximations of primary unknowns of the system.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$, $\Omega \in C^{0,1}$ and let Γ_D and Γ_N be open disjoint subsets of $\partial\Omega$ (not necessarily connected) such that $\Gamma_D \neq \emptyset$ and the $\partial\Omega \backslash (\Gamma_D \cup \Gamma_N)$ is a finite set. Let $T \in (0,\infty)$ be fixed throughout the paper, $I = (0,T)$ and $Q_T = \Omega \times I$ denotes the space-time cylinder, $\Gamma_{DT} = \Gamma_D \times I$ and $\Gamma_{NT} = \Gamma_N \times I$.

We shall study the following initial boundary value problem in Q_T,

$$\partial_t b(u) = \nabla \cdot [a(\theta)\nabla u], \tag{1.1}$$

$$\partial_t [b(u)w] = \nabla \cdot [b(u)D_w(u)\nabla w] + \nabla \cdot [wa(\theta)\nabla u], \tag{1.2}$$

$$\partial_t [b(u)\theta + \varrho\theta] = \nabla \cdot [\lambda(\theta,u)\nabla\theta] + \nabla \cdot [\theta a(\theta)\nabla u], \tag{1.3}$$

with the mixed-type boundary conditions

$$u = 0, \quad w = 0, \quad \theta = 0 \quad \text{on } \Gamma_{DT}, \tag{1.4}$$

$$\nabla u \cdot \mathbf{n} = 0, \quad \nabla w \cdot \mathbf{n} = 0, \quad \nabla\theta \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_{NT} \tag{1.5}$$

and the initial conditions

$$u(\cdot,0) = u_0, \quad w(\cdot,0) = w_0, \quad \theta(\cdot,0) = \theta_0 \quad \text{in } \Omega. \tag{1.6}$$

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System 1.1–1.6 arises from the coupled moisture movement, transport of dissolved species and heat transfer through the porous system [4, 20]. For simplicity, the gravity terms and external sources are not included since they do not affect the analysis. For specific applications we refer the reader to e.g. [19]. Here $u : Q_T \to \mathbb{R}$, $w : Q_T \to \mathbb{R}$ and $\theta : Q_T \to \mathbb{R}$ are the unknown functions. In particular, u corresponds to the Kirchhoff transformation of the matric potential [2], w represents concentration of dissolved species and θ represents the temperature of the porous system. Further, $a : \mathbb{R} \to \mathbb{R}$, $D_w : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$, $\lambda : \mathbb{R}^2 \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$, $w_0 : \Omega \to \mathbb{R}$, and $\theta_0 : \Omega \to \mathbb{R}$ are given functions, ϱ is a real positive constant and $\mathbf{n}$ is the outward unit normal vector. In this paper we study the existence of the weak solution to (1.1)–(1.6).

Nowadays, description of heat, moisture or soluble/non-soluble contaminant transport in concrete, soil or rock porous matrix is frequently based on time dependent models. Coupled transport processes (diffusion processes, heat conduction, moister flow, contaminant transport or coupled flows through porous media) are typically associated with systems of strongly nonlinear degenerate parabolic partial differential equations of type (written in terms of operators A, Ψ, F)

$$\partial_t \Psi(\mathbf{u}) - \nabla \cdot A(\mathbf{u}, \nabla \mathbf{u}) = F(\mathbf{u}), \tag{1.7}$$

where $\mathbf{u}$ stands for the unknown vector of state variables. There is no complete theory for such general problems. However, some particular results assuming special structure of operators A and Ψ and growth conditions on F can be found in the literature, see [22].

Most theoretical results on parabolic systems exclude the case of non-symmetrical parabolic parts [2, 8, 13].

Giaquinta and Modica [10] proved the local-in-time solvability of quasilinear diagonal parabolic systems with nonlinear boundary conditions (without assuming any growth condition), see also [23].

The existence of weak solutions to more general non-diagonal systems like (1.7) subject to mixed boundary conditions has been proven in [2]. The authors proved an existence result assuming the operator Ψ to be only (weak) monotone and subgradient. This result has been extended in [8], where the authors presented the local existence of the weak solutions for the system with nonlinear Neumann boundary conditions and under more general growth conditions on nonlinearities in $\mathbf{u}$. These results, however, are not applicable if Ψ does not take the subgradient structure, which is typical of coupled transport models in porous media. Thus, the analysis needs to exploit the specific structure of such problems.

The existence of a local-in-time strong solution for moisture and heat transfer in multi-layer porous structures modelling by the doubly nonlinear parabolic system is proven in [5]. In [21], the author proved the existence of the solution to the purely diffusive hygro-thermal model allowing non-symmetrical operators Ψ, but requiring non-realistic symmetry in the elliptic part. In [7, 12], the authors studied the existence, uniqueness and regularity of coupled quasilinear equations modeling evolution of fluid species influenced by thermal, electrical and diffusive forces. In [15, 16, 17], the authors studied a model of specific structure of a heat and mass transfer arising from textile industry and proved the global existence for one-dimensional problems in [15, 16] and three-dimensional problems in [17].

In the present paper we extend our previous existence result for coupled heat and mass flows in porous media [6] to more general problem (including the convection-dispersion equation) modeling coupled moisture, solute and heat transport in porous media. This leads to a fully nonlinear degenerate parabolic system with natural (critical) growths and degeneracies in all transport coefficients.

The rest of this paper is organized as follows. In Section 2, we introduce basic notation and suitable function spaces and specify our assumptions on data and coefficient functions in the problem. In Section 3, we formulate the problem in the variational sense and state the main result, the global-in-time existence of the weak solution. The main result is proved by an approximation procedure in Section 4. First we formulate the semi-discrete scheme and prove the existence of its solution. The crucial a priori estimates and uniform boundness of time interpolants are proved in part 4.2. Finally, we conclude that the solutions of semi-discrete scheme converge and the limit is the solution of the original problem (Subsection 4.3).

Remark 1.1. The present analysis can be straightforwardly extended to a setting with nonhomogeneous boundary conditions (see [6] for details). Here we work with homogeneous boundary conditions, ignoring the gravity terms and excluding external sources to simplify the presentation and avoid unnecessary technicalities in the existence result.

2. Preliminaries

2.1. **Notation and some properties of Sobolev spaces.** Vectors and vector functions are denoted by boldface letters. Throughout the paper, we will always use positive constants C, c, c_1, c_2, $\dots$, which are not specified and which may differ from line to line. Throughout this paper we suppose $s, q, s' \in [1,\infty]$, s' denotes the conjugate exponent to $s > 1$, $1/s + 1/s' = 1$. $L^s(\Omega)$ denotes the usual Lebesgue space equipped with the norm $\|\cdot\|_{L^s(\Omega)}$ and $W^{k,s}(\Omega)$, $k \geq 0$ (k need not to be an integer, see [14]), denotes the usual Sobolev-Slobodecki space with the norm $\|\cdot\|_{W^{k,s}(\Omega)}$. We define

$$W^{1,2}_{\Gamma_D}(\Omega) := \big\{ v \in W^{1,2}(\Omega) : v\big|_{\Gamma_D} = 0 \big\}.$$

By E^* we denote the space of all continuous, linear forms on Banach space E and by $\langle \cdot, \cdot \rangle$ we denote the duality between E and E^*. By $L^s(I;E)$ we denote the Bochner space (see [1]). Therefore, $L^s(I;E)^* = L^{s'}(I;E^*)$.

2.2. **Structure and data properties.** We start by introducing our assumptions on functions in (1.1)–(1.6).

(i) $b \in C^1(\mathbb{R})$, $0 < b'(\xi) < b_*$ and

$$0 < b(\xi) \leq b_2 < +\infty \quad \forall \xi \in \mathbb{R} \quad (b_2, b_* = \text{const}).$$

(ii) a, $D_w \in C(\mathbb{R})$ and $\lambda \in C(\mathbb{R}^2)$ such that

$$0 < a(\xi), \quad 0 < D_w(\xi) \quad \forall \xi \in \mathbb{R},$$
$$0 < \lambda(\xi, \zeta) \quad \forall \xi, \zeta \in \mathbb{R}.$$

(iii) (Initial data) Assume $u_0, w_0, \theta_0 \in L^\infty(\Omega)$, such that

$$-\infty < u_1 < u_0 < 0 \qquad \text{a.e. in } \Omega \quad (u_1 = \text{const}). \tag{2.1}$$

2.3. **Auxiliary results.**

Remark 2.1 ([2, Section 1.1])**.** Let us note that (i) implies that there is a (strictly) convex C^1-function $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(0) = 0$, $\Phi'(0) = 0$, such that $b(z) - b(0) = \Phi'(z)$ for all $z \in \mathbb{R}$. Introduce the Legendre transform

$$B(z) := \int_0^1 (b(z) - b(sz))z \,\mathrm{d}s = \int_0^z (b(z) - b(s)) \,\mathrm{d}s.$$

Let us present some properties of B [2]:

$$\begin{gathered} B(z) := \int_0^1 (b(z) - b(sz))z \,\mathrm{d}s \geq 0 \quad \forall z \in \mathbb{R}, \\ B(s) - B(r) \geq (b(s) - b(r))r \quad \forall r, s \in \mathbb{R}, \\ b(z)z - \Phi(z) + \Phi(0) = B(z) \leq b(z)z \quad \forall z \in \mathbb{R}. \end{gathered}$$

3. Main result

The aim of this paper is to prove the existence of a weak solution to problem (1.1)–(1.6). First we formulate our problem in a variational sense.

Definition 3.1. A weak solution of (1.1)–(1.6) is a triplet $[u, w, \theta]$ such that

$$\begin{gathered} u \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)), \quad w \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap L^\infty(Q_T), \\ \theta \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap L^\infty(Q_T), \end{gathered}$$

which satisfies

$$-\int_{Q_T} b(u)\partial_t\phi \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} a(\theta)\nabla u \cdot \nabla\phi \,\mathrm{d}x\mathrm{d}t = \int_\Omega b(u_0)\phi(\mathbf{x}, 0) \,\mathrm{d}x \tag{3.1}$$

for any $\phi \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap W^{1,1}(I; L^\infty(\Omega))$ with $\phi(\cdot, T) = 0$;

$$\begin{aligned} &-\int_{Q_T} b(u)w\partial_t\eta \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} b(u)D_w(u)\nabla w \cdot \nabla\eta \,\mathrm{d}x\mathrm{d}t \\ &+ \int_{Q_T} wa(\theta)\nabla u \cdot \nabla\eta \,\mathrm{d}x\mathrm{d}t \\ &= \int_\Omega b(u_0)w_0\eta(\mathbf{x}, 0) \,\mathrm{d}x \end{aligned} \tag{3.2}$$

for any $\eta \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap W^{1,1}(I; L^\infty(\Omega))$ with $\eta(\cdot, T) = 0$;

$$\begin{aligned} &-\int_{Q_T} [b(u)\theta + \varrho\theta]\partial_t\psi \,\mathrm{d}x\mathrm{d}t + \int_{Q_T} \lambda(\theta, u)\nabla\theta \cdot \nabla\psi \,\mathrm{d}x\mathrm{d}t \\ &+ \int_{Q_T} \theta a(\theta)\nabla u \cdot \nabla\psi \,\mathrm{d}x\mathrm{d}t \\ &= \int_\Omega [b(u_0)\theta_0 + \varrho\theta_0]\psi(\mathbf{x}, 0) \,\mathrm{d}x \end{aligned} \tag{3.3}$$

for any $\psi \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap W^{1,1}(I; L^\infty(\Omega))$ with $\psi(\cdot, T) = 0$.

The main result of this paper reads as follows.

Theorem 3.2. *Let assumptions* (i)–(iii) *be satisfied. Then there exists at least one weak solution of the system* (1.1)–(1.6).

To prove the main result of the paper we use the method of semidiscretization in time by constructing temporal approximations and limiting procedure. The proof can be divided into three steps. In the first step we approximate our problem by means of a semi-implicit time discretization scheme (which preserve the pseudo-monotone structure of the discrete problem) and prove the existence and $W^{1,s}(\Omega)$-regularity (with some $s > 2$) of temporal approximations. In the second step we construct piecewise constant time interpolants and derive suitable a priori estimates. The key point is to establish L^∞-estimates to overcome degeneracies in transport coefficients. Finally, in the third step we pass to the limit from discrete approximations.

4. Proof of the main result

4.1. **Approximations.** Let us fix $p \in \mathbb{N}$ and set $\tau := T/p$ (a time step). Further, let us consider $u_p^0 := u_0$, $w_p^0 := w_0$, $\theta_p^0 := \theta_0$ a.e. on Ω. We approximate our evolution problem by a semi-implicit time discretization scheme. Then we define, in each time step $n = 1, \dots, p$, a triplet $[u_p^n, w_p^n, \theta_p^n]$ as a solution of the following recurrence steady problem.

For a given triplet $[u_p^{n-1}, w_p^{n-1}, \theta_p^{n-1}]$, $n = 1, \dots, p$, $u_p^{n-1} \in L^\infty(\Omega)$, $w_p^{n-1} \in L^\infty(\Omega)$, $\theta_p^{n-1} \in L^\infty(\Omega)$, find $[u_p^n, w_p^n, \theta_p^n]$, such that $u_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$, $w_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$, $\theta_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$ with some $s > 2$ and

$$\int_\Omega \frac{b(u_p^n) - b(u_p^{n-1})}{\tau} \phi \,\mathrm{d}x + \int_\Omega a(\theta_p^{n-1}) \nabla u_p^n \cdot \nabla \phi \,\mathrm{d}x = 0 \tag{4.1}$$

for any $\phi \in W^{1,2}_{\Gamma_D}(\Omega)$;

$$\begin{aligned} &\int_\Omega \frac{b(u_p^n) w_p^n - b(u_p^{n-1}) w_p^{n-1}}{\tau} \eta \,\mathrm{d}x \\ &+ \int_\Omega b(u_p^{n-1}) D_w(u_p^{n-1}) \nabla w_p^n \cdot \nabla \eta \,\mathrm{d}x + \int_\Omega w_p^n a(\theta_p^{n-1}) \nabla u_p^n \cdot \nabla \eta \,\mathrm{d}x = 0 \end{aligned} \tag{4.2}$$

for any $\eta \in W^{1,2}_{\Gamma_D}(\Omega)$;

$$\begin{aligned} &\int_\Omega \frac{b(u_p^n) \theta_p^n - b(u_p^{n-1}) \theta_p^{n-1}}{\tau} \psi \,\mathrm{d}x + \int_\Omega \varrho \frac{\theta_p^n - \theta_p^{n-1}}{\tau} \psi \,\mathrm{d}x \\ &+ \int_\Omega \lambda(\theta_p^{n-1}, u_p^{n-1}) \nabla \theta_p^n \cdot \nabla \psi \mathrm{d}\Omega + \int_\Omega \theta_p^n a(\theta_p^{n-1}) \nabla u_p^n \cdot \nabla \psi \mathrm{d}\Omega = 0 \end{aligned} \tag{4.3}$$

for any $\psi \in W^{1,2}_{\Gamma_D}(\Omega)$.

Next we show the existence of the solution to (4.1)–(4.3).

Theorem 4.1. *Let $u_p^{n-1} \in L^\infty(\Omega)$, $w_p^{n-1} \in L^\infty(\Omega)$, $\theta_p^{n-1} \in L^\infty(\Omega)$ be given and the assumptions* (i)–(iii) *be satisfied. Then there exists $[u_p^n, w_p^n, \theta_p^n]$, such that $u_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$, $w_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$ and $\theta_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$ with some $s > 2$ satisfying* (4.1)–(4.3).

Proof. The proof rests on the $W^{1,p}$-regularity of elliptic problems presented in [9, 11] and the embedding $W^{1,s}_{\Gamma_D}(\Omega) \subset L^\infty(\Omega)$ if $s > 2$ (recall that Ω is a bounded domain in $\mathbb{R}^2$).

The existence of $u_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$ with some $s > 2$ and $\theta_p^n \in W^{1,2}_{\Gamma_D}(\Omega)$, solutions to problems (4.1) and (4.3), respectively, is proven in [6]. The existence of $w_p^n \in W^{1,2}_{\Gamma_D}(\Omega)$, the solution to (4.2), can be handled in the same way.

Now, with $w_p^n \in W^{1,2}_{\Gamma_D}(\Omega)$ in hand, rewrite the equation (4.2) in the form (transferring the lower-order terms to the right hand side)

$$\begin{aligned}&\int_\Omega b(u_p^{n-1})D_w(u_p^{n-1})\nabla w_p^n\cdot\nabla\eta\,\mathrm{d}x\\&= -\int_\Omega \frac{b(u_p^n)w_p^n - b(u_p^{n-1})w_p^{n-1}}{\tau}\eta\,\mathrm{d}x - \int_\Omega w_p^n a(\theta_p^{n-1})\nabla u_p^n\cdot\nabla\eta\,\mathrm{d}x.\end{aligned}$$

Since $u_p^{n-1} \in L^\infty(\Omega)$, $u_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$ with some $s > 2$, $w_p^{n-1} \in L^\infty(\Omega)$, $\theta_p^{n-1} \in L^\infty(\Omega)$, both integrals on the right hand side make sense for any $\eta \in W^{1,r'}_{\Gamma_D}(\Omega)$, $r' = r/(r-1)$ with some $r > 2$. Now we are able to apply [9, Theorem 4] to obtain $w_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$ with some $s > 2$. Analysis similar to the above implies that $\theta_p^n \in W^{1,s}_{\Gamma_D}(\Omega)$ with some $s > 2$. □

4.2. **A priori estimates.** In this part we prove some uniform estimates (with respect to p) for the time interpolants of the solution. In the following estimates, many different constants will appear. For simplicity of notation, C represents generic constants which may change their numerical value from one formula to another but do not depend on p and the functions under consideration.

4.2.1. *Construction of temporal interpolants.* With the sequences u_p^n, w_p^n, θ_p^n constructed in Section 4.1, we define the piecewise constant interpolants $\bar\phi_p(t) = \phi_p^n$ for $t \in ((n-1)\tau, n\tau]$ and, in addition, we extend $\bar\phi_p$ for $t \le 0$ by $\bar\phi_p(t) = \phi_0$ for $t \in (-\tau, 0]$. For a function φ we often use the simplified notation $\varphi := \varphi(t)$, $\varphi_\tau(t) := \varphi(t-\tau)$, $\partial_t^{-\tau}\varphi(t) := \frac{\varphi(t)-\varphi(t-\tau)}{\tau}$, $\partial_t^{\tau}\varphi(t) := \frac{\varphi(t+\tau)-\varphi(t)}{\tau}$. Then, following (4.1)–(4.3), the piecewise constant time interpolants $\bar u_p \in L^\infty(I; W^{1,s}_{\Gamma_D}(\Omega))$, $\bar w_p \in L^\infty(I; W^{1,s}_{\Gamma_D}(\Omega))$ and $\bar\theta_p \in L^\infty(I; W^{1,s}_{\Gamma_D}(\Omega))$ (with some $s > 2$) satisfy the equations

$$\int_\Omega \partial_t^{-\tau} b(\bar u_p(t))\phi\,\mathrm{d}x + \int_\Omega a(\bar\theta_p(t-\tau))\nabla\bar u_p(t)\cdot\nabla\phi\,\mathrm{d}x = 0 \tag{4.4}$$

for any $\phi \in W^{1,2}_{\Gamma_D}(\Omega)$,

$$\begin{aligned}&\int_\Omega \partial_t^{-\tau}[b(\bar u_p(t))\bar w_p(t)]\eta\,\mathrm{d}x + \int_\Omega b(\bar u_p(t-\tau))D_w(\bar u_p(t-\tau))\nabla\bar w_p(t)\cdot\nabla\eta\,\mathrm{d}x\\&+ \int_\Omega \bar w_p(t)a(\bar\theta_p(t-\tau))\nabla\bar u_p(t)\cdot\nabla\eta\,\mathrm{d}x = 0\end{aligned}\tag{4.5}$$

for any $\eta \in W^{1,2}_{\Gamma_D}(\Omega)$ and

$$\begin{aligned}&\int_\Omega \partial_t^{-\tau}\left[b(\bar u_p(t))\bar\theta_p(t) + \varrho\bar\theta_p(t)\right]\psi\,\mathrm{d}x\\&+ \int_\Omega \lambda(\bar\theta_p(t-\tau), \bar u_p(t-\tau))\nabla\bar\theta_p(t)\cdot\nabla\psi\,\mathrm{d}x\\&+ \int_\Omega \bar\theta_p(t)a(\bar\theta_p(t-\tau))\nabla\bar u_p(t)\cdot\nabla\psi\,\mathrm{d}x = 0\end{aligned}\tag{4.6}$$

for any $\psi \in W^{1,2}_{\Gamma_D}(\Omega)$.

4.2.2. *L^∞-bound for $\bar{u}_p$, $\bar{w}_p$ and $\bar{\theta}_p$.* First we prove the L^∞-estimate for $\bar{u}_p$. Let us set

$$\phi := [b(\bar{u}_p) - b(u_1)]_- = \begin{cases} b(\bar{u}_p) - b(u_1), & \bar{u}_p < u_1, \\ 0, & \bar{u}_p \geq u_1, \end{cases} \tag{4.7}$$

as a test function in (4.4). Note that ϕ vanishes on Γ_D. It is a simple matter to derive

$$\frac{1}{2}\int_\Omega [b(\bar{u}_p(t)) - b(u_1)]_-^2 \,\mathrm{d}x + \int_{Q_t} a(\bar{\theta}_p(s-\tau)) b'(\bar{u}_p(s)) |\nabla \bar{u}_p(s)|^2 \chi_{\{\bar{u}_p < u_1\}} \mathrm{d}x\mathrm{d}s \leq 0$$

for almost every $t \in I$. Hence we conclude that the set $\{x \in \Omega : \bar{u}_p(x,t) < u_1\}$ has a measure zero for almost every $t \in I$.

Now setting

$$\phi = [b(\bar{u}_p) - b(0)]_+ = \begin{cases} b(\bar{u}_p) - b(0), & \bar{u}_p > 0, \\ 0, & \bar{u}_p \leq 0, \end{cases} \tag{4.8}$$

we obtain, using similar arguments,

$$\frac{1}{2}\int_\Omega [b(\bar{u}_p) - b(0)]_+^2 \mathrm{d}x = 0 \quad \text{for almost every } t \in I.$$

Hence the set $\{x \in \Omega : \bar{u}_p(x,t) > 0\}$ has a measure zero for almost every $t \in I$. Finally, combining the previous arguments, we deduce

$$\|\bar{u}_p\|_{L^\infty(Q_T)} \leq C, \tag{4.9}$$

where C does not depend on p.

Now we prove a similar estimate for $\bar{w}_p$. Let ℓ be an odd integer. Using $\phi = [\ell/(\ell+1)](\bar{w}_p)^{\ell+1}$ as a test function in (4.4) and $\eta = (\bar{w}_p)^\ell$ in (4.5) and combining both equations we obtain

$$\begin{aligned} &\frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar{u}_p(s))[\bar{w}_p(s)]^{\ell+1}\,\mathrm{d}x \\ &- \frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar{u}_p(s-\tau))[\bar{w}_p(s-\tau)]^{\ell+1}\,\mathrm{d}x \\ &+ \frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar{u}_p(s-\tau))[\bar{w}_p(s-\tau)]^{\ell+1}\,\mathrm{d}x \\ &+ \frac{1}{\tau}\frac{\ell}{\ell+1}\int_\Omega b(\bar{u}_p(s-\tau))[\bar{w}_p(s)]^{\ell+1}\,\mathrm{d}x \\ &- \frac{1}{\tau}\int_\Omega b(\bar{u}_p(s-\tau))\bar{w}_p(s-\tau)[\bar{w}_p(s)]^{\ell}\,\mathrm{d}x \\ &+ \int_\Omega \ell[\bar{w}_p(s)]^{\ell-1} b(\bar{u}_p(s-\tau)) D_w(\bar{u}_p(s-\tau))\nabla\bar{w}_p(s)\cdot\nabla\bar{w}_p(s)\,\mathrm{d}x = 0. \end{aligned} \tag{4.10}$$

Applying the Young's inequality we can write for the term in the third line

$$\begin{aligned} &\frac{1}{\tau}\int_\Omega b(\bar u_p(s-\tau))\bar w_p(s-\tau)[\bar w_p(s)]^\ell \,\mathrm{d}x \\ &\le \frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar u_p(s-\tau))[\bar w_p(s-\tau)]^{\ell+1}\,\mathrm{d}x \\ &+ \frac{1}{\tau}\frac{\ell}{\ell+1}\int_\Omega b(\bar u_p(s-\tau))[\bar w_p(s)]^{\ell+1}\,\mathrm{d}x. \end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11) we deduce

$$\begin{aligned} &\frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar u_p(s))[\bar w_p(s)]^{\ell+1}\,\mathrm{d}x \\ &- \frac{1}{\tau}\frac{1}{\ell+1}\int_\Omega b(\bar u_p(s-\tau))[\bar w_p(s-\tau)]^{\ell+1}\,\mathrm{d}x \\ &+ \int_\Omega \ell[\bar w_p(s)]^{\ell-1} b(\bar u_p(s-\tau))D_w(\bar u_p(s-\tau))\nabla\bar w_p(s)\cdot\nabla\bar w_p(s)\,\mathrm{d}x \le 0. \end{aligned} \tag{4.12}$$

Now, integrating (4.12) over s from 0 to t we obtain

$$\begin{aligned} &\int_\Omega (\bar w_p(t))^{\ell+1} b(\bar u_p(t))\mathrm{d}x \\ &+ \int_{\Omega_t} (\ell+1)\ell[\bar w_p(s)]^{\ell-1} b(\bar u_p(s-\tau))D_w(\bar u_p(s-\tau))|\nabla\bar w_p(s)|^2\mathrm{d}x\mathrm{d}s \\ &\le \int_\Omega (w_0)^{\ell+1} b\,(u_0)\,\mathrm{d}x. \end{aligned} \tag{4.13}$$

Note that the second integral in (4.13) is nonnegative (ℓ is supposed to be the odd integer). Moreover, from (4.13) and (4.9) it follows that

$$\|\bar w_p\|_{L^\infty(0,T;L^{\ell+1}(\Omega))} \le C, \tag{4.14}$$

where the constant C is independent of ℓ and p. Now, let $\ell \to +\infty$ in (4.14), we obtain

$$\|\bar w_p\|_{L^\infty(Q_T)} \le C. \tag{4.15}$$

In the same manner we arrive at the estimate for $\bar\theta_p$, i.e.

$$\|\bar\theta_p\|_{L^\infty(Q_T)} \le C. \tag{4.16}$$

4.2.3. *Energy estimates for $\bar u_p$, $\bar w_p$ and $\bar\theta_p$.* We test (4.4) with $\phi = \bar u_p(t)$ and integrate (4.4) over t from 0 to s. For the parabolic term we can write

$$\int_0^s\int_\Omega \partial_t^{-\tau} b(\bar u_p(t))\bar u_p(t)\,\mathrm{d}x\mathrm{d}t \ge \frac{1}{\tau}\int_{s-\tau}^s\int_\Omega B(\bar u_p(t)) - B(u_0)\,\mathrm{d}x\mathrm{d}t. \tag{4.17}$$

Further, using (4.9) and (4.17), applying the usual estimates for the elliptic part (see also [2]), we obtain the a priori estimate

$$\sup_{0\le t\le T}\int_\Omega B(\bar u_p(t))\mathrm{d}x + \int_0^T\int_\Omega |\nabla\bar u_p(t)|^2\mathrm{d}x\mathrm{d}t \le C. \tag{4.18}$$

Now it follows that there exists a function $u \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega))$ such that, along a selected subsequence (letting $p \to \infty$), we have $\bar u_p(t) \rightharpoonup u$ weakly in $L^2(I; W^{1,2}_{\Gamma_D}(\Omega))$.

Now we prove similar result for $\bar{w}_p(t)$. Using $\eta(t) = 2\bar{w}_p(t)$ as a test function in (4.5) we obtain

$$\begin{aligned}
&\int_\Omega \partial_t^{-\tau} b(\bar{u}_p(t)) 2\bar{w}_p(t)^2 \,\mathrm{d}x + \int_\Omega \partial_t^{-\tau} \bar{w}_p(t) 2\bar{w}_p(t) b(\bar{u}_p(t-\tau)) \,\mathrm{d}x \\
&+ 2\int_\Omega b(\bar{u}_p(t-\tau)) D_w(\bar{u}_p(t-\tau)) \nabla \bar{w}_p(t) \cdot \nabla \bar{w}_p(t) \,\mathrm{d}x \\
&+ \int_\Omega a(\bar{\theta}_p(t-\tau)) \nabla \bar{u}_p(t) \cdot 2\bar{w}_p(t) \nabla \bar{w}_p(t) \,\mathrm{d}x = 0.
\end{aligned} \tag{4.19}$$

One is allowed to use $\phi(t) = \bar{w}_p(t)^2$ as a test function in (4.4) to obtain

$$\int_\Omega [\partial_t^{-\tau} b(\bar{u}_p(t))] \bar{w}_p(t)^2 \,\mathrm{d}x + \int_\Omega a(\bar{\theta}_p(t-\tau)) \nabla \bar{u}_p(t) \cdot \nabla \bar{w}_p(t)^2 \,\mathrm{d}x = 0. \tag{4.20}$$

Combining (4.19) and (4.20) we deduce

$$\begin{aligned}
&\int_\Omega \partial_t^{-\tau} \left[\bar{w}_p(t)^2 b(\bar{u}_p(t)) \right] \,\mathrm{d}x + \int_\Omega \frac{1}{\tau} \left[\bar{w}_p(t) - \bar{w}_p(t-\tau) \right]^2 b(\bar{u}_p(t-\tau)) \,\mathrm{d}x \\
&+ 2\int_\Omega b(\bar{u}_p(t-\tau)) D_w(\bar{u}_p(t-\tau)) \nabla \bar{w}_p(t) \cdot \nabla \bar{w}_p(t) \,\mathrm{d}x = 0.
\end{aligned} \tag{4.21}$$

In view of (4.9) we have

$$b(\bar{u}_p(t)),\ b(\bar{u}_p(t-\tau)),\ D_w(\bar{u}_p(t-\tau)) > C \qquad \text{in } \Omega \times (-\tau, T). \tag{4.22}$$

Recall that C does not depend on p. Now, integrating (4.21) with respect to time t we obtain

$$\sup_{0 \le t \le T} \int_\Omega |\bar{w}_p(t)|^2 \mathrm{d}\Omega + \int_0^T \|\bar{w}_p(t)\|^2_{W^{1,2}_{\Gamma_D}(\Omega)} \mathrm{d}\Omega \le C.$$

From this we can write

$$\|\bar{w}_p\|_{L^2(I; W^{1,2}_{\Gamma_D}(\Omega))} \le C. \tag{4.23}$$

Similarly, we use $\psi(t) = 2\bar{\theta}_p(t)$ as a test function in (4.6) to obtain

$$\begin{aligned}
&\int_\Omega \partial_t^{-\tau} b(\bar{u}_p(t)) 2\bar{\theta}_p(t)^2 \,\mathrm{d}x + \int_\Omega \partial_t^{-\tau} \bar{\theta}_p(t) 2\bar{\theta}_p(t) b(\bar{u}_p(t-\tau)) \,\mathrm{d}x \\
&+ 2\int_\Omega \lambda(\bar{\theta}_p(t-\tau), \bar{u}_p(t-\tau)) \nabla \bar{\theta}_p(t) \cdot \nabla \bar{\theta}_p(t) \,\mathrm{d}x \\
&+ \int_\Omega a(\bar{\theta}_p(t-\tau)) \nabla \bar{u}_p(t) \cdot 2\bar{\theta}_p(t) \nabla \bar{\theta}_p(t) \,\mathrm{d}x \le 0.
\end{aligned} \tag{4.24}$$

Using $\phi(t) = \bar{\theta}_p(t)^2$ as a test function in (4.4) we obtain

$$\int_\Omega \partial_t^{-\tau} b(\bar{u}_p(t)) \bar{\theta}_p(t)^2 \,\mathrm{d}x + \int_\Omega a(\bar{\theta}_p(t-\tau)) \nabla \bar{u}_p(t) \cdot \nabla \bar{\theta}_p(t)^2 \,\mathrm{d}x = 0. \tag{4.25}$$

Combining (4.24) and (4.25) we deduce

$$\begin{aligned}
&\int_\Omega \partial_t^{-\tau} \left[\left(\bar{\theta}_p(t)\right)^2 b(\bar{u}_p(t)) \right] \,\mathrm{d}x + \int_\Omega \frac{1}{\tau} \left[\bar{\theta}_p(t) - \bar{\theta}_p(t-\tau) \right]^2 b(\bar{u}_p(t-\tau)) \,\mathrm{d}x \\
&+ 2\int_\Omega \lambda(\bar{\theta}_p(t-\tau), \bar{u}_p(t-\tau)) \nabla \bar{\theta}_p(t) \cdot \nabla \bar{\theta}_p(t) \,\mathrm{d}x \le 0.
\end{aligned} \tag{4.26}$$

Integrating (4.26) with respect to time t we obtain the a priori estimate (using (4.9) and (4.16))

$$\sup_{0\le t\le T}\int_\Omega |\bar\theta_p(t)|^2 \mathrm{d}x + \int_0^T \|\bar\theta_p(t)\|^2_{W^{1,2}_{\Gamma_D}(\Omega)} \mathrm{d}t \le C. \tag{4.27}$$

From this we have

$$\|\bar\theta_p\|_{L^2(I;W^{1,2}_{\Gamma_D}(\Omega))} \le C. \tag{4.28}$$

4.2.4. *Further estimates.* To show that $\bar u_p$ converges to u almost everywhere on Q_T we follow [2]. Let $k\in\mathbb{N}$ and use

$$\phi(t) = \partial_t^{k\tau}\bar u_p(s)$$

for $j\tau \le t\le (j+k)\tau$ with $(j-1)\tau\le s\le j\tau$ and $1\le j\le \frac{T}{\tau}-k$, as a test function in (4.4). For the parabolic term, we can write

$$\begin{aligned}
&\int_{j\tau}^{(j+k)\tau}\int_\Omega \partial_t^{-\tau} b(\bar u_p(t))\,\partial_t^{k\tau}\bar u_p(t)\,\mathrm{d}x\mathrm{d}t\\
&= \frac{1}{k\tau^2}\int_{(j-1)\tau}^{j\tau}\int_\Omega \left(b(\bar u_p(t+k\tau)) - b(\bar u_p(t))\right)\left(\bar u_p(t+k\tau)-\bar u_p(t)\right)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Hence, summing over $j=1,\dots,p-k$ we obtain the estimate

$$\begin{aligned}
&\sum_{j=1}^{p-k}\int_{j\tau}^{(j+k)\tau}\int_\Omega \partial_t^{-\tau} b(\bar u_p(t))\partial_t^{k\tau}\bar u_p(t)\,\mathrm{d}x\mathrm{d}t\\
&\ge \frac{1}{k\tau^2}\int_0^{T-k\tau}\int_\Omega \left(b(\bar u_p(t+k\tau)) - b(\bar u_p(t))\right)\left(\bar u_p(t+k\tau)-\bar u_p(t)\right)\,\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.29}$$

Similarly, for the elliptic term, after a little lengthy but straightforward computation we obtain

$$\begin{aligned}
&\sum_{j=1}^{p-k}\int_{j\tau}^{(j+k)\tau}\int_\Omega a(\bar\theta_p(t-\tau))\nabla\bar u_p\cdot\nabla\partial_t^{k\tau}\bar u_p\,\mathrm{d}x\mathrm{d}t\\
&= \sum_{\ell=1}^{k}\sum_{j=1}^{p-k}\int_{(j+\ell-1)\tau}^{(j+\ell)\tau}\int_\Omega \left(a(\bar\theta_p(t-\tau))\nabla\bar u_p\right)\cdot\nabla\partial_t^{k\tau}\bar u_p\mathrm{d}x\mathrm{d}t\\
&= \sum_{\ell=1}^{k}\int_{\ell\tau}^{T-k\tau+\ell\tau}\int_\Omega a(\bar\theta_p(t-\tau))\nabla\bar u_p(t)\cdot\nabla\partial_t^{k\tau}\bar u_p(t-\ell\tau)\,\mathrm{d}x\mathrm{d}t\\
&\le \frac{c_1}{\tau}\int_{Q_T}|a(\bar\theta_p(t-\tau))\nabla\bar u_p|^2\,\mathrm{d}x\mathrm{d}t + \frac{c_2}{\tau}\int_{Q_T}|\nabla\bar u_p|^2\,\mathrm{d}x\mathrm{d}t\\
&\le \frac{C}{\tau}.
\end{aligned} \tag{4.30}$$

Combining (4.29)–(4.30) and using (4.18) we obtain

$$\int_0^{T-k\tau}\left(b(\bar u_p(s+k\tau)) - b(\bar u_p(s))\right)\left(\bar u_p(s+k\tau)-\bar u_p(s)\right)\mathrm{d}s \le Ck\tau. \tag{4.31}$$

Using the compactness argument one can show in the same way as in [2, Lemma 1.9] and [8, Eqs. (2.10)–(2.12)]

$$b(\bar u_p)\to b(u)\quad\text{in } L^1(Q_T) \tag{4.32}$$

and almost everywhere on Q_T. Since b is strictly monotone, it follows from (4.32) that [13, Proposition 3.35]

$$\bar{u}_p \to u \quad \text{almost everywhere on } Q_T. \tag{4.33}$$

Further, in much the same way as in (4.31), we arrive at

$$\int_0^{T-k\tau} |b(\bar{u}_p(s+k\tau))\bar{w}_p(s+k\tau) - b(\bar{u}_p(s))\bar{w}_p(s)|^2 \mathrm{d}s \le Ck\tau. \tag{4.34}$$

From this we conclude, using (4.15), that

$$\int_0^{T-k\tau} |\bar{w}_p(s+k\tau) - \bar{w}_p(s)|^2 \mathrm{d}s \le Ck\tau. \tag{4.35}$$

Finally, in a similar way, using (4.16), we arrive at

$$\int_0^{T-k\tau} |\bar{\theta}_p(s+k\tau) - \bar{\theta}_p(s)|^2 \mathrm{d}s \le Ck\tau. \tag{4.36}$$

4.3. **Passage to the limit.** The a priori estimates (4.15), (4.16), (4.18), (4.23), (4.28), (4.31), (4.35), (4.36) allow us to conclude that there exist $u \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega))$, $w \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap L^\infty(Q_T)$ and $\theta \in L^2(I; W^{1,2}_{\Gamma_D}(\Omega)) \cap L^\infty(Q_T)$ such that, letting $p \to +\infty$ (along a selected subsequence),

$$\begin{gathered}
\bar{u}_p \rightharpoonup u \quad \text{weakly in } L^2(I; W^{1,2}_{\Gamma_D}(\Omega)),\\
\bar{u}_p \to u \quad \text{almost everywhere on } Q_T,\\
\bar{w}_p \rightharpoonup w \quad \text{weakly in } L^2(I; W^{1,2}_{\Gamma_D}(\Omega)),\\
\bar{w}_p \rightharpoonup w \quad \text{weakly star in } L^\infty(Q_T),\\
\bar{w}_p \to w \quad \text{almost everywhere on } Q_T,\\
\bar{\theta}_p \rightharpoonup \theta \quad \text{weakly in } L^2(I; W^{1,2}_{\Gamma_D}(\Omega)),\\
\bar{\theta}_p \rightharpoonup \theta \quad \text{weakly star in } L^\infty(Q_T),\\
\bar{\theta}_p \to \theta \quad \text{almost everywhere on } Q_T.
\end{gathered}$$

The above established convergences are sufficient for taking the limit $p \to \infty$ in (4.4)–(4.6) (along a selected subsequence) to get the weak solution of the system (1.1)–(1.6) in the sense of Definition 3.1. This completes the proof of the main result stated in Theorem 3.2.

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