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SECOND-ORDER FULLY DISCRETIZED PROJECTION METHOD FOR INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. A second-order fully discretized projection method for the incompressible Navier-Stokes equations is proposed. It is an explicit method for updating the pressure field. No extra conditions of immediate velocity fields are needed. The stability and convergence are investigated.

1. Introduction

The projection method (or fractional step method) for solving the incompressible Navier-Stokes equations was originally introduced and studied independently by Chorin [1, 2] and Temam [9, 10], it has had multiple applications, see e.g., Guermond, Shen, and Yang [3], Guermond, Minev, and Shen [4], Kim and Moin [7], Temam [11], Yanenko [12], and the references therein, for the theoretical and numerical aspects. Despite many advantages and extensive uses in the past by numerous researchers, the projection method has a few major drawbacks for numerical computations. In general, the original method is only first-order accurate in time. It also needs the supplementary boundary conditions for the intermediate level velocity field and the pressure, which are not supplied in the original equations.

A fully discretized projection method was studied in Guo [5, 6] on the staggered grid. The idea was originated in the block LU decomposition, see Perot [8]. This leaded to a whole class of methods (first-order, second-order and even higher order methods). It depends on how the Navier-Stokes equations are discretized, higher order methods can be possible.

In this article, we investigate the stability and convergence of a second-order fully discretized projection method proposed in Guo [5, 6]. This article is organized as follows. In Section 2, we introduce the Navier-Stokes equations and the boundary conditions. The space discretization is listed in Section 3. The full discretization of the Navier-Stokes equations is shown in Section 4. In Section 5, we present fully discretized projection methods. The stability and convergence of second-order fully

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discretized projection method is studied in Section 6. The conclusion is in the last section.

2. Incompressible Navier-Stokes equations

We will consider the non-dimensionalized unsteady incompressible Navier-Stokes equations in space dimension two and three on a given regular domain Ω, namely

$$\begin{gathered}\frac{\partial u}{\partial t} + (u\cdot\nabla)u - \frac{1}{\mathrm{Re}}\Delta u + \nabla p = f,\\ \nabla\cdot u = 0,\\ u|_{t=0} = u_0,\end{gathered} \tag{2.1}$$

with the boundary conditions on $\partial\Omega$:

$$\eta u + (1-\eta)\frac{\partial u}{\partial \overrightarrow{n}} = 0.$$

Here Re is the Reynolds number; the parameter η has the limit values of 0 for the free-slip (no stress) condition (Neumann) and 1 for the no-slip condition (Dirichlet). In general, we will not specify η, but keep in mind that $0 \le \eta \le 1$.

For the stability and convergence, what we need are the boundary conditions that guarantee the existence of the solution for the original equations. Actually, the boundary conditions are also important in discretizing the equations near the boundary. We may need to use different methods for the points near boundary. In this article, we suppose that all discretizations have the same order on all of grid points.

3. Space discretization

Two families of finite-dimensional Hilbert spaces X_h and V_h are given, which depend on a parameter $h \in \mathbb{R}_+^d$ $(d = 2, 3)$. For finite differences, h is the mesh, i.e. $h = \{h_1, h_2\} = \{\Delta x, \Delta y\}$ in space dimension two and $h = \{h_1, h_2, h_3\} = \{\Delta x, \Delta y, \Delta z\}$ in space dimension three.

Two scalar products $((\cdot,\cdot))_h$ and $(\cdot,\cdot)_h$ with corresponding norms $\|\cdot\|_h$ and $|\cdot|_h$ are defined on each V_h. Since V_h is a finite-dimensional space the two norms $\|\cdot\|_h$ and $|\cdot|_h$ are equivalent. We assume that they are related as follows

$$|u_h|_h \le c_1\|u_h\|_h, \tag{3.1}$$

$$\|u_h\|_h \le S(h)|u_h|_h, \quad \forall u_h \in V_h. \tag{3.2}$$

where c_1 is independent of h and $S(h)$ depends on h. We assume that $S(h) \to \infty$ as $h \to 0$.

When convergence be studied, we will be interested in the passage to the limit $h \to 0$. The spaces V_h with scalar product $((\cdot,\cdot))_h$ will approximate in some sense the space V, while the spaces V_h with scalar product $(\cdot,\cdot)_h$ will approximate the space H. The spaces V and H are defined as follows

$$\begin{gathered}V = \text{ the closure of } \mathcal{V} \text{ in } H_0^1(\Omega), \quad H = \text{ the closure of } \mathcal{V} \text{ in } L^2(\Omega),\\ \mathcal{V} = \{u \in \mathcal{D}(\Omega), \operatorname{div} u = 0\}.\end{gathered}$$

A trilinear operator b_h is defined on $V_h \times V_h \times V_h$ as follows:

$$b_h(u_h, v_h, w_h) = ((u_h\cdot\nabla)v_h, w_h), \quad \forall u_h, v_h, w_h \in V_h,$$

and we have the properties:

$$|b_h(u_h, v_h, w_h)| \leq c_2 |u_h|_h^{1/2} \|u_h\|_h^{1/2} \|v_h\|_h |w_h|_h^{1/2} \|w_h\|_h^{1/2}, \quad \forall u_h, v_h, w_h \in V_h, \tag{3.3}$$

in space dimension two, and

$$|b_h(u_h, v_h, w_h)| \leq c_2 |u_h|_h^{\frac{1}{4}} \|u_h\|_h^{3/4} \|v_h\|_h |w_h|_h^{\frac{1}{4}} \|w_h\|_h^{3/4}, \quad \forall u_h, v_h, w_h \in V_h, \tag{3.4}$$

in space dimension three. The constant c_2 in (3.3) and (3.4) is independent of h.

We also assume the skewness property:

$$b_h(u_h, v_h, v_h) = 0, \quad \forall u_h, v_h \in V_h, \tag{3.5}$$

which implies

$$b_h(u_h, v_h, w_h) = -b_h(u_h, w_h, v_h), \quad \forall u_h, v_h, w_h \in V_h.$$

We also define two operators A_h and B_h that are the discrete analogs of A and B as follows:

$$\begin{gathered}(A_h u_h, v_h)_h = ((u_h, v_h))_h, \quad \forall u_h, v_h \in V_h,\\ (B_h(u_h, v_h), w_h)_h = b_h(u_h, v_h, w_h), \quad \forall u_h, v_h, w_h \in V_h.\end{gathered}$$

4. Full discretization of Navier-Stokes equations

Let $T > 0$ be fixed, and let the time step be denoted by $\Delta t = T/N$ where N is an integer. We construct recursively elements $u_h^n \in V_h$, $n = 0, 1, 2, \dots, N$; u_h^0 is an approximation of u_0; we assume that u_h^1 is constructed by a different scheme because our scheme has two levels in time.

Hence starting from u_h^2, we recursively define u_h^n, $n = 2, \dots, N$, by applying the explicit Adams-Bashforth scheme of order two to the convective term and the Crank-Nicholson scheme of order two to the viscous terms, the pressure term and the right-hand side of equation (2.1),

$$\begin{aligned}&\frac{1}{\Delta}t(u_h^{n+1} - u_h^n) + \frac{3}{2}B_h(u_h^n) - \frac{1}{2}B_h(u_h^{n-1})\\ &+ \frac{1}{\mathrm{Re}}A_h(u_h^{n+1} + u_h^n) + \frac{1}{2}G_h(p_h^{n+1} + p_h^n) = \frac{1}{2}(f_h^{n+1} + f_h^n),\end{aligned} \tag{4.1}$$

$$D_h u_h^{n+1} = 0. \tag{4.2}$$

where f_h^n is an approximation of f^n in V_h; G_h is the discrete gradient operator (grad) mapping X_h into V_h, and D_h is the discrete divergence operator (div). In fact, they are related as follows,

$$(u_h, G_h p_h) = -(D_h u_h, p_h), \quad \text{for } u_h \in V_h,\ p_h \in X_h. \tag{4.3}$$

One can easily show that (4.1) and (4.2) are a second-order scheme in time for the Navier-Stokes equations (2.1). We will assume that (4.1) and (4.2) are second-order in space. It is possible to get higher order discretizations, but other procedure would be needed. Therefore, (4.1) and (4.2) have the overall second-order accuracy. They are a fully implicit coupled system. Generally speaking, it is hard to solve this system.

Notice that if we consider the boundary conditions in equation (4.1) and (4.2), we may need to add one or more terms to the right-hand side of equations (4.1) and (4.2). For the sake of simplicity, we suppose that these terms are combined in the terms of the right-hand side. So, the idea is that during fully discretizing procedure in time and spaces, we need all boundary conditions. And the final discretized equations are a complete linear system.

For the rest of this article, we drop the subscript h. Define g^{n+1} as follows,

$$g^{n+1} = \frac{1}{2}(f^{n+1} + f^n) - \frac{3}{2}B(u^n) + \frac{1}{2}B(u^{n-1}) \tag{4.4}$$

then we rewrite (4.1) and (4.2) as

$$\frac{1}{\Delta t}(u^{n+1} - u^n) + \frac{1}{2Re}A(u^{n+1} + u^n) + \frac{1}{2}G(p^{n+1} + p^n) = g^{n+1}, \tag{4.5}$$

and

$$Du^{n+1} = 0. \tag{4.6}$$

5. Fully discretized projection method

We then introduce the following method for the equations (4.5) and (4.6),

$$\begin{gathered} \frac{1}{\Delta}t(u^{n+\frac{1}{2}} - u^n) + \frac{1}{\mathrm{Re}}A(u^{n+\frac{1}{2}} + u^n) = g^{n+1}, \\ \frac{1}{\Delta}t(u^{n+1} - u^{n+\frac{1}{2}}) + \frac{1}{2}(I - \frac{\Delta t}{\mathrm{Re}}A)G(p^{n+1} + p^n) = 0, \\ Du^{n+1} = 0. \end{gathered} \tag{5.1}$$

Observe that in (5.1), A is a known operator, say a square matrix; G and D also are two known operators, but they may be two non-square matrices. However, they satisfy the equation below,

$$(Du^n, p^n) = -(u^n, Gp^n), \quad \text{for } u^n \in V_h,\ p^n \in X_h.$$

Let us check the difference between (4.5) and (5.1). First, we add the two equations in (5.1) and we obtain:

$$\frac{1}{\Delta}t(u^{n+1} - u^n) + \frac{1}{\mathrm{Re}}A(u^{n+1} + u^n) = g^{n+1} - \frac{1}{2}G(p^{n+1} + p^n) + R^{n+1};$$

i.e.,

$$\begin{aligned} &\frac{1}{\Delta}t(u^{n+1} - u^n) + \frac{3}{2}B(u^n) - \frac{1}{2}B(u^{n-1}) \\ &+ \frac{1}{\mathrm{Re}}A(u^{n+1} + u^n) + \frac{1}{2}G(p^{n+1} + p^n) = \frac{1}{2}(f^{n+1} + f^n) + R^{n+1}, \end{aligned} \tag{5.2}$$

where

$$R^{n+1} = \frac{1}{2}(\frac{\Delta t}{\mathrm{Re}})^2 AAG(p^{n+1} + p^n).$$

Since R^{n+1} is of order $(\Delta t)^2$, this method is a second-order method to the Navier-Stokes equations (2.1). However, to update p^{n+1}, we need to solve a linear system with the same size as A at each time step. In this paper, we study the stability and convergence of the equations (5.1).

6. Stability analysis and convergence

Let
$$\Phi^{n+1} = \frac{1}{2}(p^{n+1} + p^n).$$
Then
$$R^{n+1} = (\frac{\Delta t}{\mathrm{Re}})^2 AAG\Phi^{n+1}.$$
Note that from the definition of G and D and the third equation of (5.1), it holds,
$$(G\Phi^{n+1}, u^{n+1}) = -(\Phi^{n+1}, Du^{n+1}) = 0.$$

6.1. **Stability analysis.** It is assumed that
$$\begin{gathered}|u^0| \le K_1, \quad \Delta t \sum_{n=0}^{n=N} |f^n|^2 \le K_2, \\ |u^1|^2 + |u^{1/2}|^2 + \frac{\Delta t}{Re}\{\|u^1\|^2 + \|u^{1/2}\|^2\} \le K_3,\end{gathered} \tag{6.1}$$
where K_1, K_2 and K_3 are independent of h and Δt. Let
$$\delta^{n+1} = \frac{1}{2}(u^{n+1} - u^n), \quad \delta^{n+\frac{1}{2}} = \frac{1}{2}(u^{n+\frac{1}{2}} - u^n);$$
then
$$\frac{1}{2}(u^{n+1} + u^n) = u^{n+1} - \delta^{n+1}, \quad \frac{1}{2}(u^{n+\frac{1}{2}} + u^n) = u^{n+\frac{1}{2}} - \delta^{n+\frac{1}{2}}.$$
Now rewriting the first equation in (5.1) with g^{n+1} defined in (4.4), it reads
$$\begin{aligned}&\frac{1}{\Delta t}(u^{n+\frac{1}{2}} - u^n) + \frac{1}{2Re}A(u^{n+\frac{1}{2}} + u^n) \\ &= \frac{1}{2}(f^{n+1} + f^n) - \frac{3}{2}B(u^n) + \frac{1}{2}B(u^{n-1}).\end{aligned} \tag{6.2}$$
Solve the second equation in (5.1) for $u^{n+\frac{1}{2}}$ as follows
$$u^{n+\frac{1}{2}} = u^{n+1} + \Delta t(I - \frac{\Delta t}{\mathrm{Re}}A)G\Phi^{n+1}.$$
Substituting $u^{n+\frac{1}{2}}$ in (6.2), it results,
$$\begin{aligned}&\frac{1}{\Delta t}(u^{n+1} - u^n) + \frac{1}{\mathrm{Re}}A(u^{n+1} - \delta^{n+1}) \\ &= \frac{1}{2}(f^{n+1} + f^n) - \frac{3}{2}B(u^n) + \frac{1}{2}B(u^{n-1}) - G\Phi^{n+1} + R^{n+1}.\end{aligned} \tag{6.3}$$
Multiply equation (6.3) with $2\Delta t u^{n+1}$, we obtain
$$\begin{aligned}&2(u^{n+1} - u^n, u^{n+1}) + \frac{2\Delta t}{\mathrm{Re}}((u^{n+1} - \delta^{n+1}, u^{n+1})) \\ &= \Delta t(f^{n+1} + f^n, u^{n+1}) - 2\Delta t b(u^n, u^n, u^{n+1}) \\ &\quad - \Delta t[b(u^n, u^n, u^{n+1}) - b(u^{n-1}, u^{n-1}, u^{n+1})] + 2\Delta t(R^{n+1}, u^{n+1}),\end{aligned}$$
or
$$\begin{aligned}&|u^{n+1}|^2 - |u^n|^2 + 4|\delta^{n+1}|^2 + \frac{2\Delta t}{\mathrm{Re}}\|u^{n+1}\|^2 \\ &= \frac{2\Delta t}{\mathrm{Re}}((\delta^{n+1}, u^{n+1})) + \Delta t(f^{n+1} + f^n, u^{n+1}) + 4\Delta t b(u^n, \delta^{n+1}, u^{n+1}) \\ &\quad - 2\Delta t[b(\delta^n, u^n, u^{n+1}) + b(u^{n-1}, \delta^n, u^{n+1})] + 2\Delta t(R^{n+1}, u^{n+1}).\end{aligned} \tag{6.4}$$

We now estimate each term in the right-hand side of (6.4) in space dimension two. The first one is majorized, thanks to the Schwarz and Young inequalities, and (3.2) (note, $S = S(h)$):

$$\begin{aligned}\frac{2\Delta t}{\mathrm{Re}}((\delta^{n+1}, u^{n+1})) &\le \frac{2\Delta t}{\mathrm{Re}}\|\delta^{n+1}\|\|u^{n+1}\| \\ &\le \frac{2\Delta t}{\mathrm{Re}} S|\delta^{n+1}|\|u^{n+1}\| \\ &\le \frac{1}{8}|\delta^{n+1}|^2 + \frac{8\Delta t^2}{\mathrm{Re}^2} S^2\|u^{n+1}\|^2.\end{aligned}$$

For the second term we use the Schwarz and Young inequalities again and thanks to (3.1):

$$\begin{aligned}\Delta t(f^{n+1} + f^n, u^{n+1}) &\le \Delta t(|f^{n+1}| + |f^n|)|u^{n+1}| \\ &\le c_1\Delta t(|f^{n+1}| + |f^n|)\|u^{n+1}\| \\ &\le \frac{\Delta t}{4Re}\|u^{n+1}\|^2 + c_1^2\mathrm{Re}\Delta t(|f^{n+1}|^2 + |f^n|^2).\end{aligned}$$

For the third term we use (3.1), (3.2) and (3.3) and majorize it by

$$\begin{aligned}4\Delta t b(u^n, \delta^{n+1}, u^{n+1}) &\le 4c_2\Delta t|u^n|^{1/2}\|u^n\|^{1/2}|\delta^{n+1}|^{1/2}\|\delta^{n+1}\|^{1/2}\|u^{n+1}\| \\ &\le 4c_2 S\Delta t|u^n\|\delta^{n+1}\|\|u^{n+1}\| \\ &\le \frac{1}{8}|\delta^{n+1}|^2 + 32c_2^2S^2\Delta t^2|u^n|^2\|u^{n+1}\|^2.\end{aligned}\tag{6.5}$$

Thanks to (3.3) the fourth term is approximated by

$$\begin{aligned}&2\Delta t[b(\delta^n, u^n, u^{n+1}) + b(u^{n-1}, \delta^n, u^{n+1})] \\ &\le 2c_2\Delta t|\delta^n|^{1/2}\|\delta^n\|^{\frac{1}{2}}(|u^n|^{1/2}\|u^n\|^{1/2} + |u^{n-1}|^{\frac{1}{2}}\|u^{n-1}\|^{1/2})\|u^{n+1}\| \\ &\le 2c_2S\Delta t|\delta^n|(|u^n| + |u^{n-1}|)\|u^{n+1}\| \\ &\le \frac{1}{4}|\delta^n|^2 + 4c_2^2S^2\Delta t^2(|u^n|^2 + |u^{n-1}|^2)\|u^{n+1}\|^2.\end{aligned}\tag{6.6}$$

Finally, by the definition of R^{n+1} we have

$$2\Delta t(R^{n+1}, u^{n+1}) = 2\Delta t(\frac{\Delta t}{\mathrm{Re}})^2(AAG\Phi^{n+1}, u^{n+1}).$$

Rewriting the second equation of (5.1), it results

$$\begin{aligned}G\Phi^{n+1} &= -\frac{1}{\Delta}t(I - \frac{\Delta t}{\mathrm{Re}}A)^{-1}(u^{n+1} - u^{n+\frac{1}{2}}) \\ &= -\frac{1}{\Delta}t[I + \frac{\Delta t}{\mathrm{Re}}A + (\frac{\Delta t}{\mathrm{Re}})^2AA + \dots](u^{n+1} - u^{n+\frac{1}{2}}),\end{aligned}\tag{6.7}$$

if $\frac{\Delta t}{\mathrm{Re}}$ is small enough.

Then, substituting $G\Phi^{n+1}$ in R^{n+1}, we obtain

$$\begin{aligned}&2\Delta t(R^{n+1}, u^{n+1}) \\ &= -2(\frac{\Delta t}{\mathrm{Re}})^2(AA[I + \frac{\Delta t}{\mathrm{Re}}A + (\frac{\Delta t}{\mathrm{Re}})^2AA + \dots]u^{n+1}, u^{n+1}) \\ &\quad + 2(\frac{\Delta t}{\mathrm{Re}})^2(AA[I + \frac{\Delta t}{\mathrm{Re}}A + (\frac{\Delta t}{\mathrm{Re}})^2AA + \dots]u^{n+\frac{1}{2}}, u^{n+1})\end{aligned}$$

$$
\begin{aligned}
&= -2(\frac{\Delta t}{\mathrm{Re}})^2[(AAu^{n+1}, u^{n+1}) + \frac{\Delta t}{\mathrm{Re}}(AAAu^{n+1}, u^{n+1}) + \dots] \\
&\quad + 2(\frac{\Delta t}{\mathrm{Re}})^2[(AAu^{n+\frac{1}{2}}, u^{n+1}) + \frac{\Delta t}{\mathrm{Re}}(AAAu^{n+\frac{1}{2}}, u^{n+1}) + \dots] \\
&\le 2(\frac{\Delta t}{\mathrm{Re}})^2(S^2\|u^{n+1}\|^2 + \frac{\Delta t}{\mathrm{Re}}S^4\|u^{n+1}\|^2 + \dots] \\
&\quad + 2(\frac{\Delta t}{\mathrm{Re}})^2[S^2(\frac{1}{4}\|u^{n+1}\|^2 + \|u^{n+\frac{1}{2}}\|^2) + \frac{\Delta t}{\mathrm{Re}}S^4(\frac{1}{4}\|u^{n+1}\|^2 + \|u^{n+\frac{1}{2}}\|^2) + \dots] \\
&\le 2(\frac{\Delta t}{\mathrm{Re}})^2\frac{5}{4}(S^2 + \frac{\Delta t}{\mathrm{Re}}S^4 + \dots)\|u^{n+1}\|^2 + 2(\frac{\Delta t}{\mathrm{Re}})^2(S^2 + \frac{\Delta t}{\mathrm{Re}}S^4 + \dots)\|u^{n+\frac{1}{2}}\|^2 \\
&\le 2(\frac{\Delta t}{\mathrm{Re}})^2\frac{S^2}{1 - \frac{\Delta t}{\mathrm{Re}}S^2}(\frac{5}{4}\|u^{n+1}\|^2 + \|u^{n+\frac{1}{2}}\|^2) \\
&\le \frac{\Delta t^2}{\mathrm{Re}^2}S^2(\frac{5}{4}\|u^{n+1}\|^2 + \|u^{n+\frac{1}{2}}\|^2),
\end{aligned}
$$

where we assume that $\frac{\Delta t}{\mathrm{Re}}S^2 < 1/2$.

Gathering all these estimates, we deduce from (6.4) that

$$
\begin{aligned}
&|u^{n+1}|^2 - |u^n|^2 + \frac{15}{4}|\delta^{n+1}|^2 - \frac{1}{4}|\delta^n|^2 + \frac{7}{4}\frac{\Delta t}{\mathrm{Re}}\|u^{n+1}\|^2 \\
&\le c_1^2\mathrm{Re}\Delta t(|f^{n+1}|^2 + |f^n|^2) + \frac{\Delta t^2}{\mathrm{Re}^2}S^2\|u^{n+\frac{1}{2}}\|^2 \\
&\quad + \frac{37}{4}\frac{\Delta t^2}{\mathrm{Re}^2}S^2\|u^{n+1}\|^2 + c_3S^2\Delta t^2(|u^n|^2 + |u^{n-1}|^2)\|u^{n+1}\|^2,
\end{aligned}
\tag{6.8}
$$

where $c_3 = 32c_2^2$.

We multiply equation (6.2) by $2\Delta t u^{n+\frac{1}{2}}$ to obtain

$$
\begin{aligned}
&2(u^{n+\frac{1}{2}} - u^n, u^{n+\frac{1}{2}}) + \frac{2\Delta t}{\mathrm{Re}}((u^{n+\frac{1}{2}} - \delta^{n+\frac{1}{2}}, u^{n+\frac{1}{2}})) \\
&= \Delta t(f^{n+1} + f^n, u^{n+\frac{1}{2}}) - 2\Delta t b(u^n, u^n, u^{n+\frac{1}{2}}) \\
&\quad - \Delta t[b(u^n, u^n, u^{n+\frac{1}{2}}) - b(u^{n-1}, u^{n-1}, u^{n+\frac{1}{2}})].
\end{aligned}
\tag{6.9}
$$

Thanks to (3.5), the above equation can be rewritten as

$$
\begin{aligned}
&|u^{n+\frac{1}{2}}|^2 - |u^n|^2 + 4|\delta^{n+\frac{1}{2}}|^2 + \frac{2\Delta t}{\mathrm{Re}}\|u^{n+\frac{1}{2}}\|^2 \\
&= \frac{2\Delta t}{\mathrm{Re}}((\delta^{n+\frac{1}{2}}, u^{n+\frac{1}{2}})) + \Delta t(f^{n+1} + f^n, u^{n+\frac{1}{2}}) + 4\Delta t b(u^n, \delta^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}) \\
&\quad - 2\Delta t[b(\delta^n, u^n, u^{n+\frac{1}{2}}) + b(u^{n-1}, \delta^n, u^{n+\frac{1}{2}})].
\end{aligned}
\tag{6.10}
$$

Note that (6.10) is similar to (6.4) with u^{n+1} and δ^{n+1} replaced by $u^{n+1/2}$ and $\delta^{n+1/2}$ respectively. So, we can repeat the estimation procedures as for (6.4). In space dimension two we have the following estimates:

$$
\begin{gathered}
\frac{2\Delta t}{\mathrm{Re}}((\delta^{n+\frac{1}{2}}, u^{n+\frac{1}{2}})) \le \frac{1}{8}|\delta^{n+\frac{1}{2}}|^2 + \frac{8\Delta t^2}{\mathrm{Re}^2}S^2\|u^{n+\frac{1}{2}}\|^2. \\
\Delta t(f^{n+1} + f^n, u^{n+\frac{1}{2}}) \le \frac{\Delta t}{4\mathrm{Re}}\|u^{n+\frac{1}{2}}\|^2 + c_1^2\mathrm{Re}\Delta t(|f^{n+1}|^2 + |f^n|^2) \\
4\Delta t b(u^n, \delta^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}) \le \frac{1}{8}|\delta^{n+\frac{1}{2}}|^2 + 32c_2^2S^2\Delta t^2|u^n|^2\|u^{n+\frac{1}{2}}\|^2
\end{gathered}
\tag{6.11}
$$

$$\begin{aligned}&2\Delta t[b(\delta^n, u^n, u^{n+\frac{1}{2}}) + b(u^{n-1}, \delta^n, u^{n+\frac{1}{2}})]\\&\le \frac{1}{4}|\delta^n|^2 + 4c_2^2 S^2 \Delta t^2 (|u^n|^2 + |u^{n-1}|^2)\|u^{n+\frac{1}{2}}\|^2.\end{aligned} \tag{6.12}$$

We gather these inequalities with (6.10), and obtain

$$\begin{aligned}&|u^{n+\frac{1}{2}}|^2 - |u^n|^2 + \frac{15}{4}|\delta^{n+\frac{1}{2}}|^2 - \frac{1}{4}|\delta^n|^2 + \frac{7}{4}\frac{\Delta t}{\mathrm{Re}}\|u^{n+\frac{1}{2}}\|^2\\&\le c_1^2 \mathrm{Re}\Delta t(|f^{n+1}|^2 + |f^n|^2) + \frac{8\Delta t^2}{\mathrm{Re}^2} S^2 \|u^{n+\frac{1}{2}}\|^2\\&\quad + c_3 S^2 \Delta t^2 (|u^n|^2 + |u^{n-1}|^2)\|u^{n+\frac{1}{2}}\|^2.\end{aligned} \tag{6.13}$$

Adding (6.13) to (6.8), we obtain

$$\begin{aligned}&|u^{n+1}|^2 + |u^{n+\frac{1}{2}}|^2 - 2|u^n|^2 + \frac{15}{4}|\delta^{n+1}|^2 + \frac{15}{4}|\delta^{n+\frac{1}{2}}|^2 - \frac{1}{2}|\delta^n|^2\\&+ \frac{7}{4}\frac{\Delta t}{\mathrm{Re}}[1 - \frac{37\Delta t}{7\mathrm{Re}} S^2 - c_3 \mathrm{Re}\Delta t S^2 (|u^n|^2 + |u^{n-1}|^2)](\|u^{n+1}\|^2 + \|u^{n+\frac{1}{2}}\|^2)\\&\le c_1^2 \mathrm{Re}\Delta t(|f^{n+1}|^2 + |f^n|^2),\end{aligned} \tag{6.14}$$

where c_3 is the same as in (6.8).

We now prove the following result.

Theorem 6.1. *In dimension two we assume that* (6.1) *holds. Then there exists* K_4 *independent of* h *and* Δt *such that if*

$$\Delta t < 16c_1^2 \mathrm{Re}, \quad \Delta t S(h)^2 \le \max\{\frac{\mathrm{Re}}{148}, \frac{c}{\mathrm{Re}K_4}\} \tag{6.15}$$

where c_1 *is defined in* (3.1) *and* $c = \frac{1}{56c_3}$, c_3 *the constant is defined in* (6.8). *Then*

$$|u^n|^2 \le K_4, \quad n = 0, \dots, N \tag{6.16a}$$

$$|u^{n+\frac{1}{2}}|^2 \le 14K_4, \quad n = 0, \dots, N \tag{6.16b}$$

$$\sum_{n=1}^{N} |u^n - u^{n-1}|^2 \le 3K_4, \tag{6.16c}$$

$$\sum_{n=1}^{N} |u^n - u^{n-\frac{1}{2}}|^2 \le 6K_4, \tag{6.16d}$$

$$\sum_{n=1}^{N} |u^{n-\frac{1}{2}} - u^{n-1}|^2 \le 3K_4, \tag{6.16e}$$

$$\frac{\Delta t}{\mathrm{Re}} \sum_{n=1}^{N} \|u^n\|^2 \le 17K_4, \tag{6.16f}$$

$$\frac{\Delta t}{\mathrm{Re}} \sum_{n=1}^{N} \|u^{n-\frac{1}{2}}\|^2 \le 17K_4, \tag{6.16g}$$

Proof. Taking the inner product of the equation (6.5) with u^{n+1} and in light of (3.3), we obtain

$$([I + \frac{\Delta t}{\mathrm{Re}} A + (\frac{\Delta t}{\mathrm{Re}})^2 AA + \dots](u^{n+1} - u^{n+\frac{1}{2}}), u^{n+1}) = 0;$$

i.e.,

$$(u^{n+1}-u^{n+\frac12},u^{n+1})=-\frac{\Delta t}{\mathrm{Re}}([A+(\frac{\Delta t}{\mathrm{Re}})AA+\dots](u^{n+1}-u^{n+\frac12}),u^{n+1}).$$

Thanks to (3.1), we have

$$(u^{n+1}-u^{n+\frac12},u^{n+1})\le\frac54(\frac{\Delta t}{\mathrm{Re}})\frac{1}{1-\frac{\Delta t}{\mathrm{Re}}S^2}(\|u^{n+1}\|^2+\|u^{n+\frac12}\|^2).$$

Since $\Delta tS^2\le \mathrm{Re}/148$ from (6.15), then $\frac{\Delta t}{\mathrm{Re}}S^2\le 1/6$ and it results

$$\frac12(|u^{n+1}|^2-|u^{n+\frac12}|^2+|u^{n+1}-u^{n+\frac12}|^2)\le\frac34\frac{\Delta t}{\mathrm{Re}}(\|u^{n+1}\|^2+\|u^{n+\frac12}\|^2)$$

or

$$|u^{n+1}|^2-|u^{n+\frac12}|^2+|u^{n+1}-u^{n+\frac12}|^2\le\frac32\frac{\Delta t}{\mathrm{Re}}(\|u^{n+1}\|^2+\|u^{n+\frac12}\|^2). \tag{6.17}$$

We first show (6.16a) by induction. Assuming that (6.16a) holds for $n=0,\dots,m$, we infer from (6.12), (6.13) that

$$\begin{aligned}&|u^{n+1}|^2+|u^{n+\frac12}|^2-2|u^n|^2+\frac{15}{4}|\delta^{n+1}|^2\\&+\frac{15}{4}|\delta^{n+\frac12}|^2-\frac12|\delta^n|^2+\frac{13\Delta t}{8\mathrm{Re}}\{\|u^{n+1}\|^2+\|u^{n+\frac12}\|^2\}\\&\le c_1^2Re\Delta t\{|f^{n+1}|^2+|f^n|^2\}.\end{aligned}\tag{6.18}$$

Adding (6.18) to (6.17), we obtain

$$\begin{aligned}&2|u^{n+1}|^2-2|u^n|^2+|u^{n+1}-u^{n+\frac12}|^2+\frac{15}{4}|\delta^{n+1}|^2\\&+\frac{15}{4}|\delta^{n+\frac12}|^2-\frac12|\delta^n|^2+\frac{\Delta t}{8Re}\{\|u^{n+1}\|^2+\|u^{n+\frac12}\|^2\}\\&\le c_1^2Re\Delta t\{|f^{n+1}|^2+|f^n|^2\};\end{aligned}$$

i.e.,

$$\begin{aligned}&|u^{n+1}|^2-|u^n|^2+\frac12|u^{n+1}-u^{n+\frac12}|^2+\frac{15}{8}|\delta^{n+1}|^2+\frac{15}{8}|\delta^{n+\frac12}|^2-\frac14|\delta^n|^2\\&\quad+\frac{\Delta t}{16}\{\|u^{n+1}\|^2+\|u^{n+\frac12}\|^2\}\le c_1^2Re\Delta t\{|f^{n+1}|^2+|f^n|^2\}.\end{aligned}\tag{6.19}$$

Thanks to (3.1), we obtain

$$\begin{aligned}&|u^{n+1}|^2-|u^n|^2+\frac{15}{8}|\delta^{n+1}|^2-\frac14|\delta^n|^2+\frac{\Delta t}{16c_1^2\mathrm{Re}}|u^{n+1}|^2\\&\le c_1^2Re\Delta t\{|f^{n+1}|^2+|f^n|^2\}.\end{aligned}\tag{6.20}$$

Therefore, if we define

$$\xi^n=|u^n|^2+\frac14|\delta^n|^2,\quad \gamma^n=c_1^2\mathrm{Re}\Delta t\{|f^{n+1}|^2+|f^n|^2\},$$

then, from (6.20) the following inequality holds

$$(1+\frac{\Delta t}{16c_1^2Re})\xi^{n+1}\le\xi^n+\gamma^n.$$

By applying Lemma 6.2 below with $\beta = 0$ and $\alpha = -\frac{\Delta t}{16c_1^2 Re}$ (from (6.15), $\alpha < 1$), we find, for $n = 1, 2, \dots, m$, that

$$\xi^{n+1} \le (1 + \frac{\Delta t}{16c_1^2 Re})^{-n}\{\xi^1 + c_1^2 Re\Delta t \sum_{j=1}^{n} (|f^{n+1}|^2 + |f^n|^2)\};$$

i.e.,

$$\begin{aligned} |u^{n+1}|^2 + \frac{1}{16}|u^{n+1} - u^n|^2 &\le |u^1|^2 + \frac{1}{16}|u^1 - u^0|^2 + c_1^2 \mathrm{Re}\Delta t \sum_{j=1}^{N} |f^j|^2, \\ &\le \frac{9}{8}|u^1|^2 + \frac{1}{8}|u^0|^2 + c_1^2 \mathrm{Re}\Delta t \sum_{j=1}^{N} |f^j|^2, \\ &\le \frac{9}{8}K_3 + \frac{1}{8}K_1^2 + c_1^2 \mathrm{Re} K_2. \end{aligned}$$

We choose

$$K_4 = \frac{3}{2}K_3 + K_1^2 + 2c_1^2 \mathrm{Re} K_2,$$

and conclude that (6.16a) is proved for $n = 0, 1, \dots, m + 1$, hence for all n. Then (6.17) is valid for all n and summing these relations for $n = 1, 2, \dots, N - 1$, we find

$$\begin{aligned} &|u^N|^2 + \frac{1}{2}\sum_{j=2}^{N} |u^j - u^{j-\frac{1}{2}}|^2 + \sum_{j=2}^{N} |\delta^j|^2 + \frac{15}{8}\sum_{j=2}^{N} |\delta^{j-\frac{1}{2}}|^2 \\ &+ \frac{\Delta t}{16}\sum_{j=2}^{N} \{\|u^j\|^2 + \|u^{j-\frac{1}{2}}\|^2\} \\ &\le |u^1|^2 + \frac{1}{4}|\delta^1|^2 + 2c_1^2 \mathrm{Re}\Delta t \sum_{j=1}^{N} |f^j|^2 \\ &\le |u^1|^2 + \frac{1}{16}|u^1 - u^0|^2 + 2c_1^2 \mathrm{Re}\Delta t \sum_{j=1}^{N} |f^j|^2 \le K_4. \end{aligned}$$

Hence

$$\sum_{j=2}^{N} |u^j - u^{j-\frac{1}{2}}|^2 \le 2K_4, \quad \sum_{j=2}^{N} |u^j - u^{j-1}|^2 \le K_4,$$
$$\sum_{j=2}^{N} |u^{j-\frac{1}{2}} - u^{j-1}|^2 \le K_4, \quad \frac{\Delta t}{Re}\sum_{j=2}^{N} \{\|u^j\|^2 + \|u^{j-\frac{1}{2}}\|^2\} \le 16K_4.$$

Thanks to (6.1), we know that

$$\begin{gathered} |u^1 - u^{1/2}|^2 \le 2|u^1|^2 + 2|u^{1/2}|^2 \le 4K_3 \le 4K_4, \\ |u^1 - u^0|^2 \le 2|u^1|^2 + 2|u^0|^2 \le 2K_3 + 2K_1^2 \le 2K_4, \\ |u^{1/2} - u^0|^2 \le 2|u^{1/2}|^2 + 2|u^0|^2 \le 2K_3 + 2K_1^2 \le 2K_4, \\ \frac{\Delta t}{Re}(\|u^1\|^2 + \|u^{1/2}\|^2) \le K_3. \end{gathered}$$

Then (6.16c)–(6.16g) are proved.

Finally, from the inequality

$$|u^{n+\frac{1}{2}}|^2 \le 2|u^{n+1}|^2 + 2|u^{n+1} - u^{n+\frac{1}{2}}|^2,$$

then (6.16b) holds following (6.16a) and (6.16d). $\square$

We state the Lemma used in the proof of Theorem 6.1.

Lemma 6.2. *Consider two sequences of numbers ξ^n, $\gamma_n \ge 0$ such that*

$$(1-\alpha)\xi^n \le (1+\beta)\xi^{n-1} + \gamma_n,$$

for all $n \ge 1$ and for some $\alpha < 1$ and $\beta > -1$. Then for all n:

$$\xi^n \le \Big(\frac{1+\beta}{1-\alpha}\Big)^n \xi^0 + \frac{(1+\beta)^{n-1}}{(1-\alpha)^n} \sum_{j=1}^{n} \gamma_j. \tag{6.21}$$

If $\gamma_j \le \gamma$, for all j, we also have

$$\xi^n \le \Big(\frac{1+\beta}{1-\alpha}\Big)^n \Big(\xi^0 + \frac{\gamma}{\beta+\alpha}\Big). \tag{6.22}$$

Proof. For $m = 0, \dots, n-1$, we write

$$\xi^{n-m} \le \frac{1+\beta}{1-\alpha}\xi^{n-m-1} + \frac{1}{1-\alpha}\gamma_{n-m}.$$

We multiply this relation by $((1+\beta)/(1-\alpha))^m$ and add the corresponding inequalities for $m = 0, \dots, n-1$; (6.21) and (6.22) follow easily. $\square$

6.2. **Convergence.** We introduce the approximate functions u_{1k}, u_{2k}, $\tilde{u}_k$ ($k = \Delta t$) defined as follows:

$$u_{1k} = u^{n+\frac{1}{2}} \quad \text{for } t \in [n\Delta t, (n+1)\Delta t),\ n = 0, \dots, N-1,$$
$$u_{2k} = u^{n+1} \quad \text{for } t \in [n\Delta t, (n+1)\Delta t),\ n = 0, \dots, N-1,$$

$\tilde{u}_k$ is continuous from $[0,T]$ to H, linear on each interval $((n-1)\Delta t, n\Delta t)$ and equal to u^n at $n\Delta t$, $n = 0, \dots, N$.

Then (6.14) yields the following result.

Theorem 6.3. *As $k = \Delta t \to 0$, the functions u_{1k}, u_{2k} and $\tilde{u}_k$ remain bounded in $L^\infty(0,T;L^2(\Omega)^d)$; u_{1k} and u_{2k} remain bounded in $L^\infty(0,T;H^1(\Omega)^d)$, and the same is true for $\tilde{u}_k$ if $u_0 \in V$.*

Furthermore $u_{1k} - u_{2k}$ and $u_{2k} - \tilde{u}_k$ converge to 0 in $L^\infty(0,T;L^2(\Omega)^d)$ as $k = \Delta t \to 0$, their norm being bounded.

The above theorem follows directly from Theorem 6.1.

Because of Theorem 6.3 there exists a subsequence $k' \to 0$ such that

$$u_{1k'} \to u_1 \text{ in } L^\infty(0,T;H) \text{ weak-star and } L^2(0,T;H^1(\Omega)^d) \text{ weakly},$$
$$u_{2k'} \to u_2 \text{ in } L^\infty(0,T;H) \text{ weak-star and } L^2(0,T;H^1(\Omega)^d) \text{ weakly},$$
$$\tilde{u}_{k'} \to u \text{ in } L^\infty(0,T;H) \text{ weak-star and } L^2(0,T;H^1(\Omega)^d) \text{ weakly}.$$

Theorem 6.3 also implies that $u_1 = u_2 = u$ and thus

$$u \in L^\infty(0,T;H) \cap L^2(0,T;V).$$

Conclusions. In this article, we proved the stability and convergence of a second order fully discretized projection method for the incompressible Navier-Stokes equations. We did not specify the boundary conditions, but all necessary boundary conditions should be supplied in order to have a solution. It is possible to construct higher order methods, but the numerical analysis is much more complicated.

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