Tenth MSU Conference on Differential Equations and Computational Simulations.
Electronic Journal of Differential Equations, Conference 23 (2016), pp. 155–173.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

UNIQUENESS OF POSITIVE SOLUTIONS FOR COOPERATIVE HAMILTONIAN ELLIPTIC SYSTEMS

JUNPING SHI, RATNASINGHAM SHIVAJI

Abstract. The uniqueness of positive solution of a semilinear cooperative Hamiltonian elliptic system with two equations is proved for the case of sub-linear and superlinear nonlinearities. Implicit function theorem, bifurcation theory, and ordinary differential equation techniques are used in the proof.

1. Introduction

We consider positive solutions of semilinear elliptic system

$$\begin{aligned} \Delta u + \lambda f(v) = 0, \quad x \in \Omega, \\ \Delta v + \lambda g(u) = 0, \quad x \in \Omega, \\ u(x) = v(x) = 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where $\lambda > 0$, Ω is a bounded smooth domain, and throughout the paper we always assume that f, g are continuously differentiable functions defined on $\mathbb{R}^+ := [0, \infty)$. System (1.1) is called cooperative, if f and g satisfy

$$f'(v) > 0, \text{ for } v > 0, \quad \text{and} \quad g'(u) > 0, \text{ for } u > 0. \tag{1.2}$$

For our main results, we also assume that

$$f(0) \geq 0 \quad \text{and} \quad g(0) \geq 0. \tag{1.3}$$

Hence $f(v)$ and $g(u)$ are both positone, *i.e.* positive and monotone. The word "positone"was invented by Keller and Cohen [12] in a now classical paper.

In various situations, we will obtain the uniqueness of positive solutions of (1.1) in this paper. This is motivated by the extensive study of exact multiplicity (and uniqueness) of positive solutions of the scalar semilinear elliptic equation

$$\begin{aligned} \Delta u + \lambda f(u) = 0, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.4}$$

starting from Korman, Li and Ouyang [15, 16], Ouyang and Shi [21, 22], and also the recent results on the existence and uniqueness of positive solution of semilinear

2010 *Mathematics Subject Classification.* 35J66, 35B32.
Key words and phrases. Uniqueness; semilinear elliptic equations.

Published March 21, 2016.

elliptic systems. In general, the exact number of solutions to semilinear elliptic system

$$\begin{aligned} \Delta u + \lambda f(u,v) = 0, \quad x \in \Omega, \\ \Delta v + \lambda g(u,v) = 0, \quad x \in \Omega, \\ u(x) = v(x) = 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.5}$$

is more difficult to determine. However when $f(u,v) \equiv f(v)$ and $g(u,v) \equiv g(u)$, some nice properties of scalar equations are still valid, hence some parts of the approach given in [21, 22] can be extended to obtain exact multiplicity for systems. Note that system (1.1) has been considered previously by many people, since it possesses a variational structure and in some literature it is called a Hamiltonian elliptic system (see de Figueiredo [9]).

The positone condition on f, g alone does not imply the exact multiplicity/ uniqueness of positive solutions. Even for the scalar case, a positone problem could have a unique solution or multiple solutions, see for example Shivaji et.al. [3, 27, 28] in the case of a combustion model. It has been shown that the shape of the graph of $f(u)/u$ is an important factor on the global bifurcation diagram of (1.4), see Lions [18] and Ouyang and Shi [22]. To state our results, we recall the following definitions from [22]:

Definition 1.1. Let $f \in C^1(\mathbb{R}^+)$. f is said to be *sublinear* (resp. *superlinear*) if $f(u)/u$ is strictly decreasing (resp. $f(u)/u$ is strictly increasing) for $u > 0$.

Clearly f is sublinear (or superlinear) if $f(u)/u \geq$ (or $\leq$) $f'(u)$. Note that these definitions emphasize the global convexity properties of the nonlinear function $f(u)$. In literature, the phrases sublinear and superlinear have also been defined differently, mostly on the asymptotical behavior of the function $f(u)/u$, see for example, Lions [18] for the scalar case and Sirakov [29] for the system case.

Assume that f and g are positone, *i.e.* f, g satisfy (1.2) and (1.3). Our main results can be summarized as follows:

(1) If f and g are sublinear, then for a general bounded domain Ω, (1.1) has at most one positive solution for any given $\lambda > 0$. (This result holds even without the positone condition.)
(2) If $n = 1$ (*i.e.* Ω is an interval), f and g are superlinear, then (1.1) has at most one positive solution for any given $\lambda > 0$.

Moreover in all these cases, the set of positive solutions (λ, u, v) of (1.1) consist of a smooth curve; the λ-component of the solution curve is monotone.

All these results resemble the corresponding uniqueness of positive solutions to the scalar equation (1.4), which can be found in [22]. Indeed, the sublinear case is known as early as [12], and the superlinear case also hold for n-dimensional ball domains if f, g satisfy some subcritical growth conditions (see [22]). Earlier work on the superlinear scalar equations can be found in [12], Crandall and Rabinowitz [7], and the review article of Amann [1].

While there are more exact multiplicity results for the solutions of scalar semilinear elliptic equations, there are not so many for the systems. Some known results are about the Lane-Emden system:

$$
\begin{gathered}
\Delta u+v^q=0, \quad x \in \Omega, \\
\Delta v+u^p=0, \quad x \in \Omega, \\
u(x)>0, \quad v(x)>0, \quad x \in \Omega, \\
u(x)=v(x)=0, \quad x \in \partial \Omega,
\end{gathered} \tag{1.6}
$$

where Ω is either $\mathbb{R}^n$ or B^n, the unit ball in $\mathbb{R}^n$. When (p, q) is supercritical or critical, *i.e.*

$$
p>0, \quad q>0, \text { and } \frac{1}{p+1}+\frac{1}{q+1} \leq \frac{n-2}{n}, \tag{1.7}
$$

then for each central value $u(0)>0$, there exists a $v(0)>0$ such that (1.6) has a radial ground state on $\mathbb{R}^n$ with the central value (maximum) $(u(0), v(0))$ (see Serrin and Zou [23]). On the other hand, for ball domain B^n, it is known that for any $p, q>0$ except $pq=1$, (1.6) has at most one solution (see Dalmasso [8], and Korman and Shi [17]), and the existence of a solution on a ball depends on whether (p, q) is above or below the curve $\frac{1}{p+1}+\frac{1}{q+1}=\frac{n-2}{n}$. If (1.7) holds, there is no solution of (1.6) on a ball, and if the opposite one holds, there is a unique solution. The proof of uniqueness for (1.6) heavily depends on the scaling property of (1.6), which does not hold for almost any other f, g. Our results hold for much more general nonlinearities which are defined by the monotonicity and convexity of the functions. But on the other hand, we require that both f and g satisfy such geometric properties such as sublinear and superlinear. It would be interesting to obtain results with certain combined conditions on f and g instead of individual ones. The result for $n=1$ and f, g superlinear has also been proved in Korman [13, 14]. But our approach is different.

The approach in this article includes several ingredients. First we apply abstract analytical bifurcation theory based on implicit function theorem used by Crandall and Rabinowitz [5, 6], Shi [25] and Liu, Shi and Wang [20]. Such applications were first used for scalar equation (1.4) in [15, 16, 21, 22]. In general the methods for scalar equations cannot be easily generalized to systems, but we are able to carry that over for the special system (1.1). Secondly we give a comprehensive study of the solutions to the linearized equations when f, g are positone and $n=1$, following some recent work by An, Chern and Shi [2]. This is vital for the result when f, g are superlinear and $n=1$. Finally the maximum principle for the cooperative system (satisfying (1.2)) also plays an essential role. It is used to proved the symmetry of the solutions of either nonlinear and the linearized equations. Moreover we show that in the case of $n=1$, the solution set of the cooperative system (5.1) is actually parameterized by one of $u(0)$ or $v(0)$ not the two-dimensional value $(u(0), v(0))$. This is similar to a result in [17], which holds for radially symmetric solutions on balls in $\mathbb{R}^n$. The study of the linearized equations also heavily relies on the maximum principle.

The organization of the remaining part of the paper is as follows: in Section 2, we recall some classical abstract bifurcation theorems, and we apply these abstract theorems to the equation (1.1) in general, and also derive some basic properties of the degenerate solutions; the stability property of the solution is considered in Section 3. In Section 4 we prove the result for sublinear case in general domains. Various properties for the $n=1$ case are studied in Section 5, and we prove the result for superlinear and $n=1$ case in Section 6.

2. Implicit function theorem and bifurcation theory

In this section, we recall some well-known abstract implicit function theorem and bifurcation theorems.

Theorem 2.1 (Implicit function theorem). *Let X, Y and Z be Banach spaces, and let $U \subset X \times Y$ be a neighborhood of (λ_0, u_0). Let $F : U \to Z$ be a continuously differentiable mapping. Suppose that $F(\lambda_0, u_0) = 0$ and $F_u(\lambda_0, u_0)$ is an isomorphism, i.e. $F_u(\lambda_0, u_0)$ is one-to-one and onto, and $F_u^{-1}(\lambda_0, u_0) : Z \to Y$ is a linear bounded operator. Then there exists a neighborhood A of λ_0 in X, and a neighborhood B of u_0 in Y, such that for any $\lambda \in A$, there exists a unique $u(\lambda) \in B$ satisfying $F(\lambda, u(\lambda)) = 0$. Moreover $u(\cdot) : A \to B$ is continuously differentiable, and $u'(\lambda_0) : X \to Y$ is defined as $u'(\lambda_0)[\psi] = -[F_u(\lambda_0, u_0)]^{-1} \circ F_\lambda(\lambda_0, u_0)[\psi]$.*

In the following theorem, we assume that X, Y are Banach spaces, $N(L)$, $R(L)$ are the null space and range space of a linear operator L respectively, $\langle \cdot, \cdot \rangle$ is the duality between Banach space Y and its dual space Y^*, and F_u, F_λ and F_{uu} etc. are the partial derivatives of the nonlinear operator F in u, λ and 2nd order derivative in u etc.

Theorem 2.2 (Transcritical and pitchfork bifurcation theorem). *Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and let $F : U \to Y$ be a twice continuously differentiable mapping. Assume that $F(\lambda, u_0) = 0$ for $(\lambda, u_0) \in U$. At (λ_0, u_0), F satisfies the following assumptions:*

- (A1) $\dim N(F_u(\lambda_0, u_0)) = \operatorname{codim} R(F_u(\lambda_0, u_0)) = 1$, *and* $N(F_u(\lambda_0, u_0)) = \operatorname{span}\{w_0\}$;
- (A2) $F_{\lambda u}(\lambda_0, u_0)[w_0] \notin R(F_u(\lambda_0, u_0))$.

Let Z be any complement of $\operatorname{span}\{w_0\}$ in X. Then the solution set of $F(\lambda, u) = 0$ near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $\{(\lambda(s), u(s)) : |s| < \epsilon\}$, where $s \mapsto (\lambda(s), u(s)) \in \mathbb{R} \times X$ is a continuously differentiable function, such that $u(s) = u_0 + sw_0 + sz(s)$, $\lambda(0) = \lambda_0$, $z(0) = 0$, $z(s) \in Z$ and

$$\lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0] \rangle}{2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0] \rangle}, \tag{2.1}$$

where $l \in Y^*$ satisfying $N(l) = R(F_u(\lambda_0, u_0))$.

The implicit function theorem is well-known and Theorem 2.2 appears in [5]. If $\lambda'(0) \neq 0$ in Theorem 2.2, then it is s transcritical bifurcation; if $\lambda'(0) = 0$ and F is C^3, then the solution curve is C^2 near the bifurcation point, and $\lambda''(0) \neq 0$, then a pitchfork bifurcation occurs.

To apply Theorems 2.1 and 2.2 to the semilinear system (1.1), we define

$$F(\lambda, u, v) = \begin{pmatrix} \Delta u + \lambda f(v) \\ \Delta v + \lambda g(u) \end{pmatrix}, \tag{2.2}$$

where $\lambda \in \mathbb{R}$ and $u, v \in C_0^{2,\alpha}(\overline{\Omega})$. Here we assume that f, g are at least C^1, then $F : \mathbb{R} \times X \to Y$ is continuously differentiable, where $X = [C_0^{2,\alpha}(\overline{\Omega})]^2$ and $Y = [C^\alpha(\overline{\Omega})]^2$. For weak solutions (u, v), one can also consider $X = [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]^2$ and $Y = [L^p(\Omega)]^2$ with $p > n$.

Suppose that $(\lambda, u(x), v(x))$ is a positive solution of (1.1), then this solution is degenerate if the linearized equation

$$\begin{gathered} \Delta\phi + \lambda f'(v(x))\psi = 0, \quad x \in \Omega, \\ \Delta\psi + \lambda g'(u(x))\phi = 0, \quad x \in \Omega, \\ \phi(x) = \psi(x) = 0, \quad x \in \partial\Omega. \end{gathered} \tag{2.3}$$

has a nontrivial solution (ϕ, ψ). To apply Theorem 2.2 to F defined in (2.2), we assume that (2.3) has a one-dimensional solution space spanned by (ϕ_0, ψ_0) at $(\lambda, u, v) = (\lambda_0, u_0, v_0) \in \mathbb{R} \times X$. Notice that here

$$F_{(u,v)}(\lambda, u, v) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \Delta\phi + \lambda f'(v)\psi \\ \Delta\psi + \lambda g'(u)\phi \end{pmatrix}, \tag{2.4}$$

hence $N(F_u(\lambda_0, u_0, v_0)) = \operatorname{span}\{(\phi_0, \psi_0)\}$ is one-dimensional. Suppose that (h_1, h_2) belongs to $R(F_u(\lambda_0, u_0, v_0))$, then there exists $(\phi, \psi) \in X$ such that

$$F_{(u,v)}(\lambda, u, v) \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}. \tag{2.5}$$

Hence

$$\begin{aligned} &\int_\Omega (h_1\psi_0 + h_2\phi_0) dx \\ &= \lambda \int_\Omega f'(v)\psi\psi_0 dx + \int_\Omega \Delta\phi\psi_0 dx + \lambda \int_\Omega g'(u)\phi\phi_0 dx + \int_\Omega \Delta\psi\phi_0 dx \\ &= \lambda \int_\Omega f'(v)\psi\psi_0 dx + \int_\Omega \phi\Delta\psi_0 dx + \lambda \int_\Omega g'(u)\phi\phi_0 dx + \int_\Omega \psi\Delta\phi_0 dx \\ &= \int_\Omega (\Delta\phi_0 + \lambda f'(v)\psi_0)\psi dx + \int_\Omega (\Delta\psi_0 + \lambda g'(u)\phi_0)\phi dx = 0. \end{aligned} \tag{2.6}$$

On the other hand, if $\int_\Omega (h_1\psi_0 + h_2\phi_0) dx = 0$, then $(h_1, h_2) \in R(F_u(\lambda_0, u_0, v_0))$ from Fredholm theory. Hence $\operatorname{codim} R(F_u(\lambda_0, u_0, v_0)) = 1$, and

$$R(F_u(\lambda_0, u_0, v_0)) = \big\{(h_1, h_2) \in Y : \int_\Omega (h_1\psi_0 + h_2\phi_0) dx = 0\big\}. \tag{2.7}$$

Therefore condition (A1) is satisfied.

To conclude this section, we prove a non-existence result regarding the positive solutions of (1.1). Here let (λ_1, φ_1) be the principal eigen-pair of

$$-\Delta\varphi = \lambda\varphi, \quad x \in \Omega, \quad \varphi(x) = 0, \quad x \in \partial\Omega, \tag{2.8}$$

such that $\varphi_1(x) > 0$ in Ω.

Proposition 2.3. *Suppose that a, b are positive constants, and (u, v) is a positive solution of* (1.1).

(1) *If $f(v) \le av$ for any $v \ge 0$ and $g(u) \le bu$ for any $u \ge 0$, then $\lambda \ge \frac{2\lambda_1}{a+b}$. In particular,* (1.1) *has no positive solution if $\lambda < \frac{2\lambda_1}{a+b}$.*
(2) *If $f(v) \ge av$ for any $v \ge 0$ and $g(u) \ge bu$ for any $u \ge 0$, then $\lambda \le \frac{\lambda_1}{\min\{a,b\}}$. In particular,* (1.1) *has no positive solution if $\lambda > \frac{\lambda_1}{\min\{a,b\}}$.*

Proof. First we assume that $f(v) \le av$ for any $v \ge 0$ and $g(u) \le bu$ for any $u \ge 0$. Multiplying the equation of u in (1.1) by u, multiplying the equation of v in (1.1) by v, integrating over Ω and adding the two equations, we obtain

$$\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx = \lambda \int_\Omega f(v) u dx + \int_\Omega g(u) v dx \le \lambda(a+b) \int_\Omega uv dx. \quad (2.9)$$

By using Cauchy-Schwarz inequality and Poincaré inequality, from (2.9), we have

$$\lambda_1 \Big(\int_\Omega u^2 dx + \int_\Omega v^2 dx \Big) \le \frac{\lambda(a+b)}{2} \Big(\int_\Omega u^2 dx + \int_\Omega v^2 dx \Big), \quad (2.10)$$

which implies $\lambda \ge \frac{2\lambda_1}{a+b}$.

Next we assume that $f(v) \ge av$ for any $v \ge 0$ and $g(u) \ge bu$ for any $u \ge 0$. Multiplying the equations of u and v in (1.1) by φ_1, integrating over Ω and adding the two equations, we obtain

$$\begin{aligned} \lambda_1 \Big(\int_\Omega u\varphi_1 dx + \int_\Omega v\varphi_1 dx \Big) &= \lambda \int_\Omega f(v)\varphi_1 dx + \lambda \int_\Omega g(u)\varphi_1 dx \\ &\ge \lambda a \int_\Omega v\varphi_1 dx + \lambda b \int_\Omega u\varphi_1 dx \\ &\ge \lambda \min\{a,b\} \Big(\int_\Omega u\varphi_1 dx + \int_\Omega v\varphi_1 dx \Big), \end{aligned} \quad (2.11)$$

which implies $\lambda \le \lambda_1 / \min\{a,b\}$. □

3. Stability and linearized equations

Let (u,v) be a solution of (1.1). The stability of (u,v) is determined by the linearized equation:

$$\begin{gathered} \Delta\xi + f'(v)\eta = -\mu\xi, \quad x \in \Omega, \\ \Delta\eta + g'(u)\xi = -\mu\eta, \quad x \in \Omega, \\ \xi(x) = \eta(x) = 0, \quad x \in \partial\Omega, \end{gathered} \quad (3.1)$$

which can be written as

$$L\mathbf{u} = H\mathbf{u} + \mu\mathbf{u}, \quad (3.2)$$

where

$$\mathbf{u} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad L\mathbf{u} = \begin{pmatrix} -\Delta\xi \\ -\Delta\eta \end{pmatrix}, \quad H = \begin{pmatrix} 0 & f'(v) \\ g'(u) & 0 \end{pmatrix}. \quad (3.3)$$

If we assume that (f,g) is cooperative (satisfying (1.2)), then the system (3.2) and (3.3) is a linear elliptic system of cooperative type, and the maximum principles hold for such systems. Here we recall some known results.

Lemma 3.1. *Suppose that L, H are given by* (3.3), $\mathbf{u} \in X \equiv [W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)]^2$, *and* (f,g) *satisfies* (1.2).

(1) $\mu_1 = \inf\{\mathrm{Re}(\mu) : \mu \in spt(L-H)\}$ *is a real eigenvalue of* $L-H$*, where* $spt(L-H)$ *is the spectrum of* $L-H$.
(2) *For* $\mu = \mu_1$*, there exists a unique (up a constant multiple) eigenfunction* $\mathbf{u}_1 \in Y \equiv [L^2(\Omega)]^2$*, and* $\mathbf{u}_1 > 0$ *in* Ω.
(3) *For* $\mu < \mu_1$*, the equation* $L\mathbf{u} = H\mathbf{u} + \mu\mathbf{u} + \mathbf{f}$ *is uniquely solvable for any* $\mathbf{f} \in Y$*, and* $\mathbf{u} > 0$ *as long as* $\mathbf{f} \ge 0$.
(4) *(Maximum principle) For* $\mu \le \mu_1$*, suppose that* $\mathbf{u} \in [W^{2,2}(\Omega)]^2$*, satisfies* $L\mathbf{u} \ge H\mathbf{u} + \mu\mathbf{u}$ *in* Ω*,* $\mathbf{u} \ge 0$ *on* $\partial\Omega$*, then* $\mathbf{u} \ge 0$ *in* Ω.

(5) *If there exists* $\mathbf{u} \in [W^{2,2}(\Omega)]^2$, *satisfies* $L\mathbf{u} \geq H\mathbf{u}$ *in* Ω, $\mathbf{u} \geq 0$ *on* $\partial\Omega$, *and either* $\mathbf{u} \not\equiv 0$ *on* $\partial\Omega$ *or* $L\mathbf{u} \not\equiv H\mathbf{u}$ *in* Ω, *then* $\mu_1 > 0$.

For the result and proofs, see Sweers [30, Prop. 3.1 and Thm. 1.1]. Moreover from a standard compactness argument, the eigenvalues $\{\mu_i\}$ of $L-H$ is countably many, and $|\mu_i - \mu_1| \to \infty$ as $i \to \infty$. We notice that μ_i is not necessarily real-valued. We call a solution (u,v) is *stable* if $\mu_1 > 0$, and it is *unstable* if $\mu_1 \leq 0$.

We prove the following stability result when (f,g) is sublinear or superlinear.

Proposition 3.2. *Suppose that* (u,v) *is a positive solution of* (1.1).

(1) *If* f,g *are sublinear, then* (u,v) *is stable;*
(2) *If* f,g *are superlinear, then* (u,v) *is unstable with* $\mu_1(u,v) < 0$.

Proof. Multiplying the equation of u in (1.1) by η, multiplying the equation of η by u, integrating over Ω and subtracting, we obtain

$$\lambda \int_\Omega f(v)\eta dx = \lambda \int_\Omega g'(u)u\xi dx + \mu_1 \int_\Omega u\eta dx. \tag{3.4}$$

Similarly form the equation of v and ξ, we find

$$\lambda \int_\Omega g(u)\xi dx = \lambda \int_\Omega f'(v)v\eta dx + \mu_1 \int_\Omega v\xi dx. \tag{3.5}$$

Adding (3.4) and (3.5), we obtain

$$\mu_1 \left(\int_\Omega v\xi dx + \int_\Omega u\eta dx \right) = \lambda \int_\Omega [f(v) - f'(v)v]\eta dx + \lambda \int_\Omega [g(u) - g'(u)u]\xi dx. \tag{3.6}$$

From Lemma 3.1, $\xi > 0$ and $\eta > 0$ in Ω. If f,g are sublinear, then $f(v) - vf'(v) > 0$ and $g(u) - ug'(u) > 0$ for $u,v > 0$. Hence $\mu_1 > 0$ from (3.6). If f,g are superlinear, then $f(v) - vf'(v) < 0$ and $g(u) - ug'(u) < 0$ for $u,v > 0$. Thus $\mu_1 < 0$. □

We remark that the stability for solution to sublinear problem implies the nondegeneracy of the solution, which will play an important role in the proof of uniqueness of the solution.

4. Sublinear problem

We have the following result for the existence and uniqueness of the positive solution to (1.1) with sublinear nonlinearities f and g.

Theorem 4.1. *Assume that* Ω *is a bounded domain with smooth boundary* $\partial\Omega$, $f,g : \mathbb{R}^+ \to \mathbb{R}^+$ *are sublinear, and*

$$\lim_{u\to\infty} \frac{g(u)}{u} = \lim_{v\to\infty} \frac{f(v)}{v} = 0. \tag{4.1}$$

(1) *If at least one of* $f(0)$ *and* $g(0)$ *is positive, then* (1.1) *has a unique positive solution* (u_λ, v_λ) *for all* $\lambda > 0$*;*
(2) *If* $f(0) = g(0) = 0$, *and* $f'(0) > 0$, $g'(0) > 0$, *then for some* $\lambda_* = \lambda_1/\sqrt{f'(0)g'(0)} > 0$, (1.1) *has no positive solution when* $\lambda \leq \lambda_*$, *and* (1.1) *has a unique positive solution* (u_λ, v_λ) *for* $\lambda > \lambda_*$.

Moreover, $\{(\lambda, u_\lambda, v_\lambda) : \lambda > \lambda_*\}$ *(in the first case, we assume* $\lambda_* = 0$*) is a smooth curve so that* u_λ *and* v_λ *are strictly increasing in* λ, *and* $(u_\lambda, v_\lambda) \to (0,0)$ *as* $\lambda \to \lambda_*^+$.

Note that if f is sublinear, then it is necessary that $f(0) \geq 0$. Hence f and g are positive for $u > 0$ here. If $f(0) = 0$ and f is sublinear, then we must have $f'(0) > 0$ since $f'(0) > f(u)/u$ for $u > 0$. Note that here we assume f, g are asymptotical sublinear (see (4.1)), and we also have corresponding results for asymptotical linear and asymptotical negative cases, see Theorem 4.2.

Proof of Theorem 4.1. First we assume at least one of $f(0)$ and $g(0)$ is positive. Recall the operator F defined in (2.2). Then $F_{(u,v)}(0, 0, 0)$ is an isomorphism, and the implicit function theorem (Theorem 2.1) implies that $F(\lambda, u, v) = 0$ has a unique solution $(\lambda, u_\lambda, v_\lambda)$ for $\lambda \in (0, \delta)$ for some small $\delta > 0$, and

$$(u'(0), v'(0)) = (\frac{\partial u_\lambda}{\partial \lambda}, \frac{\partial v_\lambda}{\partial \lambda})|_{\lambda=0}$$

is the unique solution of

$$\begin{gathered} \Delta\phi + f(0) = 0, \quad \Delta\psi + g(0) = 0, \quad x \in \Omega, \\ \phi(x) = \psi(x) = 0, \quad x \in \partial\Omega. \end{gathered} \tag{4.2}$$

Then $(u'(0), v'(0)) = (f(0)e, g(0)e)$ where e is the unique positive solution of

$$\Delta e + 1 = 0, \quad x \in \Omega, \quad e(x) = 0, \quad x \in \partial\Omega. \tag{4.3}$$

If $f(0) > 0$ and $g(0) > 0$, then (u_λ, v_λ) is positive for $\lambda \in (0, \delta)$. If $f(0) = 0$ and $g(0) > 0$, then $v_\lambda > 0$. But $\Delta u_\lambda = -f(v_\lambda)$ and f is positive, hence $u_\lambda > 0$ as well. Therefore (1.1) has a positive solution (u_λ, v_λ) for $\lambda \in (0, \delta)$ in this case.

Next we assume that $f(0) = g(0) = 0$, and $f'(0) > 0$, $g'(0) > 0$. In this case $(0, 0)$ is a trivial solution of (1.1) for any $\lambda > 0$. We show that there is a bifurcation point λ_* where nontrivial solutions bifurcate from the branch of trivial solutions. The linearized operator is

$$F_{(u,v)}(\lambda, 0, 0)[\Phi, \Psi] = [\Delta\Phi + \lambda f'(0)\Psi, \Delta\Psi + \lambda g'(0)\Phi],$$

and the eigenvalues are $\widetilde{\lambda}_i = \lambda_i/\sqrt{f'(0)g'(0)}$ with eigenfunction

$$[\Phi_i, \Psi_i] = [1, \sqrt{g'(0)/f'(0)}]\varphi_i,$$

where (λ_i, φ_i) is the i-th eigen-pair of $-\Delta$ on $W_0^{1,2}(\Omega)$. In particular, $\lambda_* = \widetilde{\lambda}_1$ is a bifurcation point where positive solutions of (1.1) bifurcate. Since λ_* is a simple eigenvalue, $R(F_{(u,v)}(\lambda_*, 0, 0)) = \{(h_1, h_2) \in Y : \int_\Omega (h_1\Psi_1 + h_2\Phi_1)dx = 0\}$ from (2.6), and

$$F_{\lambda(u,v)}[\Phi_1, \Psi_1] = [f'(0)\Psi_1, g'(0)\Phi_1] \notin R(F_{(u,v)}(\lambda, 0, 0)),$$

then one can apply Theorem 2.2 to obtain a curve of positive solutions of (1.1): $\{(\lambda(s), u(s), v(s)) : s \in (0, \delta)\}$. We claim that (1.1) has no positive solution when $\lambda \leq \lambda_*$. Note that (1.1) has no positive solutions when $\lambda \leq 2\lambda_1/(f'(0) + g'(0))$ from Proposition 2.3, but we can improve that lower bound here. Since $f(v) \leq f'(0)v$ and $g(u) \leq g'(0)u$ for $u, v \geq 0$, then

$$\begin{aligned} 0 &= \int_\Omega [\Delta u + \lambda f(v)]\Psi_1 dx + \int_\Omega [\Delta v + \lambda g(u)]\Phi_1 dx \\ &= \int_\Omega \Phi_1[-\lambda_* g'(0)u + \lambda g(u)]dx + \int_\Omega \Psi_1[-\lambda_* f'(0)v + \lambda f(v)] < 0, \end{aligned} \tag{4.4}$$

if $\lambda \leq \lambda_*$ and $u, v > 0$. Hence (1.1) has no positive solution when $\lambda \leq \lambda_*$. In particular, $\lambda(s) > \lambda_*$ for $s \in (0, \delta)$.

In the two cases above, we obtain a curve of solutions to (1.1) for $\lambda \in (\lambda_*, \lambda_*+\delta)$ and (u,v) is close to $(0,0)$. From Proposition 3.2, each positive solution (u,v) of (1.1) with f,g sublinear is stable thus non-degenerate, then implicit function theorem implies that the solution set is always locally a C^1 curve near a positive solution (u,v). Thus in the second case, the solution $(\lambda(s), u(s), v(s))$ can also be parameterized as $(\lambda, u_\lambda, v_\lambda)$ for $\lambda \in (\lambda_*, \lambda_*+\delta)$. With implicit function theorem, we can extend this curve to a largest λ^*. Let $\Gamma = \{(\lambda, u_\lambda, v_\lambda) : \lambda_* < \lambda < \lambda^*\}$. We show that (u_λ, v_λ) is strictly increasing in λ for $\lambda \in (\lambda_*, \lambda^*)$. In fact, $(\partial u_\lambda/\partial\lambda, \partial v_\lambda/\partial\lambda)$ satisfies the equation:

$$\Delta \frac{\partial u_\lambda}{\partial \lambda} + \lambda f'(v_\lambda)\frac{\partial v_\lambda}{\partial \lambda} + f(v_\lambda) = 0, \quad \Delta \frac{\partial v_\lambda}{\partial \lambda} + \lambda g'(u_\lambda)\frac{\partial u_\lambda}{\partial \lambda} + g(u_\lambda) = 0,$$

hence $(\partial u_\lambda/\partial\lambda, \partial v_\lambda/\partial\lambda) > 0$ from the maximum principle (Lemma 3.1 part 3) and the fact that $\mu_1((u_\lambda, v_\lambda)) > 0$ from Proposition 3.2. If $\lambda^* < \infty$, and $\|u_\lambda\|_X + \|v_\lambda\|_X < \infty$, then one can show that the curve Γ can be extended to $\lambda = \lambda^*$ from some standard elliptic estimates; if $\lambda^* < \infty$, and $\|u_\lambda\|_X + \|v_\lambda\|_X = \infty$, a contradiction can be derived with the asymptotical sublinear condition (4.1) (see similar arguments for scalar equation in [26]). Hence we must have $\lambda^* = \infty$.

If there is another positive solution for some $\lambda > \lambda_*$, then the arguments above show this solution also belongs to a solution curve defined for $\lambda \in (\lambda_*, \infty)$, and the solutions on the curve are increasing in λ, but the nonexistence of positive solutions for $\lambda < \lambda_*$ and the local bifurcation at $\lambda = \lambda_*$ excludes the possibility of another solution curve. Hence the positive solution is unique for all $\lambda > \lambda_*$. This completes the proof. □

Some examples of sublinear functions satisfying the conditions in Theorem 4.1 are $f(u) = \ln(u+1) + k$, $f(u) = 1 - e^{-u} + k$, $f(u) = (1+u)^p - 1 + k$ $(0 < p < 1)$ with $k \geq 0$. Indeed the proof of Theorem 4.1 can also be used to prove similar results when asymptotical sublinear condition (4.1) is not satisfied. We state the following theorem without proof.

Theorem 4.2. *Assume that $f, g : \mathbb{R}^+ \to \mathbb{R}$ are sublinear.*

(1) *If f, g are asymptotically linear, that is*
$$\lim_{u\to\infty} \frac{g(u)}{u} = k_1 > 0, \quad \lim_{v\to\infty} \frac{f(v)}{v} = k_2 > 0, \tag{4.5}$$
then the results in Theorem 4.1 still hold except the solutions only exist for $\lambda \in (\lambda_*, \lambda^*)$, where $\lambda^* = \lambda_1/\sqrt{k_1 k_2}$, and (1.1) has no positive solutions for $\lambda \geq \lambda^*$;

(2) *If f, g are negative for u, v large, then the results in Theorem 4.1 still hold for $\lambda \in (\lambda_*, \infty)$.*

In the first case, we have a bifurcation from infinity at $\lambda = \lambda^*$, and examples of such functions are $f(u) = \sqrt{u^2 + 2u} + k$, $f(u) = 2u + k - \sqrt{u^2+1}$ for $k \geq 0$. In the second case, a typical example is the logistic function $f(u) = u - u^p$ for $p > 1$. Hence the second case of Theorem 4.2 describes the solution set of the following diffusive logistic system:

$$\begin{gathered} \Delta u + \lambda(av - bv^2) = 0, \quad x \in \Omega, \\ \Delta v + \lambda(cu - du^2) = 0, \quad x \in \Omega, \\ u(x) = v(x) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{4.6}$$

where $a, b, c, d > 0$. One can show that as $\lambda \to \infty$, the unique solution (u_λ, v_λ) tends to $(a/b, c/d)$ (the carrying capacity) on any compact subset of Ω as $\lambda \to \infty$.

5. One-dimensional problem

In this section we establish some results for the equation (1.1) and related linearized equation when $n = 1$ and $\Omega = (-1, 1)$. Hence we consider

$$\begin{gathered} u'' + \lambda f(v) = 0, \quad x \in (-1,1), \\ v'' + \lambda g(u) = 0, \quad x \in (-1,1), \\ u(\pm 1) = v(\pm 1) = 0, \end{gathered} \tag{5.1}$$

and for a degenerate solution (u, v) of (5.1), the equation

$$\begin{gathered} \phi'' + \lambda f'(v(x))\psi = 0, \quad x \in (-1,1), \\ \psi'' + \lambda g'(u(x))\phi = 0, \quad x \in (-1,1), \\ \phi(\pm 1) = \psi(\pm 1) = 0 \end{gathered} \tag{5.2}$$

has a nontrivial solution. We first prove the following symmetry result.

Theorem 5.1. *Assume that $f, g : \mathbb{R}^+ \to \mathbb{R}$ are C^1 satisfy* (1.2). *Let $(\lambda, u(x), v(x))$ be a positive solution of* (5.1). *Then*

1. *u and v are symmetric with respect to $x = 0$, i.e. $u(-x) = u(x)$ and $v(-x) = v(x)$, and $u'(x) > 0$ and $v'(x) > 0$ in $(-1, 0)$;*
2. *If (λ, u, v) is a degenerate solution of* (5.1), *and $f(0) \ge 0$ or $g(0) \ge 0$, then for any solution (ϕ, ψ) of* (5.2), *ϕ and ψ are symmetric with respect to $x = 0$, i.e. $\phi(-x) = \phi(x)$ and $\psi(-x) = \psi(x)$.*

Proof. The symmetry of u and v follows from a result of Troy [31]. We prove that ϕ and ψ are symmetric with respect to $x = 0$ if $f(0) \ge 0$ or $g(0) \ge 0$. From Theorem 1 in [31], $u'(x) > 0$ and $v'(x) > 0$ in $(-1, 0)$. Then $(L - H)(u', v') = 0$ in $(-1, 0)$ and $u'(x) \ge 0$, $v'(x) \ge 0$ for $x = -1$ and $x = 0$. We claim that if $f(0) \ge 0$ or $g(0) \ge 0$, then either $u'(-1) > 0$ or $v'(-1) > 0$. Suppose not, then $u'(-1) = v'(-1) = 0$, and $x = -1$ is a local minimum of $u(x)$ and $v(x)$. So $g(0) \le 0$ and $f(0) \le 0$. If $f(0) = 0$, then $f(v) > 0$ for $v > 0$ since $f'(v) > 0$. But integrating $v'' + f(v) = 0$ over $(-1, 0)$, we obtain $\int_0^1 f(v(x))dx = 0$ since $v'(-1) = v'(0) = 0$, that is a contradiction. Hence $f(0) < 0$ and $g(0) < 0$. But we assume $f(0) \ge 0$ or $g(0) \ge 0$, that is a contradiction again. Thus either $u'(-1) \ne 0$ or $v'(-1) \ne 0$, then from Lemma 3.1 part 5, $\mu_1((L - H)|_{(-1,0)}) > 0$. and the maximum principle holds from Lemma 3.1 part 4. Now let $\xi(x) = \phi(-x) - \phi(x)$ and $\eta(x) = \psi(-x) - \psi(x)$ for $x \in (-1, 0)$. Then $(L - H)(\xi, \eta) = 0$ for $x \in (-1, 0)$ and $\xi(-1) = \xi(0) = 0$, $\eta(-1) = \eta(0) = 0$. Then from Lemma 3.1 part 4, $\xi(x) \equiv 0$ and $\eta(x) \equiv 0$, hence ϕ and ψ are symmetric with respect to $x = 0$. □

The results in Theorem 5.1 are well-known for scalar equations, see Gidas, Ni, Nirenberg [11] and Lin and Ni [19], and they have played an important role in proving the exact multiplicity of positive solutions of scalar semilinear elliptic equations (see [15, 16, 21, 22]). We remark that the condition $f(0) \ge 0$ or $g(0) \ge 0$ in Theorem 5.1 part 2 cannot be removed. Indeed if $f(0) < 0$ and $g(0) < 0$, then it is possible that (5.1) has a positive solution (u, v) such that $u'(\pm 1) = v'(\pm 1) = 0$, and in that case $(\phi, \psi) = (u', v')$ is not symmetric but odd with respect to $x = 0$.

An example is when $f(v) = v - 1$ and $g(u) = u - 1$, then $u(x) = v(x) = 1 + \cos(\pi x)$ is such a solution when $\lambda = \pi^2$.

Next we show a variational identity satisfied by solutions of (5.1). Let

$$H(x) = u'(x)v'(x) + \lambda F(v(x)) + \lambda G(u(x)), \tag{5.3}$$

where $F(v) = \int_0^v f(t)dt$ and $G(u) = \int_0^u g(t)dt$. Then for a solution $(\lambda, u(x), v(x))$ of (5.1), $H'(x) \equiv 0$ for $x \in [-1, 1]$ from the equations. This implies that

$$u'(\pm 1)v'(\pm 1) = \lambda[F(v(0)) + G(u(0))]. \tag{5.4}$$

Hence

$$F(v(0)) + G(u(0)) \geq 0. \tag{5.5}$$

Moreover if (u, v) is a solution such that $u'(\pm 1)v'(\pm 1) = 0$, then it is necessary that $F(v(0)) + G(u(0)) = 0$. Note that such a property is well-known for the scalar equation $u'' + \lambda f(u) = 0$. (5.5) gives a restriction on the possible value of $(u(0), v(0))$. Another restriction is

$$f(v(0)) \geq 0, \quad g(u(0)) \geq 0, \quad \text{and } f(v(0)) + g(u(0)) > 0. \tag{5.6}$$

This follows from the fact that $u(0)$ and $v(0)$ are the maximum values of $u(x)$ and $v(x)$ respectively.

From Theorem 5.1, we could consider the systems on the interval $(0, 1)$ instead of $(-1, 1)$. Hence we consider

$$\begin{gathered} u'' + \lambda f(v) = 0, \quad x \in (0,1), \\ v'' + \lambda g(u) = 0, \quad x \in (0,1), \\ u'(0) = v'(0) = 0, \quad u(1) = v(1) = 0, \end{gathered} \tag{5.7}$$

and, (at least when $f(0) \geq 0$ or $g(0) \geq 0$)

$$\begin{gathered} \phi'' + \lambda f'(v(x))\psi = 0, \quad x \in (0,1), \\ \psi'' + \lambda g'(u(x))\phi = 0, \quad x \in (0,1), \\ \phi'(0) = \psi'(0) = 0, \quad \phi(1) = \psi(1) = 0. \end{gathered} \tag{5.8}$$

To further study the solution set of (5.1) (or equivalently (5.7)), we consider the initial value problem

$$\begin{gathered} u'' + f(v) = 0, \quad x > 0, \\ v'' + g(u) = 0, \quad x > 0, \\ u'(0) = v'(0) = 0, \\ u(0) = \alpha > 0, \quad v(0) = \beta > 0. \end{gathered} \tag{5.9}$$

We denote the solution of (5.9) by $(u(x; \alpha, \beta), v(x; \alpha, \beta))$ or simply $(u(x), v(x))$ when there is no confusion. The solution $(u(x), v(x))$ can be extended to a maximal interval $(0, R(\alpha, \beta))$ so that $u(x) > 0$ and $v(x) > 0$ in $(0, R(\alpha, \beta))$. In the following we will use $R = R(\alpha, \beta)$ when there is no confusion.

Any positive solution of (5.1) with $(u(0), v(0)) = (\alpha, \beta)$ satisfies $f(\beta) > 0$ and $g(\alpha) > 0$. Hence we define $I = \{(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+ : f(\beta) > 0, g(\alpha) > 0\}$. For

$(\alpha,\beta)\in I$, $u'<0$ and $v'<0$ in $(0,\delta)$, and we partition I into the following classes:

$$
\begin{aligned}
&U=\big\{(\alpha,\beta)\in I: R<\infty, u>0, v>0, u'<0, v'<0,\ x\in(0,R),\\
&\quad u(R)=0, v(R)>0\big\},\\
&V=\big\{(\alpha,\beta)\in I: R<\infty, u>0, v>0, u'<0, v'<0,\ x\in(0,R),\\
&\quad u(R)>0, v(R)=0\},\\
&N=\big\{(\alpha,\beta)\in I: R<\infty, u>0, v>0, u'<0, v'<0,\ x\in(0,R),\\
&\quad u(R)=0, v(R)=0\big\},\\
&G=\big\{(\alpha,\beta)\in I: R=\infty, u>0, v>0, u'<0, v'<0,\ x\in(0,\infty),\\
&\quad \lim_{x\to\infty}u(x)=\lim_{x\to\infty}v(x)=0\big\},\\
&P=I\backslash(U\cup V\cup N\cup G)\,.
\end{aligned}
\tag{5.10}
$$

If $(\alpha,\beta)\in N$, then a rescaled $(u(x/R;\alpha,\beta), v(x/R;\alpha,\beta))$ satisfies (5.1) with $\lambda=[R(\alpha,\beta)]^2$. Since R is uniquely determined by (α,β) from the uniqueness of solution of ODEs, then λ is uniquely determined by $(\alpha,\beta)\in N$.

We consider the linearized equation of (5.9). Assume that (u,v) is a positive solution of (5.9). Let (ϕ_1,ψ_1) satisfy

$$
\begin{gathered}
\phi_1''+f'(v)\psi_1=0,\quad 0<x<R,\\
\psi_1''+g'(u)\phi_1=0,\quad 0<x<R,\\
\phi_1(0)=1,\quad \phi_1'(0)=0,\\
\psi_1(0)=0,\quad \psi_1'(0)=0;
\end{gathered}
\tag{5.11}
$$

and let (ϕ_2,ψ_2) satisfy

$$
\begin{gathered}
\phi_2''+f'(v)\psi_2=0,\quad 0<x<R,\\
\psi_2''+g'(u)\phi_2=0,\quad 0<x<R,\\
\phi_2(0)=0,\quad \phi_2'(0)=0,\\
\psi_2(0)=1,\quad \psi_2'(0)=0.
\end{gathered}
\tag{5.12}
$$

The following oscillatory result is similar to Sturm comparison lemma for scalar equation: (this is a special case of [2, Lemma 2.2])

Lemma 5.2. *Let (u,v) be a solution of (5.9) such that $u(x)>0$, $v(x)>0$, $u'(x)<0$ and $v'(x)<0$ for $x\in(0,R)$, and let ϕ_i, ψ_i $(i=1,2)$ be defined as in (5.11) and (5.12). Assume that (f,g) satisfies (1.2). Then*

1. *$\phi_1(x)>0$ and $\psi_1(x)<0$ for $x\in(0,R]$;*
2. *$\psi_2(x)>0$ and $\phi_2(x)<0$ for $x\in(0,R]$.*

Proof. We only prove it for ϕ_1 and ψ_1, and the proof for the other case is similar. Since $\phi_1(0)>0$, $\psi_1(0)=0$, $\psi_1'(0)=0$, and $\psi_1''(0)=-\lambda g'(u)\phi_1(0)<0$. Then for some $x_0>0$, $\phi_1(x)>0$ and $\psi_1(x)<0$ in $(0,x_0)$. Define

$$
x_1=\sup\{0<x<R:\phi_1(x)>0 \text{ and } \psi_1(x)<0 \text{ in } (0,x_0)\}. \tag{5.13}
$$

If $x_1=R$, then the result holds. So we assume $x_1<R$. Then either $\phi_1(x_1)=0$ or $\psi_1(x_1)=0$. If $\psi_1(x_1)=0$, then $\psi_1(x)<0$ in $(0,x_1)$ and $\psi_1'(x_1)\geq 0$.

Then multiplying the equation of ψ_1 in (5.11) by v', multiplying the equation of v' ($(v')'' + \lambda g'(u)u' = 0$) by ψ_1, subtracting and integrating on $(0, x_1)$, we obtain

$$[\psi_1' v' - \psi_1 v''] \,|_0^{x_1} = -\lambda \int_0^{x_1} g'(u)(\phi_1 v' - \psi_1 u')dx. \tag{5.14}$$

The left hand side is $\psi_1'(x_1)v'(x_1) \le 0$, and the right hand side is positive since $g'(u) > 0$, $\phi_1 > 0$, $\psi_1 < 0$, $u' < 0$ and $v' < 0$. That is a contradiction. Similarly we can show that $\phi_1(x_1) = 0$ also leads to a contradiction. Hence $x_1 = R$, and from the equations in (5.11), it is also clear that $\phi_1(R) > 0$ and $\psi_1(R) < 0$. □

Next we show that Lemma 5.2 implies the solution set can be parameterized by a single parameter. This is another key ingredient for our exact multiplicity results. The following result is similar to the one in Korman and Shi [17] (see Lemma 1 and Proposition 1):

Lemma 5.3. *Suppose that f, g are C^1 and satisfy* (1.2). *Then for each $\alpha > 0$, there exists at most one $\beta = \beta(\alpha) > 0$ such that* (5.1) *has a positive solution with $u(0) = \alpha$ and $v(0) = \beta$, and the value of parameter λ is determined by $(\alpha, \beta(\alpha))$.*

Proof. We claim that for each $\alpha > 0$, there exists at most one $\beta > 0$ such that $(\alpha, \beta) \in N$. Suppose that $(\alpha_0, \beta_0) \in N$, we show that $(\alpha_0, \beta) \in U$ if $\beta > \beta_0$, and $(\alpha_0, \beta) \in V$ if $0 < \beta < \beta_0$. Assume that $R_0 = R(\alpha_0, \beta_0)$ and $0 < \beta < \beta_0$. Define $R_1 = \sup\{r > 0 : u(x; \alpha_0, \beta) > 0,\ v(x; \alpha_0, \beta) > 0\}$. We claim $R_1 < R_0$. Define $\phi(x) = u(x; \alpha_0, \beta_0) - u(x; \alpha_0, \beta)$ and $\psi(x) = v(x; \alpha_0, \beta_0) - v(x; \alpha_0, \beta)$, then (ϕ, ψ) satisfies

$$\begin{gathered} \phi'' + f'(V)\psi = 0, \quad x \in (0, R_*), \\ \psi'' + g'(U)\phi = 0, \quad x \in (0, R_*), \\ \phi(0) = 0, \ \ \phi'(0) = 0, \\ \psi(0) = \beta_0 - \beta > 0, \ \ \psi'(0) = 0, \end{gathered}$$

where $R_* = \min(R_0, R_1)$, $U(x) = t_1(x)u(x; \alpha_0, \beta_0) + (1 - t_1(x))u(x; \alpha_0, \beta) > 0$, and $U_2 = t_2(x)v(x; \alpha_0, \beta_0) + (1 - t_2(x))v(x; \alpha_0, \beta) > 0$. Then from Lemma 5.2, $\psi(x) > 0$ and $\phi(x) < 0$ for $x \in (0, R_*]$, and ϕ has at most one zero in $(0, R_*]$. This implies that at R_*, $u(R_*; \alpha_0, \beta_0) < u(R_*; \alpha_0, \beta)$ and $v(R_*; \alpha_0, \beta_0) > v(R_*; \alpha_0, \beta)$. Thus $R_* = R_1$, $v(R_1; \alpha_0, \beta) = 0$, and $(\alpha_0, \beta) \in V$. Similarly $(\alpha_0, \beta) \in U$ if $\beta > \beta_0$. □

From the discussions above, the subset N defined in (5.10) can be parameterized by either α or β. Here we write

$$N = \{(\alpha, \beta(\alpha)) : \alpha \in N_1\}, \tag{5.15}$$

where $N_1 \subset \mathbb{R}^+$. Thus N or N_1 is the admissible maximum values of solutions to (5.1). For each $\alpha \in N_1$, let $R(\alpha) = R(\alpha, \beta(\alpha))$ be defined as in the proof of Lemma 5.3. Then $(u(x/R(\alpha); \alpha, \beta(\alpha)), v(x/R(\alpha); \alpha, \beta(\alpha)))$ satisfies (5.1) with $\lambda(\alpha) = [R(\alpha)]^2$. We have the following properties:

Lemma 5.4. *Assume that f, g satisfy* (1.3). *Let N, N_1, $R(\alpha)$ and $\beta(\alpha)$ and $\lambda(\alpha)$ be defined as above, let $\alpha \in N_1$, and let $(\lambda_0, U(x), V(x))$ be the solution of* (5.1) *with $(U(0), V(0)) = (\alpha, \beta(\alpha))$. Then $\beta'(\alpha) > 0$.*

Proof. We see that $(u(x), v(x)) = (U(x/R(\alpha)), V(x/R(\alpha)))$ is the solution of (5.9) with initial value $(\alpha, \beta(\alpha))$.

Differentiating $u(R(\alpha);\alpha,\beta(\alpha))=0$ and $v(R(\alpha);\alpha,\beta(\alpha))=0$, we obtain

$$\begin{aligned} u'(R(\alpha))R'(\alpha)+\phi_1(R(\alpha))+\beta'(\alpha)\phi_2(R(\alpha))=0,\\ v'(R(\alpha))R'(\alpha)+\psi_1(R(\alpha))+\beta'(\alpha)\psi_2(R(\alpha))=0, \end{aligned} \tag{5.16}$$

where ϕ_i and ψ_i are defined in (5.11) and (5.12). Suppose that $\beta'(\alpha)\le 0$. Then $\phi_1(R(\alpha))+\beta'(\alpha)\phi_2(R(\alpha))>0$ and $\psi_1(R(\alpha))+\beta'(\alpha)\psi_2(R(\alpha))<0$ from Lemma 5.2. But $u'(\alpha)\le 0$ and $v'(\alpha)\le 0$. That is a contradiction. Therefore we must have $\beta'(\alpha)>0$. □

The results proved so far in this section hold when either (1.2) or (1.3) is satisfied. The last result is our main result for the structure of solutions of initial value problem (5.9) when both (1.2) and (1.3) are satisfied.

Theorem 5.5. *Assume that $f,g:\mathbb{R}^+\to\mathbb{R}^+$ satisfy* (1.2) *and* (1.3)*. Then for each $\alpha>0$, there exists a unique $\beta=\beta(\alpha)>0$ such that* (5.1) *has a positive solution (u,v) with $u(0)=\alpha$, $v(0)=\beta$, with $\beta'(\alpha)>0$ and $\lim_{\alpha\to\infty}\beta(\alpha)=\infty$. Moreover, let U,V,N,G and P be defined as in* (5.10)*, then $P=G=\emptyset$, $I=\mathbb{R}^+\times\mathbb{R}^+=U\cup V\cup N$, and*

$$\begin{aligned} U=\{(\alpha,\beta):\alpha>0,\beta>\beta(\alpha)\},\\ V=\{(\alpha,\beta):\alpha>0,0<\beta<\beta(\alpha)\},\\ N=\{(\alpha,\beta):\alpha>0,\beta=\beta(\alpha)\}. \end{aligned} \tag{5.17}$$

Proof. We fix $\alpha>0$. First we show that if $\beta>0$ is small enough, then $(\alpha,\beta)\in V$, which is defined in (5.10). If $v(0)=\beta$, then $0\le v(x)\le\beta$ for $x\in[0,R]$. Hence $u''=-f(v)\ge -f(\beta)$, and $u(x)-\alpha\ge -(1/2)f(\beta)x^2$ for $x\in[0,R]$. We choose $\beta_1>0$ such that $2\beta_1 f(\beta_1)\le \alpha g(\alpha/2)$. Then for $x_0=\sqrt{2\beta_1/g(\alpha/2)}$, $u(x)\ge u(x_0)\ge\alpha-\beta_1 f(\beta_1)/g(\alpha/2)\ge\alpha/2$. On the other hand, for $x\in[0,x_0]$, $v''=-g(u)\le -g(\alpha/2)$ hence $v(x)\le\beta-(1/2)g(\alpha/2)x^2$. In particular, $v(x_0)\le 0$. Therefore $R\le x_0$ and $(\alpha,\beta)\in V$ for each $\beta\in(0,\beta_1)$. Similarly one can show that if we choose $\beta_2>0$ such that $2\alpha f(\alpha)\le\beta_2 g(\beta_2/2)$, then $(\alpha,\beta)\in U$ for each $\beta\ge\beta_2$.

Similar to the proof of Lemma 5.3, one can show that if $(\alpha,\beta_3)\in V$, then $(\alpha,\beta)\in V$ for $0<\beta<\beta_3$; and if $(\alpha,\beta_4)\in U$, then $(\alpha,\beta)\in U$ for $\beta>\beta_4$. On the other hand, U and V are both open subsets of $\mathbb{R}^+\times\mathbb{R}^+$ from the continuous dependence of solutions on the initial values. Define $\beta_0=\sup\{\beta>0:(\alpha,\beta)\in V\}$. Again from the proof of Lemma 5.3, $R(\alpha,\beta)$ is increasing in β for $\beta\in(0,\beta_0)$. Hence $R_*=\lim_{\beta\to\beta_0}R(\alpha,\beta)$ exists. If $R_*=\infty$, then $(\alpha,\beta_0)\in G$ and it is necessary that $u',v'\to 0$ as $x\to\infty$. But from (1.3), $F(\beta_0)+G(\alpha)>0$, thus (5.4) leads to a contradiction. Hence $R_*<\infty$, and $(\alpha,\beta_0)\in N$ since U and V are both open. From Lemma 5.3, for any $\beta>\beta_0$, we have $(\alpha,\beta)\in U$. Hence $\beta(\alpha)=\beta_0$ is the unique value inducing a positive solution of (5.1). This also proves that $P=G=\emptyset$, $I=\mathbb{R}^+\times\mathbb{R}^+=U\cup V\cup N$, and (5.17) holds. □

Theorem 5.5 shows that when (1.2) and (1.3) are satisfied (hence it is a positone problem), then the admissible initial condition (α,β) for positive solutions of (5.1) are on a monotone increasing curve $N=\{(\alpha,\beta(\alpha):\alpha>0\}$ connecting $(0,0)$ and (∞,∞). Hence we only need to consider the solutions of (5.9) with initial conditions in N, and the function $R(\alpha)$ determines the uniqueness or multiplicity of positive solutions to (5.1). The set N_1 defined above is the entire $\mathbb{R}^+$ for this case, but this may not be true when either (1.2) or (1.3) is not satisfied.

For the sublinear case, the uniqueness of positive solution of (5.1) has been proved in Section 4 even for the general bounded domains. Hence we will not use the approach given in this section for that case again. We point out for the sublinear case, $R'(\alpha) > 0$ always hold. For the result in Theorem 4.1, when at least one of $f(0)$ and $g(0)$ is positive, then $\lim_{\alpha\to 0^+} R(\alpha) = 0$ and $\lim_{\alpha\to\infty^+} R(\alpha) = \infty$; when $f(0) = g(0) = 0$, and $f'(0) > 0$, $g'(0) > 0$, then $\lim_{\alpha\to 0^+} R(\alpha) = \pi^4/(16f'(0)g'(0))$ and $\lim_{\alpha\to\infty^+} R(\alpha) = \infty$. Note that $\lambda_1((-1,1)) = \pi^2/4$. For the result in Theorem 4.2, when (4.5) is satisfied, then $\lim_{\alpha\to 0^+} R(\alpha) = \pi^4/(16f'(0)g'(0))$ and $\lim_{\alpha\to\infty^+} R(\alpha) = \pi^4/(16k_1k_2)$; when f and g are negative for large $u, v > 0$, then $\lim_{\alpha\to 0^+} R(\alpha) = \pi^4/(16f'(0)g'(0))$ and $\lim_{\alpha\to\infty^+} R(\alpha) = \infty$.

6. One-dimensional superlinear problem

In this section we apply the theory developed in Section 5 for the solutions of (5.1) with superlinear f and g. Our main result is the following.

Theorem 6.1. *Assume that $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ are superlinear, $f(0) = g(0) = 0$, f, g satisfy* (1.2)*, and*

$$\lim_{u\to\infty} \frac{g(u)}{u} = \lim_{v\to\infty} \frac{f(v)}{v} = \infty. \tag{6.1}$$

(1) *If at least one of $f'(0)$ and $g'(0)$ equals to* 0*, then* (5.1) *has a unique positive solution (u_λ, v_λ) for all $\lambda > 0$;*
(2) *If $f'(0) > 0$ and $g'(0) > 0$, then for $\lambda_* = \lambda_1/\sqrt{f'(0)g'(0)}$,* (5.1) *has no positive solution when $\lambda \ge \lambda_*$, and* (5.1) *has a unique positive solution (u_λ, v_λ) for $0 < \lambda < \lambda_*$.*

Moreover, $\{(\lambda, u_\lambda, v_\lambda) : 0 < \lambda < \lambda_*\}$ (in the first case, we assume $\lambda_* = \infty$) is a smooth curve such that $u_\lambda(0)$ and $v_\lambda(0)$ are decreasing in λ.

Proof. If $f'(0) > 0$, $g'(0) > 0$, then by using $f(v) \ge f'(0)v$ and $g(u) \ge g'(0)u$ for $u, v \ge 0$, and similar to (4.4), we have

$$\begin{aligned} 0 &= \int_{-1}^{1} [u'' + \lambda f(v)]\Psi_1 dx + \int_{-1}^{1} [v'' + \lambda g(u)]\Phi_1 dx \\ &= \int_{-1}^{1} \Phi_1[-\lambda_* g'(0)u + \lambda g(u)]dx + \int_{-1}^{1} \Psi_1[-\lambda_* f'(0)v + \lambda f(v)]dx > 0, \end{aligned} \tag{6.2}$$

if $\lambda \ge \lambda_*$ and $u, v > 0$, where (Φ_1, Ψ_1) is same as in the proof of Theorem 4.1. Hence (5.1) has no positive solution when $\lambda \ge \lambda_*$. On the other hand, if (i) $f'(0) > 0$, $g'(0) > 0$, and $\lambda < \lambda_*$, or (ii) at least one of $f'(0)$ and $g'(0)$ equals to 0, then one can use the results in [4, 10] to obtain the existence of a positive solution of (1.1) for any bounded smooth domain in $\mathbb{R}^n$ by using degree theory or variational method.

For the one-dimensional problem (5.1), each positive solution corresponds to an $(\alpha, \beta(\alpha)) \in N$, from Theorem 5.5. To prove the uniqueness, we prove that $R'(\alpha) < 0$ for any $\alpha > 0$. For that purpose, we consider the solution (A_c, B_c) of the initial value problem

$$\begin{gathered} A_c'' + f'(v)B_c = 0, \quad 0 < x < R, \\ B_c'' + g'(u)A_c = 0, \quad 0 < x < R, \\ A_c(0) = 1, \quad A_c'(0) = 0, \\ B_c(0) = c > 0, \quad B_c'(0) = 0, \end{gathered} \tag{6.3}$$

for $c \geq 0$. Apparently $(A_c, B_c) = (\phi_1, \psi_1) + c(\phi_2, \psi_2)$. From (5.16), it is sufficient to determine the sign of $A_c(R(\alpha))$ (which also equals to the sign of $B_c(R(\alpha))$), where $c = \beta'(\alpha)$. To the contrary, we assume that $A_c(R(\alpha)) \geq 0$ and $B_c(R(\alpha)) \geq 0$. First we prove that $A_c(x)$ and $B_c(x)$ must change sign. Suppose that $A_c(x) > 0$ and $B_c(x) > 0$ for $x \in [0, R(\alpha))$. By integrating the equation $B_c(u'' + f(v)) = 0$ and $u(B_c'' + g'(u)A_c) = 0$ on $(0, R(\alpha))$, we obtain

$$\int_0^{R(\alpha)} [g'(u)uA_c - f(v)B_c]dx = B_c(R(\alpha))u'(R(\alpha)). \tag{6.4}$$

Similarly we also have

$$\int_0^{R(\alpha)} [f'(v)vB_c - g(u)A_c]dx = A_c(R(\alpha))v'(R(\alpha)). \tag{6.5}$$

Then by adding (6.4) and (6.5), we obtain

$$\begin{aligned}
&\int_0^{R(\alpha)} [g'(u)u - g(u)]A_c dx + \int_0^{R(\alpha)} [f'(v)v - f(v)]B_c dx \\
&= B_c(R(\alpha))u'(R(\alpha)) + A_c(R(\alpha))v'(R(\alpha)).
\end{aligned}$$

Since f and g are superlinear and $A_c(x) > 0$ and $B_c(x) > 0$ for $x \in [0, R(\alpha))$, then the left hand side is positive. On the other hand, the right hand side is non-positive since $u'(R(\alpha)) < 0$ and $v'(R(\alpha)) < 0$. This is a contradiction.

Therefore $A_c(x)$ and $B_c(x)$ must change sign in $(0, R(\alpha))$. To track the sign-changing of A_c and B_c, we define

$$c_1(x) = -\frac{\phi_1(x)}{\phi_2(x)}, \quad 0 < x \leq R, \quad c_2(x) = -\frac{\psi_1(x)}{\psi_2(x)}, \quad 0 < x \leq R, \tag{6.6}$$

where (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are solutions of (5.11) and (5.12) respectively. Then from proof above, the graphs of $c_1(x)$ and $c_2(x)$ must intersect in $(0, R)$. Let x_* be the smallest $x > 0$ such that $c_1(x) = c_2(x) = c^* > 0$. From the definition of $c_i(x)$, we have

$$\begin{aligned}
c_1'(x) &= \frac{\phi_1\phi_2' - \phi_2\phi_1'}{\phi_2^2} = \frac{\int_0^x f'(v)(\psi_1\phi_2 - \psi_2\phi_1)ds}{\phi_2^2} \\
&= \frac{\int_0^x f'(v)\phi_2\psi_2(c_1 - c_2)ds}{\phi_2^2}, \\
c_2'(x) &= \frac{\psi_1\psi_2' - \psi_2\psi_1'}{\psi_2^2} = \frac{\int_0^x g'(u)(\psi_2\phi_1 - \psi_1\phi_2)ds}{\psi_2^2} \\
&= \frac{\int_0^x g'(u)\phi_2\psi_2(c_2 - c_1)ds}{\psi_2^2}.
\end{aligned} \tag{6.7}$$

Since $c_1(x) - c_2(x) > 0$ for $x \in (0, x_*)$, then $c_1'(x) < 0$, $c_2'(x) > 0$, and $c_1(x) > c^* > c_2(x)$ for $x \in (0, x_*)$. This also implies that for $c = c^*$, $A_{c^*}(x) > 0$ and $B_{c^*}(x) > 0$ for $x \in [0, x_*)$ and $A_{c^*}(x_*) = B_{c^*}(x_*) = 0$.

Suppose that the graphs of $c_1(x)$ and $c_2(x)$ have another intersection point. Let x^{**} be the smallest $x \in (x^*, R]$ such that $c_1(x) = c_2(x) = c^{**} > 0$. Then

$c_1(x) < c_2(x)$ for $x \in (x^*, x^{**})$. Note that

$$\begin{aligned} c_1''(x) &= \frac{f'(v)\phi_2\psi_2(c_1-c_2)\phi_2^2 - 2\phi_2\phi_2' \int_0^x f'(v)\phi_2\psi_2(c_1-c_2)ds}{\phi_2^4}, \\ c_2''(x) &= \frac{g'(u)\phi_2\psi_2(c_2-c_1)\psi_2^2 - 2\psi_2\psi_2' \int_0^x g'(u)\phi_2\psi_2(c_2-c_1)ds}{\psi_2^4}. \end{aligned} \tag{6.8}$$

If there exists $x_1 \in (x^*, x^{**})$ such that $c_1'(x_1) = 0$, then

$$c_1''(x_1) = \frac{f'(v)\phi_2\psi_2(c_1-c_2)}{\phi_2^2}\Big|_{x=x_1} > 0,$$

since $c_1(x_1) < c_2(x_1)$ and $f'(v(x)) > 0$, $\phi_2(x) < 0$, $\psi_2(x) > 0$ for $x \in (0, R]$. Similarly, if there exists $x_2 \in (x^*, x^{**})$ such that $c_2'(x_2) = 0$, then

$$c_2''(x_2) = \frac{g'(u)\phi_2\psi_2(c_2-c_1)}{\phi_2^2}\Big|_{x=x_2} < 0.$$

This combining with $c_1'(x) < 0$ and $c_2'(x) > 0$ in $(0, x^*]$ implies that $c_1(x)$ has at most one critical point which is a local minimum, and $c_2(x)$ has at most one critical point which is a local maximum. In particular, the horizontal line $c = c^{**}$ intersects each of $c = c_1(x)$ and $c = c_2(x)$ at most once for $x \in [0, x^{**})$, or equivalently, each of $A_{c^{**}}(x)$ and $B_{c^{**}}(x)$ changes sign in $(0, x^{**})$ at most once. Then there are following three possible cases:

(i) Both of $A_{c^{**}}(x)$ and $B_{c^{**}}(x)$ change sign in $(0, x^{**})$ exactly once.
(ii) $A_{c^{**}}(x)$ changes sign in $(0, x^{**})$ exactly once, and $B_{c^{**}}(x)$ does not change sign in $(0, x^{**})$.
(iii) $B_{c^{**}}(x)$ changes sign in $(0, x^{**})$ exactly once, and $A_{c^{**}}(x)$ does not change sign in $(0, x^{**})$.

If case (i) occurs, then

$$A_{c^{**}}(x^{**}) = B_{c^{**}}(x^{**}) = 0, \quad A'_{c^{**}}(x^{**}) \ge 0, \quad B'_{c^{**}}(x^{**}) \ge 0. \tag{6.9}$$

Using the equation of A_c with $c = c^{**}$ and the equation of v', we obtain

$$A''_{c^{**}} v' - (v')'' A_{c^{**}} + f'(v)v' B_{c^{**}} - g'(u)u' A_{c^{**}} = 0. \tag{6.10}$$

Similarly we have

$$B''_{c^{**}} u' - (u')'' B_{c^{**}} + g'(u)u' A_{c^{**}} - f'(v)v' B_{c^{**}} = 0. \tag{6.11}$$

Adding (6.10) and (6.11), we obtain

$$A''_{c^{**}} v' - (v')'' A_{c^{**}} + B''_{c^{**}} u' - (u')'' B_{c^{**}} = 0. \tag{6.12}$$

Define a function

$$P(x) = A'_{c^{**}}(x)v'(x) - v''(x)A_{c^{**}}(x) + B'_{c^{**}}(x)u'(x) - u''(x)B_{c^{**}}(x),$$

$x \in [0, R]$. Then (6.12) implies that $P'(x) \equiv 0$ for $x \in (0, R)$. Hence, for $x \in [0, R]$,

$$P(x) \equiv P(0) = -v''(0)A_{c^{**}}(0) - u''(0)B_{c^{**}}(0) = g(u(0)) + f(v(0))c^{**} > 0. \tag{6.13}$$

However, from (6.9), we have

$$P(x^{**}) = A'_{c^{**}}(x^{**})v'(x^{**}) + B'_{c^{**}}(x^{**})u'(x^{**}) \le 0, \tag{6.14}$$

which is a contradiction with (6.13).

If case (ii) occurs, suppose the unique zero of $A_{c^{**}}$ in $(0, x^{**})$ is x_1, then

$$A_{c^{**}}(x_1) = A_{c^{**}}(x^{**}) = B_{c^{**}}(x^{**}) = 0, \quad A_{c^{**}}(x) < 0, \quad B_{c^{**}}(x) > 0 \tag{6.15}$$

in (x_1, x^{**}), and

$$A'_{c^{**}}(x_1) \le 0, \quad A'_{c^{**}}(x^{**}) \ge 0. \tag{6.16}$$

Then multiplying the equation of $A_{c^{**}}$ by u', multiplying the equation of u' by $A_{c^{**}}$, subtracting and integrating on (x_1, x^{**}), we obtain

$$A'_{c^{**}}(x^{**})u'(x^{**}) - A'_{c^{**}}(x_1)u'(x_1) = \int_{x_1}^{x^{**}} f'(v)v'(A_{c^{**}} - B_{c^{**}})dx. \tag{6.17}$$

The left hand side of (6.17) is non-positive since $u'(x) < 0$ in $(0, R)$ and (6.16) holds, while the right hand side of (6.17) is positive since $f'(v) > 0$, $v' < 0$ in $(0, R)$ and (6.15) holds. That is a contradiction. If case (iii) occurs, we can derive a similar contradiction as case (ii).

This proves that $c_1(x)$ and $c_2(x)$ cannot intersect again in $(0, R)$, and we have $c_1(R) < c_2(R)$. If $c = \beta'(\alpha) \le c_1(R)$ or $c \ge c_2(R)$, we have $\mathrm{sgn}(A_c(R(\alpha)) \ne \mathrm{sgn}(B_c(R(\alpha))$, which is not possible. Hence we must have $c = \beta'(\alpha)$ belongs to $(c_1(R), c_2(R))$, which implies that $A_c(R(\alpha)) < 0$ and $B_c(R(\alpha)) < 0$. This shows that $R(\alpha) < 0$ for any $\alpha > 0$.

This shows that the function $\alpha \mapsto R(\alpha)$ is a one-to-one and onto function from $(0, \infty)$ to its range. For the case that at least one of $f'(0)$ and $g'(0)$ equals to 0, the range is $(0, \infty)$, and for the case that $f'(0) > 0$, $g'(0) > 0$, and $\lambda < \lambda_*$, the range is $(0, \lambda_*^2)$. This proves the uniqueness of positive solution to (5.1) for both cases. Let the unique positive solution of (5.1) be $(u_\lambda(x), v_\lambda(x))$. Since $\lambda(\alpha) = \sqrt{R(\alpha)}$ and $R(\alpha)$ is decreasing in α, then $\alpha = u_\lambda(0)$ is decreasing in λ. One can also obtain that $\beta = v_\lambda(0)$ is decreasing in λ since $\beta'(\alpha) > 0$. This completes the proof. $\square$

Acknowledgments. This work was partially supported by a grant from the Simon's Foundation (# 317872) to Ratnasingham Shivaji, and an NSF grant DMS-1022648 to Junping Shi. We thank anonymous reviewers for careful reading and helpful comments which improve the manuscript.

References

[1] Amann, Herbert; Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. *SIAM Rev.* **18** (1976), no. 4, 620–709.
[2] An, Yulian; Chern, Jann-Long; Shi, Junping; Uniqueness of positive solution to a coupled coorperative system on an interval. Preprint.
[3] Brown, K. J.; Ibrahim, M. M. A.; Shivaji, R.; S-shaped bifurcation curves. *Nonlinear Anal.* **5** (1981), no. 5, 475–486.
[4] Clément, Ph.; de Figueiredo, D. G.; Mitidieri, E.; Positive solutions of semilinear elliptic systems. *Comm. Partial Differential Equations* **17** (1992), no. 5-6, 923–940.
[5] Crandall, Michael G.; Rabinowitz, Paul H.; Bifurcation from simple eigenvalues. *J. Functional Analysis*, **8** (1971), 321–340.
[6] Crandall, Michael G.; Rabinowitz, Paul H.; Bifurcation, perturbation of simple eigenvalues and linearized stability. *Arch. Rational Mech. Anal.* **52** (1973), 161–180.
[7] Crandall, Michael G.; Rabinowitz, Paul H.; Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems. *Arch. Rational Mech. Anal.* **58** (1975), no. 3, 207–218.
[8] Dalmasso, Robert; Existence and uniqueness of positive radial solutions for the Lane-Emden system. *Nonlinear Anal.* **57** (2004), no. 3, 341–348.
[9] de Figueiredo, Djairo G.; Semilinear elliptic systems. Nonlinear functional analysis and applications to differential equations (Trieste, 1997), 122–152, World Sci. Publ., River Edge, NJ, 1998.
[10] de Figueiredo, Djairo G.; Felmer, Patricio L.; On superquadratic elliptic systems. *Trans. Amer. Math. Soc.* **343** (1994), no. 1, 99–116.

[11] Gidas, B.; Ni, Wei Ming; Nirenberg, L.; Symmetry and related properties via the maximum principle. *Comm. Math. Phys.* **68** (1979), no. 3, 209–243.
[12] Keller, Herbert B.; Cohen, Donald S.; Some positone problems suggested by nonlinear heat generation. *J. Math. Mech.* **16** (1967) 1361–1376.
[13] Korman, Philip; Global solution curves for semilinear systems. *Math. Methods Appl. Sci.* **25** (2002), no. 1, 3–20.
[14] Korman, Philip; Global solution curves for a class of elliptic systems. *Appl. Anal.* **89** (2010), no. 6, 849–860.
[15] Korman, Philip; Li, Yi; Ouyang, Tiancheng; Exact multiplicity results for boundary value problems with nonlinearities generalising cubic. *Proc. Roy. Soc. Edinburgh Sect. A* **126** (1996), no. 3, 599–616.
[16] Korman, Philip; Li, Yi; Ouyang, Tiancheng; An exact multiplicity result for a class of semilinear equations. *Comm. Partial Differential Equations* **22** (1997), no. 3-4, 661–684.
[17] Korman, Philip; Shi, Junping; On Lane-Emden type systems. *Discrete Contin. Dyn. Syst.* (2005), suppl., 510–517.
[18] Lions, P.-L.; On the existence of positive solutions of semilinear elliptic equations. *SIAM Review* **24** (1982), no. 4, 441–467.
[19] Lin, Chang-Shou; Ni, Wei-Ming; A counterexample to the nodal domain conjecture and a related semilinear equation. *Proc. Amer. Math. Soc.* **102** (1988), no. 2, 271–277.
[20] Liu, Ping; Shi, Junping; Wang, Yuwen; Imperfect transcritical and pitchfork bifurcations. *Jour. Func. Anal.* **251** (2007), no. 2, 573–600.
[21] Ouyang, Tiancheng; Shi, Junping; Exact multiplicity of positive solutions for a class of semilinear problem. *J. Differential Equations* **146** (1998), no. 1, 121–156.
[22] Ouyang, Tiancheng; Shi, Junping; Exact multiplicity of positive solutions for a class of semilinear problem:II. *J. Differential Equations* **158** (1999), no. 1, 94–151.
[23] Serrin, James; Zou, Henghui; Existence of positive solutions of the Lane-Emden system. *Atti Sem. Mat. Fis. Univ. Modena* **46** (1998), suppl., 369–380.
[24] Serrin, James; Zou, Henghui; Existence of positive entire solutions of elliptic Hamiltonian systems. *Comm. Partial Differential Equations* **23** (1998), no. 3-4, 577–599.
[25] Shi, Junping; Persistence and bifurcation of degenerate solutions, *Jour. Func. Anal.* **169**, (1999), No. 2, 494–531.
[26] Shi, Junping; Shivaji, Ratnasingham; Global bifurcation for concave semipositon problems. *Advances in Evolution Equations: Proceedings in honor of J.A.Goldstein's 60th birthday*, Edited by G.R. Goldstein, R. Nagel, and S. Romanelli, Marcel Dekker, Inc., New York, Basel, 385–398, (2003).
[27] Shivaji, Ratnasingham; Uniqueness results for a class of positone problems. *Nonlinear Anal.* **7** (1983), no. 2, 223–230.
[28] Shivaji, Ratnasingham; Remarks on an S-shaped bifurcation curve. *J. Math. Anal. Appl.* **111** (1985), no. 2, 374–387.
[29] Sirakov, Boyan; Notions of sublinearity and superlinearity for nonvariational elliptic systems. *Discrete Contin. Dyn. Syst.* **13** (2005), no. 1, 163–174.
[30] Sweers, Guido; Strong positivity in $C(\overline{\Omega})$ for elliptic systems. *Math. Z.* **209** (1992), no. 2, 251–271.
[31] Troy, William C.; Symmetry properties in systems of semilinear elliptic equations. *J. Differential Equations* **42** (1981), no. 3, 400–413.

Junping Shi
Department of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795, USA
E-mail address: shij@math.wm.edu

Ratnasingham Shivaji
Department of Mathematics & Statistics, University of North Carolina at Greensboro, Greensboro, NC 27412, USA
E-mail address: shivaji@uncg.edu