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MULTIPLICITY OF SOLUTIONS OF RELATIVISTIC-TYPE SYSTEMS WITH PERIODIC NONLINEARITIES: A SURVEY

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Abstract. We survey recent results on the multiplicity of T-periodic solutions of differential systems of the form

$$\Big(\frac{u'}{\sqrt{1-|u'|^2}}\Big)' + \nabla_u F(t,u) = e(t)$$

when $F(t,u)$ is ω_i-periodic with respect to u_i $(i = 1,\dots,N)$. Several techniques of critical point theory are used.

1. Motivation: the forced pendulum and corresponding systems

The periodic problem for the forced pendulum equation

$$u'' + a\sin u = e(t), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{1.1}$$

has been for almost one century a source of inspiration for ordinary differential equations and nonlinear functional analysis, and a cornerstone for most nonlinear techniques (see e.g. [16, 18]). In particular its solutions are the critical points of the Lagrangian action functional

$$\mathcal{L}(u) := \int_0^T \Big[\frac{{u'}^2}{2} + a\cos u + eu\Big]\,dt$$

in the Sobolev space $H^1_T = \{u \in H^1([0,T]) : u(0) = u(T)\}$.

In 1922, Hamel [12] proved that *for each* $e \in C([0,T])$ *such that*

$$\overline{e} := T^{-1}\int_0^T e(t)\,dt = 0,$$

there exists at least one solution of (1.1) *minimizing* $\mathcal{L}(u)$ *over* T*-periodic* C^1 *functions.* This result was rapidly forgotten and, following a renewal of interest for the problem, in 1980, due to Castro's application [6] of some minimax method to (1.1), Hamel's theorem was rediscovered independently around 1981 by Willem [27] and Dancer [10] in the more natural framework of H^1_T. Because of the structure of the equation, if u is a solution, the same is true for $u + 2k\pi$ for all $k \in \mathbb{Z}$, so that two

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solutions of (1.1) are called *geometrically distinct* if they do not differ by an integer multiple of 2π.

As shown by the special case of the unforced pendulum, Hamel's existence conclusion is not optimal and, in 1984, the following multiplicity result was proved in [20].

Theorem 1.1. *For each $e \in L^1(0,T)$ such that $\overline{e} = 0$, problem (1.1) has at least two geometrically distinct solutions.*

The second solution was obtained by a mountain pass type argument between a minimizing solution u_0 and the other one $u_0+2\pi$. The unforced case shows that this multiplicity result is optimal if no restriction is made upon a and T. An immediate generalization of Theorem 1.1, based upon the same arguments, holds for $a\sin u$ replaced by a Carathéodory function $f(t,u)$ such that $F(t,u) := \int_0^u f(t,s)\,ds$ is ω periodic in u for a.e. fixed $t \in [0,T]$, and some $\omega > 0$.

The solutions of the N-dimensional corresponding problem

$$u'' + \nabla_u F(t,u) = e(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{1.2}$$

where $e \in L^1(0,T;\mathbb{R}^N)$, $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ and $\nabla_u F : [0,T] \times \mathbb{R}^N \to \mathbb{R}^N$ are Carathéodory functions such that

$$F(t, u + \omega_j e_j) = F(t,u) \quad (j = 1,\dots,N) \tag{1.3}$$

for a.e. $t \in [0,T]$, all $u \in \mathbb{R}^N$, and some $\omega_i > 0$ $(i = 1,\dots,N)$, are the critical points of the Lagrangian action functional

$$\mathcal{L}_N(u) := \int_0^T \big[\frac{|u'|^2}{2} - F(t,u) + (e|u)\big]\,dt$$

in the Sobolev space $H^1_T = \{u \in H^1([0,T],\mathbb{R}^N) : u(0) = u(T)\}$. Here and in the whole paper, $(\cdot|\cdot)$ denotes the inner product in $\mathbb{R}^N$ and $|\cdot|$ the corresponding norm. In 1984, the following result was proved in [21].

Theorem 1.2. *If F satisfies assumption (1.3), then, for each $e \in L^1(0,T;\mathbb{R}^N)$ such that $\overline{e} = 0$, problem (1.2) has at least two geometrically distinct solutions.*

Geometrically distinct solutions of (1.2) are of course solutions whose differences are not of the form $\sum_{i=1}^N k_i\omega_i$ for some $(k_1,\dots,k_N) \in \mathbb{Z}^N$. The proof of Theorem 1.2 is an easy extension of the argument of the scalar case. Such a multiplicity result is not optimal, as easily seen, and was improved around 1988 independently by Rabinowitz [24], Chang [7], and the author [17], who got the following multiplicity conclusion.

Theorem 1.3. *If F satisfies assumption (1.3), then, for each $e \in L^1(0,T;\mathbb{R}^N)$ such that $\overline{e} = 0$, problem (1.2) has at least $N+1$ geometrically distinct solutions.*

Although they present technical differences, the three proofs of this result use the fact that $\mathcal{L}_N(u + \omega_j e^j) = \mathcal{L}_N(u)$ $(j = 1,\dots,n)$ and some Ljusternik-Schnirelmann category arguments.

2. The relativistic forced pendulum and corresponding systems

In 2010, it was shown in [4] that the solutions of the 'relativistic forced pendulum equation', i.e. the solutions of the problem

$$\Big(\frac{u'}{\sqrt{1-u'^2}}\Big)' + a\sin u = e(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{2.1}$$

namely the functions u of class C^1 on $C([0,T]$ such that $\|u'\|_\infty < 1$, $\frac{u'}{\sqrt{1-u'^2}}$ is absolutely continuous on $[0,T]$ and which verify the differential equation in (2.1) almost everywhere, and the periodic boundary conditions, can be associated to the critical points of the action defined by

$$\mathcal{R}(u) := \int_0^T \left[1 - \sqrt{1-|u'|^2} + a\cos u + eu\right] dt$$

on the closed convex set

$$K = \{u \in W^{1,\infty}([0,T]) : u(0) = u(T),\ \|u'\|_\infty \le 1\},$$

where $\|\cdot\|_\infty$ denotes the L^∞-norm. In 2011, it was shown in [2] that those solutions could be seen as well be associated to the critical points in the sense of Szulkin [25] of the functional given on $C_T = \{u \in C([0,T]) : u(0) = u(T)\}$ by

$$\mathcal{S}(u) = \Phi(u) + \mathcal{G}(u),$$

where Φ is defined on $C([0,T])$ by

$$\Phi(u) := \begin{cases} \int_0^T [1 - \sqrt{1-|u'|^2}]\, dt & \text{if } u \in W^{1,\infty}([0,T]) \\ +\infty & \text{if } u \in C([0,T]) \setminus W^{1,\infty}([0,T]) \end{cases}$$

and $\mathcal{G}$ is defined on $C([0,T])$ by

$$\mathcal{G}(u) = \int_0^T [a\cos u + eu]\, dt.$$

Φ is convex, proper, lower semi-continuous, and $\mathcal{G}$ of class C^1, so that $\mathcal{S}$ has the structure required by Szulkin's critical point theory [25]. When $\overline{e} = 0$, $\mathcal{R}$ and $\mathcal{S}$ are bounded from below, and satisfy a suitable version of Palais-Smale condition on their set of definition. Consequently, they reach there a minimum, and one can show that such minimum corresponds to a solution of (2.1) (this is less trivial than in the case of (1.1)). Hence, the following extension of Hamel's result to (2.1) follows [4, 2] : *for each $e \in L^1(0,T)$ such that $\overline{e} = 0$, problem* (2.1) *has at least one solution minimizing $\mathcal{R}$ on K (or $\mathcal{S}$ on C_T).* Another proof of this existence result, based upon some Hamiltonian equivalent formulation (described later in a different context) and a saddle point theorem, has been given in 2012 by Manásevich and Ward [14].

Like in the classical case, such a conclusion is not optimal for (2.1) and, in 2012, Bereanu and Torres [3] have proved the following multiplicity result.

Theorem 2.1. *For each $e \in C([0,T])$ such that $\overline{e} = 0$, problem* (2.1) *has at least two geometrically distinct solutions.*

Their proof is modeled on the one of [20] for the classical pendulum, but technically more involved. One first shows the existence of two positive minimizers u_0, $u_0 + 2\pi$ of $\mathcal{S}$ on C_T, then considers a modified problem like in the method of lower and upper solutions with lower solution $\alpha = u_0$, and upper solution $\beta = u_1$. Finally one shows that Szulkin's critical points of the corresponding modified action are solutions of (2.1), and obtains the second solution of the modified action by a mountain pass argument.

In 2012, Fonda and Toader [11] and, in 2013, Marò [15] have proved Theorem 2.1 by applying a Poincaré-Birkhoff type fixed point theorem to the equivalent Hamiltonian formulation mentioned above, and Marò has obtained the supplementary information that *one of the solutions is unstable.* An extension of Theorem 2.1 is easily obtained when $a\sin u$ is replaced by $\partial_u F(t,u)$, with $F(t,u)$ ω-periodic in u.

If $e \in L^1(0,T;\mathbb{R}^N)$, if the functions $F(t,u)$ and $\nabla_u F(t,u)$ are defined and continuous on $[0,T]\times\mathbb{R}^N$, and if assumption (1.3) holds, one can consider the periodic problem for a relativistic system

$$\Big(\frac{u'}{\sqrt{1-|u'|^2}}\Big)' + \nabla_u F(t,u) = e(t), \quad u(0)=u(T), \quad u'(0)=u'(T). \tag{2.2}$$

Its concept of solution is defined in an analogous way as for (2.1). With now

$$C_T := \{u \in C([0,T],\mathbb{R}^N) : u(0)=u(T)\},$$

one defines $\mathcal{S} : C_T \to (-\infty,+\infty]$ by

$$\mathcal{S}(u) = \Phi(u) + \mathcal{G}(u), \tag{2.3}$$

where

$$\Phi(u) := \begin{cases} \int_0^T [1-\sqrt{1-|u'|^2}]\,dt & \text{if } u \in W^{1,\infty}([0,T],\mathbb{R}^N) \\ +\infty & \text{if } u \in C_T \setminus W^{1,\infty}([0,T],\mathbb{R}^N), \end{cases}$$

and

$$\mathcal{G}(u) = \int_0^T [-F(t,u) + (e|u)]\,dt.$$

In 2011, using the approach of [4], the following existence result was proved in [5].

Theorem 2.2. *If F satisfies assumption* (1.3)*, then, for each $e \in L^1(0,T;\mathbb{R}^N)$ such that $\overline{e}=0$, problem* (2.2) *has at least one solution.*

The extension to systems of the methods used in [3, 11, 15] for a scalar equation seeming difficult, the obtention of multiplicity results similar to Theorem 1.3 for system (2.2) has required different approaches, that we now describe.

3. A Hamiltonian approach

The first result was given in 2012 in [19]. It is assumed that F and $\nabla_u F$ exist and are continuous, and, for simplicity, we extend them as well as e, by T-periodicity, to $\mathbb{R}\times\mathbb{R}^N$ and to $\mathbb{R}$ respectively. Setting

$$v = \frac{u'}{\sqrt{1-|u'|^2}}$$

in (2.2), which is equivalent to

$$u' = \frac{v}{\sqrt{1+|v|^2}},$$

we immediately see that (2.2) is equivalent to the first order problem

$$v' = -\nabla_u F(t,u) + e(t), \quad u' = \frac{v}{\sqrt{1+|v|^2}}, \quad u,v\ T\text{-periodic}. \tag{3.1}$$

Defining $H : \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ by

$$H(t,u,v) := \sqrt{1+|v|^2} - 1 \ + F(t,u) - (e(t)|u),$$

we see that (3.1) has the Hamiltonian form

$$v' = -\nabla_u H(t,u,v), \quad u' = \nabla_v H(t,u,v), \quad u,v \quad T\text{-periodic}. \tag{3.2}$$

The action functional naturally associated to (3.2) is given by

$$\mathcal{H}(v,u) = \int_0^T \big[-(v|u') + F(t,u) - (e|u) + \sqrt{1+|v|^2} - 1)\big]\, dt.$$

If we define the Sobolev space

$$H_T^{1/2} = \{(v,u) \in H^{1/2}([0,T],\mathbb{R}^{2N}) : v \text{ and } u \text{ are } T\text{-periodic}\},$$

then a standard result (see e.g. [23]) implies that if $e \in L^s(0,T;\mathbb{R}^N)$ for some $s > 1$, then $\mathcal{H} \in C^1(H_T^{1/2},\mathbb{R})$ and its critical points solve (3.2), or, explicitly, (3.1). Furthermore, it is easy to check that if $\overline{e} = 0$, and F satisfies condition (1.3), then

$$\mathcal{H}(v, u_1 + k_1\omega_1, \dots, u_N + k_N\omega_N) = \mathcal{H}(v,u)$$

for all $(v,u) \in H_T^{1/2}$ and all $(k_1,\dots,k_N) \in \mathbb{Z}^N$. Consequently, we can consider $\mathcal{H}$ as defined on $\mathbb{T}^N \times E$, where

$$E = \{(v,u) \in H_T^{1/2} : \overline{u} = 0\}$$

and $\mathbb{T}^N$ is the n-torus. It can be shown that $E = E^- \oplus E^0 \oplus E^+$ where $E^0 \simeq \mathbb{R}^N$, the linear operator associated to the quadratic form $(v,u) \mapsto \int_0^T [-(v|u')]\, dt$ is negative definite on E^-, positive definite on E^+, and

$$\mathcal{H}(v,\overline{u}) \to +\infty \quad \text{for all } \overline{u} \in \mathbb{R}^N \text{ when } |v| \to \infty \text{ in } E^0.$$

Therefore $\mathcal{H}$ satisfies the conditions of an abstract saddle point theorem for indefinite functionals proved in 1990 by Szulkin [25], and based upon the concept of relative Ljusternik-Schnirelmann category, which implies the following multiplicity result for (2.2).

Theorem 3.1. *If F satisfies assumption* (1.3)*, then, for each $e \in L^s(0,T;\mathbb{R}^N)$ for some $s > 1$, such that $\overline{e} = 0$, problem* (2.2) *has at least $N+1$ geometrically distinct solutions.*

As one can see, the proof of Theorem 3.1 is technically sophisticated, both from the critical point theory side, because $\mathcal{H}$ is an indefinite functional, and from the topological side, because the relative category is a more involved and delicate concept than Ljusternik-Schnirelmann category. Hence the result of [19] raises the following natural questions:

(1) Can $e \in L^s$ for some $s > 1$ be replaced by the more natural assumption $e \in L^1$?
(2) Can the result be proved using Lagrangian action and classical category?

4. A Lagrangian approach

In 2013, Bereanu and Jebelean [1] proved Theorem 3.1, when F, $\nabla_u F$ and e are continuous, through an extension to convex, lower semicontinuous perturbations of a C^1-functional on a Banach space, i.e. to functionals of Szulkin type [25], of an abstract multiplicity result for some symmetric C^1 functionals given in [22, Theorem 4.12], and motivated by Rabinowitz' approach in [24].

Let X be a Banach space with dual X^* and duality mapping $\langle\cdot,\cdot\rangle$, G a discrete additive subgroup of X such that span(G) has finite dimension N, $\pi : X \to X/G$ the canonical projection. So $G \simeq \mathbb{Z}^N$, $X = \mathbb{R}^N \oplus Y$ for some closed subspace Y, $u = \overline{u} + \widetilde{u}$, with $\overline{u} \in \mathbb{R}^N$, $\widetilde{u} \in Y$. $A \subset X$ is *G-invariant* if $u + g \in A$ for all $u \in A$

and $g \in G$, $f : X \to M$ is *G-invariant* if $f(u+g) = f(u)$ for all $u \in X$ and $g \in G$. The following assumptions are made:

- (H1) $\mathcal{G} \in C^1(X, \mathbb{R})$ is G-invariant, $\mathcal{G}'$ takes bounded sets into bounded sets.
- (H2) $\Psi : X \to (-\infty, +\infty]$ is G-invariant, convex, lower semicontinuous, with closed non-empty domain $D(\Psi) \supset \{u \in X : \|\widetilde{u}\| \le \rho,\ |\Psi(u)| \le \rho\}$, $\Psi(0) = 0$, $\Psi(u) = \Psi(\widetilde{u})$ for all $u \in X$.
- (H3) Any sequence (u_n) in X with $(\overline{u_n})$ bounded has a convergent subsequence.

According to [25], $u \in X$ is a *critical point* of $\mathcal{S} = \Psi + \mathcal{G}$ if

$$\langle \mathcal{G}'(u), v - u \rangle + \Psi(v) - \Psi(u) \ge 0 \quad \text{for all } v \in X.$$

Let

$$K = \{u \in X : u \text{ is a critical point}\}$$

be the *critical set* of $\mathcal{S}$, and let $K_c = \{u \in K : \mathcal{S}(u) = c\}$. It is easy to see that $\mathcal{S}, \mathcal{S}', K, K_c$ are G-invariant. Hence, if u is a critical point of $\mathcal{S}$, the same is true for $u + g$ for all $g \in G$, and the set $\{u + g : g \in G\}$ is called a *critical orbit* of $\mathcal{S}$.

If $\mathcal{N}$ is an open neighborhood of K_c and $\varepsilon > 0$, we set

$$\mathcal{N}_\varepsilon = \{u \in X \setminus \mathcal{N} : |\overline{u}| \le 2,\ \mathcal{S}(u) \le c + \varepsilon\}.$$

The following equivariant deformation lemma, which combines similar results in [22, 25], is essential to prove the multiplicity result.

Lemma 4.1. *Let $c \in \mathbb{R}$ and $\mathcal{N}$ be a G-invariant neighborhood of K_c. Then, for each $\varepsilon \in (0,1]$, there exists $\varepsilon \in (0, \overline{\varepsilon}]$, $d_\varepsilon > 0$, $\varepsilon' \in (0, \varepsilon]$ and $\eta \in C([0, \overline{t}] \times \mathcal{N}_\varepsilon, X)$, with the following properties:*

- (i) $\eta(0, \cdot) = id_{\mathcal{N}_\varepsilon}$.
- (ii) $\eta(t, u + g) = \eta(t, u) + g$ *for all* $(t, u) \in [0, \overline{t}] \times \mathcal{N}_\varepsilon$ *and all* $g \in G$ *with* $u + g \in \mathcal{N}_\varepsilon$.
- (iii) $\|\eta(t, u) - u\| \le d_\varepsilon t$ *for all* $(t, u) \in [0, \overline{t}] \times \mathcal{N}_\varepsilon$.
- (iv) $\mathcal{S}(\eta(t, u)) - \mathcal{S}(u) \le d_\varepsilon t$ *for all* $(t, u) \in [0, \overline{t}] \times \mathcal{N}_\varepsilon$.
- (v) $\mathcal{S}(\eta(t, u)) - \mathcal{S}(u) \le -\varepsilon' t/2$ *for all* $(t, u) \in [0, \overline{t}] \times (\mathcal{N}_\varepsilon \cap \mathcal{S}^{-1}([c - \varepsilon, +\infty)))$.
- (vi) *if* $A \subset \mathcal{N}_\varepsilon$ *with* $c \le \sup_A \mathcal{S}$, *then, for all* $t \in [0, \overline{t}]$,

$$\sup_A \mathcal{S}(\eta(t, \cdot)) - \sup_A \mathcal{S} \le -\varepsilon' t/2.$$

From this lemma, one can construct a deformation in the quotient space $\pi(X)$. Defining, like in [22],

$$\mathcal{A}_j = \{A \subset X : A \text{ is compact and } \operatorname{cat}_{\pi(X)}(\pi(A)) \ge j\},$$

one can check that $\mathcal{A}_j \ne \emptyset$ for each $j = 1, \dots, N + 1$ and $\mathcal{A}_j$ is a complete metric space for the Hausdorff distance. Furthermore, the function $\sigma : \mathcal{A}_j \to (-\infty, +\infty]$ defined by

$$\sigma(A) = \sup_{A \in \mathcal{A}_j} \mathcal{S}$$

is lower semicontinuous and bounded from below. Ekeland's variational principle and a rather standard argument of Ljusternik-Schnirelmann type give the following multiplicity result.

Proposition 4.2. *Under assumptions* (H1)–(H3)*, the functional $\mathcal{S} = \Psi + \mathcal{G}$ has at least $N + 1$ critical orbits.*

By applying Proposition 4.2 to the functional $\mathcal{S} : C_T \to (-\infty, \infty]$ defined in (2.3) with the group

$$G = \Big\{ \sum_{k=1}^{N} k_i \omega_i e^i : k_i \in \mathbb{Z},\ i = 1, \dots, N \Big\}, \tag{4.1}$$

one obtains easily the following multiplicity result.

Theorem 4.3. *If F satisfies assumption (1.3), then, for each $e \in C([0,T], \mathbb{R}^N)$ such that $\overline{e} = 0$, problem (2.2) has at least $N+1$ geometrically distinct solutions.*

The proof of Proposition 4.2 given in [1] is quite complicated and technical, but only uses classical Ljusternik-Schnirelmann category. In the following section, we describe a recent approach given in [13], which answers positively the two questions of the end of Section 3, by obtaining the requested multiplicity result through the use of a modified equivalent problem, whose action functional is defined in the classical Sobolev space H^1_T, and to which Theorem 4.12 of [22] can be directly applied.

5. A modified Lagrangian approach

Let $e \in L^1(0,T;\mathbb{R}^N)$, and assume that $F(t,\cdot)$ and $\nabla_u F(t,\cdot)$ are continuous for a.e $t \in [0,T]$, that $F(\cdot,u)$ and $\nabla_u F(\cdot,u)$ are measurable for each $u \in \mathbb{R}^N$, and that there exists some $\alpha \in L^1(0,T)$ such that

$$|F(t,u)| + |\nabla_u F(t,u)| \le \alpha(t)$$

for a.e. $t \in [0,T]$ and all $u \in \mathbb{R}^N$. Define, for $v \in B(1) \subset \mathbb{R}^N$,

$$\varphi(v) := \frac{v}{\sqrt{1-|v|^2}},$$

so that

$$\varphi^{-1}(w) = \frac{w}{\sqrt{1+|w|^2}} \quad \text{for all } w \in \mathbb{R}^N.$$

Let us introduce a modification of φ inspired by recent papers of Coelho *et al* [8, 9] in problems of positive solutions with Dirichlet conditions, but technically different, by setting

$$K := \varphi^{-1}\left(\overline{B}(\sqrt{n}\|\alpha\|_{L^1})\right) \subset B(1),$$

fixing $R \in (0,1)$ in such a way that

$$\frac{R}{\sqrt{1-R^2}} \ge \sqrt{n}\|\alpha\|_{L^1},\ K \subset \overline{B}(R),$$

and defining the homeomorphism $\psi : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\psi(y) := (1 - \min\{|y|^2, R^2\})^{-1/2} y,$$

in such a way that

$$\psi^{-1}(v) = \max\Big\{(1-R^2)^{1/2}, (1+|v|^2)^{-1/2}\Big\}\, v.$$

Lemma 5.1. *For all $y, z \in \mathbb{R}^N$, one has*

$$(\psi(z) - \psi(y)|z-y) \ge |z-y|^2,\ |\psi(y)| \le \frac{1}{\sqrt{1-R^2}}|y|.$$

With this ψ, let us consider the modified problem

$$(\psi(u'))' + \nabla_u F(t,u) = e(t), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{5.1}$$

The choice of R above allows us to prove the following equivalence result.

Lemma 5.2. *$u \in C^1$ is solution of* (2.2) *if and only if it is a solution of* (5.1).

If we define now $\Psi : \mathbb{R}^N \to \mathbb{R}$ by

$$\Psi(y) := 1 - \frac{1 - \min\{|y|^2, R^2\} + 1 - |y|^2}{2\sqrt{1 - \min\{|y|^2, R^2\}}},$$

it is easy to show that Ψ is of class C^1 and that

$$\psi(y) = \nabla\Psi(y), \quad (1/2)|y|^2 \le \Psi(y) \le \frac{1}{\sqrt{1-R^2}}|y|^2$$

for all $y \in \mathbb{R}^N$. Consequently the functional $\mathcal{M}$ given by

$$\mathcal{M}(u) := \int_0^T [\Psi(u') - F(t,u) + (e|u)]\, dt$$

is well defined and of class C^1 on H^1_T, and

$$\langle \mathcal{M}'(u), v\rangle = \int_0^T [(\psi(u')|v') - (\nabla_u F(t,u) - e|v)]\, dt,$$

for all $u, v \in H^1_T$, so that its critical points correspond to the weak, and hence to the Carathéodory solutions of (5.1). On the other hand, the following version of Palais-Smale condition can be proved.

Lemma 5.3. *Each sequence (u_n) in H^1_T such that $(\mathcal{M}(u_n))$ is bounded, $\mathcal{M}'(u_n) \to 0$, and $(\overline{u}_n)$ is bounded, contains a convergent subsequence.*

As mentioned above, [22, Theorem 4.12] is just the version of Proposition 4.2 for a C^1 functional, and we keep the notations of Section 4. If $\mathcal{I} \in C^1(X,\mathbb{R})$ is G-invariant, we introduce the following type of Palais-Smale condition, called the *$(PS)_G$-condition* : for each sequence (u_n) in X with $(\mathcal{G}(u_n))$ bounded and $\mathcal{G}'(u_n) \to 0$, $(\pi(u_n))$ contains a convergent subsequence. Theorem 4.12 in [22] goes as follows.

Proposition 5.4. *If the vector space spanned in X by G has finite dimension N, and if $\mathcal{I} \in C^1(X,\mathbb{R})$ is G-invariant, satisfies $(PS)_G$-condition, and is bounded from below, then $\mathcal{I}$ has at least $N+1$ critical orbits.*

The proof of Proposition 5.4 given in [22] is based upon Ekeland variational principle and classical Ljusternik-Schnirelmann category. As shown in [22], it provides a proof of Theorem 1.3 for the classical pendulum system. It also implies the corresponding result for (2.2).

Theorem 5.5. *If F satisfies condition* (1.3)*, then, for each $e \in L^1(0,T;\mathbb{R}^N)$, problem* (2.2) *has at least $N+1$ geometrically distinct solutions.*

Sketch of the proof. By Lemma 5.2, it suffices to prove that the modified action function $\mathcal{M}$ satisfies the conditions of Proposition 5.4. It is easy to see that

$$\mathcal{M}(u) = \int_0^T [\Psi(u') - F(t,u) + (e(t)|\widetilde{u})]\, dt = \mathcal{M}(u + \omega_i e^i)$$

for all $i = 1, \dots, n$ and $u \in H_T^1$. If we define G by (4.1), using Lemma 5.1 and the Wirtinger and Sobolev inequalities, and denoting the L^p-norm by $\|\cdot\|_p$, one can show that the inequality

$$\mathcal{M}(u) \geq \frac{1}{2}\|\widetilde{u}'\|_2^2 - C_2\|\alpha\|_1 - C_1\|h\|_1\|\widetilde{u}'\|_2$$

holds, and that $\mathcal{M}$ satisfies the $(PS)_G$-condition. Then the result follows from Proposition 5.4. □

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