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EXISTENCE OF THREE SOLUTIONS FOR A TWO-POINT SINGULAR BOUNDARY-VALUE PROBLEM WITH AN UNBOUNDED WEIGHT

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Abstract. We show the existence of three solution for the singular boundary-value problem

$$\begin{gathered} -z'' = h(t)\frac{f(z)}{z^\beta} \quad \text{in } (0,1),\\ z(t) > 0 \quad \text{in } (0,1),\\ z(0) = z(1) = 0, \end{gathered}$$

where $0 < \beta < 1$, $f \in C^1([0,\infty),(0,\infty))$ and $h \in C((0,1],(0,\infty))$ is such that $h(t) \le C/t^\alpha$ on $(0,1]$ for some $C > 0$ and $0 < \alpha < 1-\beta$. When there exist two pairs of sub-supersolutions (ψ_1,ϕ_1) and (ψ_2,ϕ_2) where $\psi_1 \le \psi_2 \le \phi_1, \psi_1 \le \phi_2 \le \phi_1$ with $\psi_2 \not\le \phi_2$, and ψ_2,ϕ_2 are strict sub and super solutions. The establish the existence of at least three solutions z_1,z_2,z_3 satisfying $z_1 \in [\psi_1,\phi_2]$, $z_2 \in [\psi_2,\phi_1]$ and $z_3 \in [\psi_1,\phi_1] \setminus ([\psi_1,\phi_2] \cup [\psi_2,\phi_1])$.

1. Introduction

In this article we consider the two point boundary-value problem

$$\begin{gathered} -z'' = h(t)\frac{f(z)}{z^\beta} \quad \text{in } (0,1),\\ z(t) > 0 \quad \text{in } (0,1),\\ z(0) = z(1) = 0, \end{gathered} \tag{1.1}$$

where $0 < \beta < 1$, $f \in C^1([0,\infty),(0,\infty))$ is nondecreasing, $h \in C((0,1],(0,\infty))$ is such that $h(t) \le C/t^\alpha$ on $(0,1]$ for some $C > 0$ and $0 < \alpha < 1-\beta$. In particular, we are interested in weights h which are unbounded at the origin. This makes the reaction term in (1.1) singular at $t = 0$ not only due to the term z^β in the denominator but also due to this unbounded weight h.

Our main focus in this paper is to establish the existence of three positive solutions in $C^1[0,1] \cap C^2(0,1)$ when certain pair of sub-super solutions can be constructed for (1.1). By a sub solution ψ of (1.1) we mean a function $\psi \in$

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$C^2(0,1)\cap C[0,1]$ such that

$$
\begin{gathered}
-\psi'' \leq h(t)\frac{f(\psi)}{\psi^\beta} \quad \text{in } (0,1),\\
\psi(t)>0 \quad \text{in } (0,1),\\
\psi(0)=\psi(1)=0,
\end{gathered}
$$

and by a super solution of (1.1) we mean a function $\phi \in C^2(0,1)\cap C[0,1]$ such that

$$
\begin{gathered}
-\phi'' \geq h(t)\frac{f(\phi)}{\phi^\beta} \quad \text{in } (0,1),\\
\phi(t)>0 \quad \text{in } (0,1),\\
\phi(0)=\phi(1)=0.
\end{gathered}
$$

Let $g(z)=(f(z)-f(0))/z^\beta$, then problem (1.1) can be equivalently re-formulated as

$$
\begin{gathered}
-z'' - \frac{h(t)f(0)}{z^\beta} = h(t)g(z) \quad \text{in } (0,1),\\
z(0)=z(1)=0.
\end{gathered} \tag{1.2}
$$

By the mean value theorem $g(t)=f'(s)t^{1-\beta}$ for some $s\in(0,t)$. Since $0<\beta<1$ and $\lim_{s\to 0}|f'(s)|<\infty$, we have $g(0)=0$. Thus we can treat g as a Hölder continuous function on $[0,\infty)$ with $g(0)=0$ and extend g to be identically zero on the negative x-axis. We assume that

- (G1) There exists a non-negative constant $\tilde{k}$ such that $\tilde{g}(t)=g(t)+\tilde{k}t$ is strictly increasing in $[0,\infty)$.

Remark 1.1. Without lost of generality, we assume throughout this article that g is strictly increasing in $\mathbb{R}^+$(i.e $\tilde{k}=0$ in (G1)). If not, we can study

$$
\begin{gathered}
-z'' - \frac{h(t)f(0)}{z^\beta} + \tilde{k}z = h(t)\tilde{g}(z) \quad \text{in } (0,1),\\
z(0)=z(1)=0
\end{gathered} \tag{1.3}
$$

instead of (1.2) and establish our results.

In this article, we prove the following results:

Theorem 1.2 (Minimal and maximal solutions)**.** *Let ψ,ϕ be positive sub and super solution of* (1.1) *satisfying $\psi\leq\phi$. Then there exists a minimal as well as a maximal solution for* (1.1) *in the ordered interval $[\psi,\phi]$.*

Theorem 1.3 (Three solution theorem)**.** *Suppose there exists two pairs of ordered sub and super solutions (ψ_1,ϕ_1) and (ψ_2,ϕ_2) of* (1.1) *with the property that $\psi_1\leq\psi_2\leq\phi_1$, $\psi_1\leq\phi_2\leq\phi_1$ and $\psi_2\not\leq\phi_2$. Additionally assume that ψ_2,ϕ_2 are not solutions of* (1.1)*. Then there exists at least three solutions z_1,z_2,z_3 for* (1.1) *where $z_1\in[\psi_1,\phi_2]$, $z_2\in[\psi_2,\phi_1]$ and $z_3\in[\psi_1,\phi_1]\setminus([\psi_1,\phi_2]\cup[\psi_2,\phi_1])$.*

We first note that when a sub solution ψ and a super solution ϕ exists such that $\psi\leq\phi$ there are results in the history which establish a solution z for (1.1) such that $z\in[\psi,\phi]$ (see [2]). Here the author establishes the result by first studying the non singular boundary value problem

$$
\begin{gathered}
-z'' = h(t)\frac{f(z)}{z^\beta} \quad \text{in } [\epsilon,1-\epsilon],\\
z(\epsilon)=\psi(\epsilon), \quad z(1-\epsilon)=\psi(1-\epsilon)
\end{gathered} \tag{1.4}
$$

for $\epsilon > 0$ and then by analyzing the limit of the solution $u_\epsilon \in [\psi, \phi]$ of (I_ϵ) as $\epsilon \to 0$. In this paper we will provide a direct method and in Theorem 1.2 we establish also the existence of maximal and minimal solutions of (1.1) in $[\psi, \phi]$.

Next, in [6] the authors study positive radial solutions on exterior domains to problem of the form

$$\begin{gathered} -\Delta u = \lambda K(|x|) \frac{f(u)}{u^\beta} \quad \text{in } \Omega_E, \\ u(x) = 0 \text{ on } |x| = r_0, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{gathered} \tag{1.5}$$

where $\lambda > 0$, $\Omega_E = \{x \in \mathbb{R}^N : |x| > r_0, r_0 > 0, N > 2\}$ and $K \in C((r_0, \infty), (0, \infty))$ such that $K(|x|) \to 0$ as $|x| \to \infty$. Assume $0 \le K(r) \le \frac{\tilde{C}}{r^{N+\mu_1}}$ where $\beta(N-2) < \mu_1 < N-2$. Then the change of variables $r = |x|$ and $t = (\frac{r}{r_0})^{2-N}$ transform (1.5) into

$$\begin{gathered} -u'' = \lambda \tilde{h}(t) \frac{f(u)}{u^\beta} \quad \text{in } (0,1), \\ u(0) = u(1) = 0, \end{gathered} \tag{1.6}$$

where

$$\tilde{h}(t) \le \frac{\tilde{C}}{(N-2)^2 r_0^{N-2+\mu_1}} t^{-1+\frac{\mu_1}{N-2}}$$

and hence this study reduces to study of positive solutions to the boundary-value problem of the form (1.1). In [6], the authors study classes of nonlinearities f where for certain range of λ they are able to construct the two pairs of sub-super solutions as in Theorem 1.3. Using the result by Cui [2] they were able to conclude the existence of two positive solutions for these ranges of λ. However, now using Theorem 1.3 we conclude that there are in fact three positive solutions in this range of λ. This three solutions theorem (Theorem 1.3) is motivated by earlier work for non-singular problems by Amann[1], Shivaji[8], and by our recent work in [3] for singular problems on bounded domain without the burden of an unbounded weight in the reaction term. We will establish some preliminaries and then prove Theorems .

2. Proofs of main theorems

Now onwards instead of looking for a positive solution of (1.1) we work with the equivalent formulation (1.2).

Lemma 2.1. *There exists a unique positive weak solution $\theta \in H_0^1(0,1) \cap C^2(0,1)$ to $-\theta'' - \frac{h(t)}{\theta^\beta} = 0$ in $(0,1)$ and $\theta(0) = \theta(1) = 0$.*

Proof. Let us define the functional

$$E_1(u) = \frac{1}{2} \int_0^1 |u'|^2 - \frac{1}{1-\beta} \int_0^1 h(t)(u^+)^{1-\beta}. \tag{2.1}$$

By the Sobolev embedding $H_0^1(0,1) \hookrightarrow C^\gamma[0,1]$ for some $\gamma \in (0,1)$ and the fact that $\alpha < 1 - \beta$ we have E_1 is a well-defined map in the entire space $H_0^1(0,1)$. The functional E_1 restricted to the set $H^+ = \{u \in H_0^1(0,1) : u \ge 0\}$ is convex. Let φ_1

be a positive eigenfunction corresponding to the first eigenvalue λ_1 of $-u'' = \lambda u$ with $u(0) = u(1) = 0$. Then, we note that for all ϵ small enough

$$E_1(\epsilon\varphi_1) = \frac{\epsilon^2\lambda_1}{2}\int_0^1 {\varphi_1}^2 - \frac{\epsilon^{1-\beta}}{1-\beta}\int_0^1 h(t)\varphi_1^{1-\beta} < 0 = E_1(0).$$

Hence coercive and weakly lower semicontinuous functional E_1 admits a non-zero global minimizer θ in $H_0^1(0,1)$. We observe that $E_1(|\theta|) \le E_1(\theta)$. Unless $\theta^- \equiv 0$, this would contradict the fact that θ is a global minimizer of E_1. Hence $\theta(t) \ge 0$ in $[0,1]$. One can repeat the arguments in [5, Lemma A.2] and infer that the functional E_1 is Gateaux differentiable at any $u \ge \epsilon\varphi_1(t)$, and for $w \in H_0^1(0,1)$ we have

$$\langle E_1'(u), w\rangle = \int_0^1 u'w' - \int_0^1 h(t)u^{-\beta}w.$$

Thus to prove that global minimizer θ is a weak solution, it suffices to prove the following claim.

Claim: $\theta(t) \ge \epsilon_0\varphi_1(t)$ for some positive constant ϵ_0.

For a given $\epsilon > 0$, let $v = (\epsilon\varphi_1 - \theta)^+$ and $\Omega_+ = \{x \in (0,1) : v(x) > 0\}$. On the contrary, suppose that v does not vanish in Ω for ϵ small enough, and then we derive a contradiction. Clearly, $\theta + v \ge \epsilon\varphi_1$ and

$$\begin{aligned}
\langle E_1'(\theta+v), v\rangle &= \int_{\Omega_+} \epsilon\varphi_1'(\epsilon\varphi_1' - \theta') - \int_{\Omega_+} h(t)(\epsilon\varphi_1)^{-\beta}(\epsilon\varphi_1 - \theta) \\
&= \lambda_1\int_{\Omega_+}(\epsilon\varphi_1)(\epsilon\varphi_1 - \theta) - \int_{\Omega_+} h(t)(\epsilon\varphi_1)^{-\beta}(\epsilon\varphi_1 - \theta) \qquad (2.2)\\
&= \int_{\Omega_+}(\epsilon\varphi_1)^{-\beta}[\lambda_1(\epsilon\varphi_1)^{1+\beta} - h(t)](\epsilon\varphi_1 - \theta).
\end{aligned}$$

If we choose ϵ_0 small enough so that $\lambda_1(\epsilon\varphi_1)^{1+\beta} - \inf_{t\in(0,1)} h(t) < 0$, then we have

$$(E_1'(\theta+v), v) < 0 \quad \text{for all } \epsilon \le \epsilon_0. \tag{2.3}$$

To complete the claim we need to show that Ω_+ is empty or $v \equiv 0$ for ϵ small enough. Let $\xi(s) = E_1(\theta + sv)$ for $s \in [0,\infty)$. The function $\xi : [0,\infty) \to \mathbb{R}$ is convex, since it is a composition of the convex function E_1 restricted to H^+ with a linear function. We already know that θ is a global minimizer of E_1 and hence we have $\xi(s) \ge \xi(0)$. Also $\theta + sv \ge \max\{\theta, \epsilon s\varphi_1\} \ge s\epsilon\varphi_1$ whenever $0 < s \le 1$. Thus ξ is differentiable for all $s \in (0,1]$. Also we note that ξ' is nondecreasing and is non-negative since $\xi(s) \ge \xi(0)$. Thus,

$$0 \le \xi'(1) - \xi'(s) = (E_1'(\theta+v), v) - \xi'(s) \le (E_1'(\theta+v), v). \tag{2.4}$$

From (2.3) and (2.4) we have a contradiction for all $\epsilon \le \epsilon_0$. Hence $v = 0$, or in other words $\theta \ge \epsilon_0\varphi_1$. Finally, we conclude that $\theta(t)$ is a weak solution of $-\theta'' - \frac{h(t)}{\theta^\beta} = 0$ and by the interior regularity $\theta \in C^2(0,1)$. Since $h(t) > 0$ we can prove the uniqueness of weak solution by a standard test function approach(for e.g. see [3, Lemma 3.2]). ☐

Lemma 2.2. *For a given nonnegative function $v \in C[0,1]$ there exists a unique weak solution $w \in C^{1,\epsilon}[0,1]$ of $-w'' - \frac{h(t)}{w^\beta} = v(t)$ in $(0,1)$ and $w(0) = w(1) = 0$. Also there exists a constant $C = C(\|v\|_\infty, \beta, \alpha)$ such that*

$$\|w\|_{C^{1,\epsilon}[0,1]} \le C \quad \textit{where } \epsilon = 1 - \beta - \alpha.$$

Proof. The existence of a unique solution $w \in H_0^1(0,1)$ follows exactly as in [4, Lemma 4.1] or [3, Lemma 3.2]. Note that $-w'' - \frac{h(t)}{w^\beta} = v(t) \geq 0 = -\theta'' - \frac{h(t)}{\theta^\beta}$ and hence by comparison principle $w(t) \geq \theta(t) \geq \epsilon_0\varphi_1$. By the Greens representation formula:

$$w(t) = \int_0^1 G(t,\xi)\Big(\frac{h(\xi)}{w^\beta} + v(\xi)\Big)d\xi, \tag{2.5}$$

where

$$G(t,\xi) = \begin{cases} (1-t)\xi & 0 \leq \xi \leq t \\ (1-\xi)t & t \leq \xi \leq 1. \end{cases}$$

If we write $h_1(\xi) = \frac{h(\xi)}{w^\beta} + v(\xi)$, then from the lower estimate on w there exists C_0 such that

$$h_1(\xi) \leq \frac{C_0\xi^{-\alpha}}{\varphi_1^\beta(\xi)} \leq C_0\xi^{-\alpha}d(\xi)^{-\beta},$$

where the distance function $d(\xi)$ is

$$d(\xi) = \begin{cases} \xi & 0 \leq \xi \leq \frac{1}{2} \\ (1-\xi) & \frac{1}{2} \leq \xi \leq 1. \end{cases}$$

For $0 < t < s < 1$ we have

$$w'(t) - w'(s) = \int_t^s h_1(\xi)d\xi.$$

We can write $h_1(\xi) \leq \tilde{C}d(\xi)^{-\alpha-\beta}$ where $\tilde{C}$ depends on $\|v\|_\infty$ and thus there exists a constant $C = C(\|v\|_\infty, \beta, \alpha)$ such that

$$|w'(t) - w'(s)| \leq C|t-s|^{1-\beta-\alpha} \tag{2.6}$$

which completes the proof. □

We now recall some results from Amann[1]. Let $e \in C^2[0,1]$ denote the unique positive solution of

$$\begin{gathered} -e''(s) = 1 \quad \text{in } (0,1), \\ e(0) = e(1) = 0. \end{gathered}$$

Then $e(s) = \frac{1}{2}s(1-s)$ and $e'(1) = -e'(0) = -1$. Also $e(s) \geq k\,d(s)$ for some constant $k > 0$. Let $C_e[0,1]$ be the set of functions in $u \in C_0[0,1]$ such that $-se \leq u \leq se$ for some $s > 0$. $C_e[0,1]$ equipped with $\|u\|_e = \inf\{s > 0 : -se \leq u \leq se\}$ is a Banach space. Also the following continuous embedding holds

$$C_0^1[0,1] \hookrightarrow C_e[0,1] \hookrightarrow C_0[0,1]. \tag{2.7}$$

Further $C_e[0,1]$ is an ordered Banach space(OBS) whose positive cone $P_e = \{u \in C_e[0,1] : u(s) \geq 0\}$ is normal and has non empty interior. In particular the interior P_e^0 consists of all those functions $u \in C_e[0,1]$ with $s_1e \leq u \leq s_2e$ for some $s_1, s_2 > 0$.

For a given $v \in C_e[0,1]$, let $\tilde{v}(t) = h(t)g(v(t))$. Using the assumptions on g and the blow up estimate on h, we have $|\tilde{v}(t)| \leq Ct^{-\alpha}|v(t)|^{1-\beta}$. This upper bound implies that $\tilde{v} \in C_0[0,1]$ and by Lemma 2.2 there exists a unique solution $w \in C^{1,\epsilon}[0,1] \cap H_0^1(0,1)$ solving $-w'' - \frac{f(0)h(t)}{w^\beta} = h(t)g(v(t))$. We make the following definition.

Definition 2.3. We define the operator $A_g : C_e[0,1] \to C_0^{1,\epsilon}[0,1]$ as $A_g(v) = w$, where w is the unique positive weak solution of $-w'' - \frac{h(t)f(0)}{w^\beta} = h(t)g(v(t))$ in $(0,1)$ and $w(0) = w(1) = 0$.

We first establish the following result.

Proposition 2.4. *The map $A_g : C_e[0,1] \to C_e[0,1]$ is completely continuous and strictly increasing.*

Proof. Clearly A_g maps $C_e[0,1]$ into $C_0^{1,\epsilon}[0,1]$ and we need to prove that the mapping is continuous. For a proof consider $v, v_0 \in C_e[0,1]$ and $\|v - v_0\|_{C_e} < \epsilon$. Let $A_g(v_0) = w_0$ and $A_g(v) = w$. Then from the weak formulation of the solution

$$\begin{aligned}
&\int_0^1 |w' - w_0'|^2 \\
&= \int_0^1 h(t)f(0)\left(\frac{1}{w^\beta} - \frac{1}{w_0^\beta}\right)(w - w_0) + \int_0^1 h(t)(g(v) - g(v_0))(w - w_0) \\
&\le \int_0^1 h(t)(g(v) - g(v_0))(w - w_0).
\end{aligned}$$

Since g is assumed to be Hölder continuous of exponent $1-\beta$ and by the continuous embedding (2.7) we have the estimate

$$\begin{aligned}
|g(v(t)) - g(v_0(t))| &\le C_0 |v(t) - v_0(t)|^{1-\beta} \\
&\le C_0 \|v - v_0\|_{C_0(\overline{\Omega})}^{1-\beta} \\
&\le C_1 \|v - v_0\|_{C_e(\Omega)}^{1-\beta} < C_1 \epsilon^{1-\beta}.
\end{aligned}$$

Hence,

$$\int_0^1 |w' - w_0'|^2 \le C_2\, \epsilon^{1-\beta} \int_0^1 \frac{|w - w_0|}{t} t^{1-\alpha}.$$

Now by Hardy's inequality we have

$$\int_0^1 |w' - w_0'|^2 \le C\epsilon^{1-\beta} \|w - w_0\|_{H_0^1(0,1)}.$$

Therefore if $v_n \to v_0$ in $C_e[0,1]$ then the corresponding solutions $A_g(v_n) = w_n \to w_0 = A_g(v_0)$ in $H_0^1(0,1)$. Next we note that if $v_n \to v_0$ in $C_e[0,1]$ then $\tilde{v_n} = h(t)g(v_n)$ is uniformly bounded in $C_0[0,1]$ and hence by Lemma 2.2, $\|w_n\|_{C^{1,\epsilon}[0,1]}$ is bounded. By Ascoli-Arzela theorem w_n has a convergent subsequence in $C^{1,\epsilon'}[0,1]$ for every $\epsilon' < \epsilon$ and the from the previous discussion the limit has to be w_0. Hence $A_g : C_e[0,1] \to C_0^{1,\epsilon'}[0,1]$ is continuous. Since $C_0^{1,\epsilon}[0,1] \subset\subset C_0^1[0,1]$ (compact imbedding), we have $A_g : C_e[0,1] \to C_0^1[0,1]$ is completely continuous. Therefore, $A_g : C_e[0,1] \to C_e[0,1]$ is completely continuous.

To prove the map A_g is strictly increasing we assume that g is strictly increasing (otherwise see Remark 1.1). We need to show that if $v_1 \le v_2$, $v_1 \ne v_2$ then $A_g(v_1) < A_g(v_2)$. By the test function approach one can easily show that $0 < w_1 = A_g(v_1) \le A_g(v_2) = w_2$. Let $\rho : (0,1) \to \mathbb{R}$ be such that $w_1(t) \le \rho(t) \le w_2(t)$ is defined by mean value theorem. Then we have

$$-(w_2 - w_1)'' + \frac{h(t)f(0)}{\rho(t)^{1+\beta}}(w_2 - w_1) = h(t)\left(g(v_2) - g(v_1)\right) \ge 0. \tag{2.8}$$

Since $(w_1 - w_2)|_{\{t=0,1\}} = 0$, we have by Theorem 3 in Chapter 1 of Protter and Weinberger [7], $w_1 < w_2$, or in other words A_g is strictly increasing. □

Lemma 2.5. *The map $A_g : C_e[0,1] \to C_e[0,1]$ is strongly increasing, i.e. $A_g(v_2) - A_g(v_1) \in P_e^0$.*

Proof. The idea of the proof is same as in [3, Theorem 3.6] with a slight change in the singular exponent β. Let us write $\tilde{w} = (w_2 - w_1)$, then from equation (2.8) and by the upper bound on $\frac{h(t)}{\rho^{\beta+1}}$ we have $-\tilde{w}'' + c\,\tilde{w}\,d(t)^{-1-\beta-\alpha} \geq 0$. I f we denote $\beta' = \alpha + \beta$, then $\beta' \in (0,1)$. Thus we obtain

$$
\begin{gathered}
-\tilde{w}'' + \frac{c\tilde{w}}{d(t)^{1+\beta'}} \geq 0 \quad \text{in } (0,1), \\
\tilde{w} > 0 \text{ in } (0,1), \\
\tilde{w}(0) = \tilde{w}(1) = 0.
\end{gathered}
$$

Rest of the proof is exactly as in [3, Theorem 3.6] and hence we skip the details. □

Now Proposition 2.4 combined with [1, Corollary 6.2] easily establishes the proof of Theorem 1.2. Further, since Lemma 2.5 holds, repeating the arguments in [3] the proof of Theorem 1.3 follows.

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