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SOME BIFURCATION RESULTS FOR QUASILINEAR DIRICHLET BOUNDARY VALUE PROBLEMS

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Abstract. This article reviews some bifurcation results for quasilinear problems in bounded domains of $\mathbb{R}^N$, with Dirichlet boundary conditions. Some of these are natural extensions of classical theorems in 'semilinear bifurcation theory' from the 1970's, based on topological arguments. In the radial setting, a recent contribution of the present author is also presented, which yields smooth solution curves, bifurcating from the first eigenvalue of the p-Laplacian.

1. Introduction

The typical problem we will consider in this review has the form

$$\begin{gathered} -\Delta_p(u) = \lambda|u|^{p-2}u + h(x,u,\lambda) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $p > 1$, $\lambda \in \mathbb{R}$, and Ω is a bounded domain in $\mathbb{R}^N$, with a smooth enough boundary, as will be specified later. We will suppose that $h(x,\xi,\lambda) : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is (at least) a Caratheodory function in its first two arguments (i.e. measurable in x and continuous in ξ), with

$$h(x,\xi,\lambda) = o(|\xi|^{p-1}) \quad \text{as } \xi \to 0, \tag{1.2}$$

uniformly for almost every $x \in \Omega$ and all λ in bounded subsets of $\mathbb{R}$.

A (weak) solution of (1.1) is a couple $(\lambda,u) \in \mathbb{R} \times W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\,dx - \lambda\int_\Omega |u|^{p-2}uv + h(x,u,\lambda)v\,dx = 0 \quad \forall\, v \in W_0^{1,p}(\Omega).$$

Under assumption (1.2), we have a line of *trivial solutions*, $\{(\lambda,0) : \lambda \in \mathbb{R}\}$. Of course, we will be interested in the existence of non-trivial solutions (i.e. with $u \not\equiv 0$). It turns out that the $(p-1)$-subhomogeneity condition (1.2) is the minimum requirement to get bifurcation from the line of trivial solutions. Additional hypotheses will be stated in due course, e.g. growth conditions as $|\xi| \to \infty$, in order to state various bifurcation results. A particular attention will be given to the radial case, i.e. the case where Ω is a ball centred at the origin and $h(x,\xi,\lambda) = h(|x|,\xi,\lambda)$.

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Some variants of (1.1) will also be considered, where Δ_p is replaced by a more general quasilinear operator.

The bifurcation theory for the semilinear case, i.e. for the Laplacian, $\Delta \equiv \Delta_2$, has been well known since the work of Rabinowitz [16], Crandall-Rabinowitz [7] and Dancer [8], in the 70's (just to mention the most relevant works in the present context, amongst an extensive literature). One of the first important contributions to the general, quasilinear case, $p > 1$, is due to del Pino and Manásevich [10]. In the spirit of [16], they use degree theoretic arguments to obtain a continuum — i.e. a connected set — of solutions bifurcating from the first eigenvalue $\lambda_1(p)$ of the homogeneous problem

$$\begin{aligned} -\Delta_p(u) &= \lambda |u|^{p-2}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

The eigenvalue $\lambda_1 = \lambda_1(p)$ is characterized variationally by

$$\lambda_1(p) = \inf\Big\{ \int_\Omega |\nabla u|^p dx : u \in W_0^{1,p}(\Omega) \quad \text{with } \int_\Omega |u|^p dx = 1 \Big\}, \tag{1.4}$$

and it was proved in [1] that, for any $p > 1$, $\lambda_1(p)$ is a simple isolated eigenvalue of (1.3). In order to evaluate the Leray-Schauder degree of the relevant operator in [10], the authors use a clever homotopic deformation along p (whence the explicit dependence on p in their notation for λ_1), already introduced in [9], that allows them to relate the general case $p > 1$ to the semilinear case $p = 2$, where the results are well known. They obtain a theorem analogous to Rabinowitz's global bifurcation theorem [16, Theorem 1.3] for problem (1.1). This result (Theorem 2.1 below) yields a component (i.e. a maximal connected subset) $\mathcal{C}$ of the set of nontrivial solutions of (1.1), which bifurcates from $(\lambda_1(p), 0)$ and is either unbounded or meets another point $(\bar\lambda, 0)$, for some eigenvalue $\bar\lambda \neq \lambda_1(p)$ of (1.3).

In the radial case, (1.1) is reduced to a one-dimensional problem — see [6] for symmetry results in the quasilinear context —, which is thoroughly investigated in [10]. It was previously known from [2] that the radial form of (1.3) has a strictly increasing sequence of positive simple eigenvalues, $0 < \mu_{1,p} < \mu_{2,p} < \dots$, such that an eigenfunction corresponding to $\mu_{n,p}$ has exactly $n - 1$ nodal zeros in $(0, 1)$, for all $n \in \mathbb{N}$. Using this information, together with similar degree theoretic arguments to those used to obtain bifurcation from the first eigenvalue in the general (nonradial) case, it is shown in [10] that an unbounded continuum of nodal solutions $\mathcal{C}_n$ bifurcates from each eigenvalue $\mu_{n,p}$, $n \in \mathbb{N}$. The main results of [10] will be presented in Section 2.

In the context of abstract semilinear bifurcation theory[1], a few years after Rabinowitz's celebrated paper [16], Dancer made an important contribution [8] showing that Rabinowitz's results could be substantially improved. Following ideas already put forth in [16], he proved that the continuum of solutions $\mathcal{C}$ obtained in Theorem 1.3 of [16] — bifurcating from a point $(\mu, 0)$ in $\mathbb{R} \times E$, with E a Banach space — can be decomposed as $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$, where the sets $\mathcal{C}^\pm$ bifurcate from $(\mu, 0)$ in 'opposite directions', and either are both unbounded, or meet each other outside a neighbourhood of $(\mu, 0)$ (see the proof of [8, Theorem 2] and the remarks following it).[2] This was an important improvement of Theorem 1.40 in [16].

[1] i.e. the abstract, functional analytic, theory that is naturally suited for semilinear equations

[2] The notation for the sets $\mathcal{C}^\pm$ comes from concrete problems where (at least locally around $(\mu, 0)$) the bifurcating solutions are either positive or negative — see Sections 3-5 below.

The quasilinear counterpart of Dancer's theorem was given by Girg and Takáč in [15] for a large class of Dirichlet problems (containing (1.1)). Their result, which will be presented in Section 3, essentially follows from Dancer's proof, albeit with a fairly technical asymptotic analysis required by the quasilinear setting. The very general result of Girg and Takáč (Theorem 3.2 below) completes the discussion of bifurcation from the first eigenvalue, from the topological point of view.

Global bifurcation being established by topological arguments, it is natural to seek conditions for the bifurcating continua to enjoy some regularity properties. A fundamental local result was proved by Crandall and Rabinowitz in [7], which is well-suited for applications in the semilinear case. Let $p = 2$ and λ_0 be a simple eigenvalue of the *linear* problem (1.3). Provided the nonlinearity in (1.1) is continuously differentiable, and a suitable transversality condition is satisfied by the eigenspace corresponding to λ_0, the Crandall-Rabinowitz theorem yields a unique continuous curve of non-trivial solutions of (1.1), bifurcating from the line of trivial solutions at $(\lambda_0, 0)$. In fact their abstract result, Theorem 1.7 in [7], applies to much more general (semilinear) problems than (1.1). A version of the Crandall-Rabinowitz theorem for (1.1) with $p > 2$ was stated by García-Melián and Sabina de Lis [13] in the radial case, where all the eigenvalues of (1.3) are simple. (However, their proof contains a gap, and a slightly more general version of this result was finally proved in [14].) Using this local result, further properties of the global continua $\mathcal{C}_n$ obtained in [10] are also discussed in [13]. In particular it is shown that, for each $n \in \mathbb{N}$, $\mathcal{C}_n$ splits into two unbounded pieces $\mathcal{C}_n^{\pm}$, that only meet at the bifurcation point $(\lambda_n, 0)$. Furthermore, the solutions in $\mathcal{C}_n^{\pm}$ retain the nodal structure of the eigenvectors $\pm v_n$ corresponding to λ_n. The main results of [13] will be described in Section 4.

A major difficulty in studying bifurcation for (1.1) with $p \neq 2$ is that the problem cannot be linearized at the trivial solution $u = 0$, since Δ_p is not differentiable at this point. Nevertheless, in the radial case, the inverse operator Δ_p^{-1} can be expressed explicitly by an integral formula, and its differentiability properties can be obtained. This program was carried out by Binding and Rynne [5] while studying the spectrum of the one-dimensional periodic p-Laplacian, and subsequently used by Rynne [17] to prove the existence of a smooth curve of solutions to a one-dimensional version of (1.1) (without radial symmetry).

Our contribution [14] was mainly motivated by [17] and [13], when we realized that, in the radial setting, a differentiability analysis similar to that of [17] would allow us to go beyond the local bifurcation result of [13]. In fact, under appropriate regularity and monotonicity assumptions, we obtain a complete characterization of the sets of positive and negative solutions of (1.1), as C^1 curves parametrized by $\lambda > 0$, bifurcating from the *first eigenvalue* of (1.3). Furthermore, we have precise information about the asymptotic behaviour of these curves. A key ingredient of [14] is the non-degeneracy of positive/negative solutions with respect to the integral form of (1.1), allowing for global continuation via the implicit function theorem. This non-degeneracy property requires a careful analysis of the inverse operator Δ_p^{-1}. Such an analysis was already attempted in [13] in order to prove the local bifurcation result [13, Theorem 1], but there seems to be a mistake in the proof of [13, Theorem 5] (see Remark 5.4 in Section 5 below) dealing with the differentiability of Δ_p^{-1}. Thus, in addition to extending the local bifurcation of [13] to a global one, our work also fills in this gap. The main results of [14] are presented in Section 5.

Notation. For the sake of homogeneity, we have allowed ourselves to change the original notations of the works reviewed here. For brevity, we will often refer to properties of solutions (λ, u) (such as positivity) by actually meaning that u possesses these properties. Throughout the paper, $\|\cdot\|$ will denote the usual norm of $W_0^{1,p}(\Omega)$. Finally, we will always use the same notation for a function and its associated Nemitskii mapping, e.g. $h(u,\lambda)(x) \equiv h(x,u(x),\lambda)$, $x \in \Omega$, $\lambda \in \mathbb{R}$.

2. del Pino and Manásevich

The bifurcation analysis presented in [10] is split into two parts: the general case and the radial case.

2.1. **The general case.** The main result in the general case is the following.

Theorem 2.1 ([10, Theorem 1.1]). *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with $C^{2,\alpha}$ boundary, for some $\alpha \in (0,1)$. Suppose that $h : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function in the first two variables, and satisfies:*

- (a) *(1.2) holds, uniformly for a.e. $x \in \Omega$ and all λ in bounded subsets of $\mathbb{R}$;*
- (b) *there exists $q \in (1, p^*)$ such that $\lim_{|\xi|\to\infty} h(x,\xi,\lambda)/|\xi|^{q-1} = 0$, uniformly for a.e. $x \in \Omega$ and all λ in bounded subsets of $\mathbb{R}$.*

There is a component[3] $\mathcal{C}$ of the set of non-trivial solutions of (1.1) in $\mathbb{R} \times W_0^{1,p}(\Omega)$, such that its closure $\overline{\mathcal{C}}$ contains the point $(\lambda_1(p), 0)$, and $\overline{\mathcal{C}}$ is either unbounded or contains a point $(\bar\lambda, 0)$, for some eigenvalue $\bar\lambda \neq \lambda_1(p)$ of (1.3).

In assumption (b), p^* denotes the Sobolev conjugate of $p > 1$, i.e.

$$p^* = \begin{cases} Np/(N-p) & \text{if } p < N, \\ \infty & \text{if } p \geq N. \end{cases}$$

In particular, Theorem 2.1 implies that $(\lambda_1(p), 0)$ is a *bifurcation point* of (1.1) in $\mathbb{R}\times W_0^{1,p}(\Omega)$, in the sense that, in any neighbourhood of $(\lambda_1(p), 0)$ in $\mathbb{R}\times W_0^{1,p}(\Omega)$, there exists a non-trivial solution of (1.1). Furthermore, there holds *global bifurcation* in the sense of Rabinowitz [16].

Proof. The proof of Theorem 2.1 follows in the same way as that of Rabinowitz's global bifurcation theorem [16, Theorem 1.3], provided the 'jump' of the Leray-Schauder degree when λ crosses λ_1 — used to contradict identity (1.11) in the last part of the proof of [16, Theorem 1.3] — holds when $p \neq 2$. More precisely, define $T_p^\lambda : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ by

$$T_p^\lambda(u) = S_p(\lambda\phi_p(u)), \tag{2.1}$$

where

$$\phi_p(s) = |s|^{p-2}s, \quad s \in \mathbb{R}, \tag{2.2}$$

and $S_p : W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega)$ (with $1/p' + 1/p = 1$) denotes the inverse of $-\Delta_p$; i.e. for each $v \in W^{-1,p'}(\Omega)$, $S_p(v) \in W_0^{1,p}(\Omega)$ is the unique weak solution

$$\begin{gathered} -\Delta_p(u) = v \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.3}$$

[3] i.e. a maximal (with respect to the order relation defined by set inclusion) connected subset

Problem (1.1) is now equivalent to

$$u = S_p(\lambda\phi_p(u) + h(u,\lambda)),$$

while the eigenvalue problem (1.3) becomes

$$u = T_p^\lambda(u).$$

Using the invariance of the degree under completely continuous homotopies, the proof of Theorem 2.1 can then be reduced to checking that, for $r > 0$ and $\lambda \in \mathbb{R}$,

$$\deg(I - T_p^\lambda, B(0,r), 0) = \begin{cases} 1 & \text{if } \lambda < \lambda_1(p), \\ -1 & \text{if } \lambda_1(p) < \lambda < \lambda_2(p). \end{cases} \tag{2.4}$$

Here, $I : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ is the identity, $B(0,r)$ the ball of radius r centred at the origin in $W_0^{1,p}(\Omega)$, and[4]

$$\lambda_2(p) = \inf\{\lambda > \lambda_1(p) : \lambda \text{ is an eigenvalue of (1.3)}\}.$$

Note that the Leray-Schauder degree in (2.4) is well defined since $T_p^\lambda : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ is a completely continuous mapping, as explained on p. 229 of [10].

Property (2.4) is [10, Proposition 2.2]. The proof of this result uses a clever procedure, first introduced by del Pino et al. [9] in the one-dimensional setting. The idea is to deform homotopically the operator T_p^λ to an operator $T_2^{\lambda'}$ for which (2.4) is known to hold, and then to use the invariance of the degree. This involves a number of technical difficulties, and relies upon the continuous dependence of λ_1 on $p > 1$. In particular, one needs to check that the first eigenvalue $\lambda_1(p)$ is isolated, uniformly for p in bounded subsets of $(1,\infty)$ (see Lemma 2.3 of [10]), in order to be able to choose the deformation in such way that $\lambda_1(2) < \lambda' < \lambda_2(2)$ provided one starts with $\lambda_1(p) < \lambda < \lambda_2(p)$. □

2.2. **The radial case.** In Section 4 of [10], it is assumed that Ω is the unit ball centred at the origin in $\mathbb{R}^N$, and that $h(x,u,\lambda) \equiv h(|x|,u,\lambda)$. The hypotheses of Theorem 2.1 are also supposed to hold.

In the radial variable $r = |x|$, problems (1.1) and (1.3) respectively become

$$\begin{gathered} -(r^{N-1}\phi_p(u'))' = r^{N-1}(\lambda\phi_p(u) + h(r,u,\lambda)), \quad 0 < r < 1, \\ u'(0) = u(1) = 0, \end{gathered} \tag{2.5}$$

and

$$\begin{gathered} -(r^{N-1}\phi_p(u'))' = \mu r^{N-1}\phi_p(u), \quad 0 < r < 1, \\ u'(0) = u(1) = 0. \end{gathered} \tag{2.6}$$

We now use μ as an eigenvalue parameter in (2.6) since, a priori, there could be more eigenvalues of (1.3) than those of (2.6).[5] It follows from standard regularity theory that the radial eigenfunctions of $-\Delta_p$ are such that $u \in C^1[0,1]$ — we slightly abuse the notation here, writing $u(x) \equiv u(|x|)$ — and satisfies (2.6).

It is known since [2] (and proved in the appendix of [10]) that, for any $p > 1$, the eigenvalues of (2.6) form an increasing sequence, $0 < \mu_{1,p} < \mu_{2,p} < \dots$, with $\lim_{n\to\infty} \mu_{n,p} = \infty$. Furthermore, $\mu_{n,p}$ is simple for all $n \in \mathbb{N}$, with an eigenfunction

[4]It is known from [3] that $\lambda_2(p)$ is, in fact, the second eigenvalue of (1.3).

[5]Note that the spherical symmetry of positive solutions of (1.1) (and hence of (1.3)) is known from [6]. However, higher order eigenvalues can have non-symmetric eigenfunctions, see [4].

v_n having exactly $n-1$ zeros in $(0,1)$, all of them simple. The following result now extends the global bifurcation from Theorem 2.1 to all the eigenvalues of (2.6).

Theorem 2.2 ([10, Theorem 4.1])**.** *For each $n \in \mathbb{N}$, there exists a component $\mathcal{C}_n \subset \mathbb{R} \times C[0,1]$ of the set of non-trivial solutions of* (2.5)*, such that $(\mu_{n,p}, 0) \in \overline{\mathcal{C}_n}$, the closure of $\mathcal{C}_n$. Furthermore, $\mathcal{C}_n$ is unbounded in $\mathbb{R} \times C[0,1]$, and all $v \in \mathcal{C}_n$ has exactly $n-1$ zeros in $(0,1)$.*

Proof. Theorem 2.2 is proved similarly to Theorem 2.1. Since the eigenvalues $\mu_{n,p}$, $n \in \mathbb{N}$, are isolated and simple, a homotopic deformation to the case $p = 2$ yields

$$\deg(I - \widetilde{T}_p^\mu, B(0,r), 0) = \begin{cases} 1 & \text{if } \mu < \mu_{1,p}, \\ (-1)^n & \text{if } \mu_{n,p} < \mu < \mu_{n+1,p}, \end{cases} \tag{2.7}$$

where $\widetilde{T}_p^\mu$ is the radial version of (2.1). Since $\widetilde{T}_p^\mu$ now has an explicit integral representation, its analytic properties are much more easily established than in the general case. Moreover, since the eigenvalues $\mu_{n,p}$, $n \in \mathbb{N}$, are isolated for all $p > 1$, and depend continuously on $p > 1$ (which is proved in the appendix of [10]), the discussion about the isolation of the first eigenvalue (Lemma 2.3 of [10]) is not required any more to construct a suitable homotopic deformation. □

3. Girg and Takáč

The more general problem considered by Girg and Takáč in [15] has the form

$$\begin{gathered} -\operatorname{div}(\mathbf{a}(x, \nabla u)) = \lambda B(x)|u|^{p-2}u + h(x,u,\lambda) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

with $\Omega \subset \mathbb{R}^N$ a bounded domain, such that the boundary $\partial\Omega$ is a compact $C^{1,\alpha}$ manifold for some $\alpha \in (0,1)$, and Ω satisfies the interior sphere condition at every point of $\partial\Omega$ (Ω is just a bounded open interval if $N = 1$).

The various coefficients in the equation are supposed to satisfy the following hypotheses.

(A) The function $\mathbf{a}$ can be written as $\mathbf{a}(x,\zeta) = \frac{1}{p}\nabla_\zeta A(x,\zeta)$, with $A \in C^1(\Omega \times \mathbb{R}^N)$ such that $a_i := \frac{1}{p}\frac{\partial A}{\partial \zeta_i} \in C^1(\Omega \times (\mathbb{R}^N \setminus \{0\}))$ for $i = 1, \dots, N$. Furthermore:

- (A1) $A(x, t\zeta) = |t|^p A(x,\zeta)$ for all $t \in \mathbb{R}$ and all $(x,\zeta) \in \Omega \times \mathbb{R}^N$;
- (A2) there exist constants $\gamma, \Gamma > 0$ such that, for all $(x,\zeta) \in \Omega \times (\mathbb{R}^N \setminus \{0\})$ and all $\eta \in \mathbb{R}^N$,

$$\sum_{i,j=1}^N \frac{\partial a_i}{\partial \zeta_j}(x,\zeta)\eta_i\eta_j \ge \gamma|\zeta|^{p-2}|\eta|^2,$$

$$\sum_{i,j=1}^N \Big|\frac{\partial a_i}{\partial \zeta_j}(x,\zeta)\Big| \le \Gamma|\zeta|^{p-2} \quad \text{and} \quad \sum_{i,j=1}^N \Big|\frac{\partial a_i}{\partial x_j}(x,\zeta)\Big| \le \Gamma|\zeta|^{p-1}.$$

(B) The weight $B \in L^\infty(\Omega, \mathbb{R}_+)$, and $B \not\equiv 0$ a.e. in Ω.

(H) The function $h : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, in the sense that $h(\cdot, \xi, \lambda) : \Omega \to \mathbb{R}$ is measurable for all fixed $(\xi, \lambda) \in \mathbb{R}^2$ and $h(x, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ is continuous for a.e. fixed $x \in \Omega$. Furthermore, there exists a constant $C > 0$ such that

$$|h(x,\xi,\lambda)| \le C|\xi|^{p-1}, \quad \text{a.e. } x \in \Omega,\ (\xi,\lambda) \in \mathbb{R}^2, \tag{3.2}$$

and (1.2) holds, for a.e. $x \in \Omega$, uniformly for λ in bounded subsets of $\mathbb{R}$.

Remark 3.1. (i) Assumption (A) is trivially satisfied by the p-Laplacian, with $\mathbf{a}(x,\zeta) = |\zeta|^{p-2}\zeta$ (i.e. $A(x,\zeta) = |\zeta|^p$) for all $(x,\zeta) \in \Omega \times \mathbb{R}^N$.

(ii) Note that the subhomogeneity condition (3.2) yields a stronger growth restriction as $|\xi| \to \infty$ than hypothesis (b) in Theorem 2.1.

We now extend the definition of the first eigenvalue λ_1 in (1.4) to the context of the more general $(p-1)$-homogeneous eigenvalue problem

$$\begin{gathered} -\operatorname{div}(\mathbf{a}(x,\nabla u)) = \lambda B(x)|u|^{p-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.3}$$

Hence λ_1 is now defined as

$$\lambda_1(p) = \inf\Big\{ \int_\Omega A(x,u)dx : u \in W_0^{1,p}(\Omega) \text{ with } \int_\Omega B(x)|u|^p dx = 1\Big\}. \tag{3.4}$$

It follows by the compactness of the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ that the above infimum is attained, satisfies $0 < \lambda_1 < \infty$, and is a simple eigenvalue of (3.3). Furthermore, a corresponding eigenfunction φ_1 can be chosen so that $\varphi_1 > 0$ in Ω and $\int_\Omega B(x)|\varphi_1|^p dx = 1$ (see the references given in Remark 2.1 of [15]).

In order to formulate the main result of [15], we still need to define the sets $\mathcal{C}^\pm$ that were briefly mentioned in the introduction. Consider the functional $\ell \in W^{-1,p'}(\Omega)$ defined by

$$\ell(\phi) = \|\varphi_1\|_{L^2(\Omega)}^{-2} \int_\Omega \varphi_1 \phi\, dx \quad \forall\, \phi \in W_0^{1,p}(\Omega).$$

Then for a fixed, small enough, $\eta > 0$, two convex cones can be defined by

$$K_\eta^\pm = \{(\lambda,u) \in \mathbb{R} \times W_0^{1,p}(\Omega) : \pm\ell(u) > \eta\|u\|\},$$

so that $K_\eta := K_\eta^+ \cup K_\eta^- = \{(\lambda,u) \in \mathbb{R} \times W_0^{1,p}(\Omega) : |\ell(u)| > \eta\|u\|\}$. Careful local a priori estimates show that all non-trivial solutions of (3.1) in a sufficiently small neighbourhood of $(\lambda_1,0)$ lie in K_η (see [15, Lemma 3.6]). Furthermore, local solutions (λ,u) in K_η can be represented as

$$u = \tau(\varphi_1 + v^\top), \tag{3.5}$$

where $\tau = \ell(u)$, $\ell(v^\top) = 0$, with $\lambda \to \lambda_1$ and $v^\top \to 0$ as $\tau \to 0$ (see [15, Lemma 3.6] for more precise statements). Hence, the component $v^\top$ is in some sense 'transverse' to φ_1; this transverse direction will also play an important role in the local results of Sections 4 and 5.

Proposition 3.5 of [15] yields a continuum $\mathcal{C}$ of non-trivial solutions (λ,u) of (3.1) bifurcating from the point $(\lambda_1,0)$ in $\mathbb{R}\times W_0^{1,p}(\Omega)$, in the spirit of Rabinowitz's global bifurcation theorem [16, Theorem 1.3] (this result reduces to Theorem 2.1 for the p-Laplacian). Now the sets $\mathcal{C}^\pm$ are essentially defined as follows: for $\nu = \pm$, $\mathcal{C}^\nu$ is the component of $\mathcal{C}$ such that $\mathcal{C}^\nu \cap N \subset K_\eta^\nu$, for any sufficiently small neighbourhood N of $(\lambda_1,0)$, and $\mathcal{C}^{-\nu} = \mathcal{C} \setminus \mathcal{C}^\nu$. A more precise contruction is given on p. 287 of [15], which is proved to be independent of η. Hence, $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$, and it follows from the properties of the decomposition (3.5) mentioned above that, roughly speaking, the subcontinua $\mathcal{C}^{\pm}$ emerge from $(\lambda_1, 0)$ with u tangential to $\pm\varphi_1$ (i.e. with $\pm\ell(u) > 0$) respectively. Since $\varphi_1 > 0$, they are referred to as the 'positive' and 'negative' parts of $\mathcal{C}$. Of course, this terminology does not mean that all solutions in these sets are positive or negative. In fact, we have the following result.

Theorem 3.2 ([15, Theorem 3.7]). *Either $\mathcal{C}^+$ and $\mathcal{C}^-$ are both unbounded, or $\mathcal{C}^+ \cap \mathcal{C}^- \neq \{(\lambda_1, 0)\}$.*

Proof. To prove the main results of [15], the authors use the Browder-Petryshyn and Skrypnik degree for perturbation of monotone operators, which is described in Section 5.1 of [15]. In particular, in this approach, the crucial 'jump' property of the degree (equation (5.13) in [15], corresponding to (2.4) in the context of Section 2) is established directly for (3.1) with an arbitrary $p > 1$, without resorting to the homotopic deformation procedure outlined in Section 2. Using this property, the proof of the Rabinowitz-type bifurcation theorem [15, Proposition 3.5] follows Rabinowitz's original proof almost verbatim.

The proof of Theorem 3.2 is also almost identical to its semilinear counterpart, Theorem 2 in Dancer [8]. However, the passage from Lemma 5.7 to Lemma 5.8 in [15] (which correspond to Lemmas 2 and 3 of [8] respectively) is particularly difficult in the quasilinear setting. Without going into the degree theoretic details of these results, let us just mention that Lemma 5.7 is established under the assumption that, for $\lambda = \lambda_1$ fixed, $u = 0$ is an isolated solution of (3.1), while this condition is removed in Lemma 5.8. The truncation procedure used in this step turns out to be much more involved in the quasilinear case, and requires sharp estimates on the difference $\lambda - \lambda_1$, for solutions (λ, u) close to $(\lambda_1, 0)$. This significant technical contribution is carried out in Section 4.2 of [15]. □

Remark 3.3. (i) A result similar to Theorem 3.2 was already stated by Drábek [12, Theorem 14.20] in the context of (1.1), under the condition that, for $\lambda = \lambda_1$, $u = 0$ is an isolated solution of (1.1).

(ii) Using the inversion $u \mapsto v = u/\|u\|^2$, asymptotic bifurcation results (i.e. with $\|u\| \to \infty$ as $\lambda \to \lambda_1$) are also derived in [15], from the results mentioned above. We will not comment further on this here.

4. García-Melián and Sabina de Lis

From now on, and until the end of the paper, we will suppose that Ω is the unit ball of $\mathbb{R}^N$, centred at the origin, and that the function $h : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is spherically symmetric, that is, $h(x, \xi, \lambda) \equiv h(|x|, \xi, \lambda)$ (with the obvious abuse of notation). In particular, the assumptions on the smoothness of the domain in Section 2 and 3 are trivially satisfied. For differentiability reasons[6], we will also suppose that $p \geq 2$ throughout the rest of the paper.

We are then interested in solutions of (2.5), bifurcating from the simple eigenvalues $0 < \mu_{1,p} < \mu_{2,p} < \dots$ of (2.6). In addition to the properties of the eigenvalues $\mu_{n,p}$, $n \in \mathbb{N}$, summarized in Section 2.2, note that the eigenfunctions v_n can be chosen so that $v_n'(1) < 0$. (To simplify the notation we will omit the index p from the eigenfunctions — this should cause no confusion here since p will be fixed.)

[6]See the proof of Lemma 5.5.

In order to state the local bifurcation results of [13], for each $n \in \mathbb{N}$ we define a set $Z_n \subset C^1[0,1]$ by

$$Z_n = \Big\{u \in C^1[0,1] : u'(0) = u(1) = 0 \text{ and } \int_0^1 |v_n(r)|^{p-2} v_n(r) u(r)\, dr = 0\Big\}. \quad (4.1)$$

Theorem 4.1 ([13, Theorem 1])**.** *Suppose that $h \in C^1([0,1] \times \mathbb{R} \times \mathbb{R})$ satisfies*

(h) *$h(r, 0, \lambda) = 0$ for all $(r, \lambda) \in [0,1] \times \mathbb{R}$, and $\frac{\partial h}{\partial \xi}(r, \xi, \lambda) = o(|\xi|^{p-2})$ as $\xi \to 0$, uniformly for $x \in [0,1]$ and λ in bounded subsets of $\mathbb{R}$.*

Then for every $n \in \mathbb{N}$ there exist $\varepsilon = \varepsilon(n) > 0$ and two continuous mappings $\mu_n : (-\varepsilon, \varepsilon) \to \mathbb{R}$, $z_n : (-\varepsilon, \varepsilon) \to Z_n$ such that $\mu_n(0) = \mu_{n,p}$, $z_n(0) = 0$, and every solution (μ, u) of (2.5) in a neighbourhood of $(\mu_{n,p}, 0)$ in $\mathbb{R} \times C^0[0,1]$ has the form $(\mu_n(s), s[v_n + z_n(s)])$, for some $s \in (-\varepsilon, \varepsilon)$.

We will only say a few words here about this result, and Section 5 will present more details about the proof in the case $n = 1$, the other cases being treated similarly. First, it is interesting to remark that the structure of the solutions given by Theorem 4.1 is analogous to (3.5), although generalized to higher order eigenvalues, and with the important difference that

$$(-\varepsilon, \varepsilon) \ni s \mapsto (\mu_n(s), s[v_n + z_n(s)])$$

now defines a *continuous local curve*, passing through $(\mu_{n,p}, 0)$. Of course, for $n \geq 1$, this should come as no surprise since Theorem 4.1 pertains to a special case of the general problem considered in Section 3. We now observe a similar local structure about each eigenvalue $\mu_{n,p}$, with two branches emerging from $(\mu_{n,p}, 0)$, one corresponding to $s \in (0, \varepsilon)$, the other one to $s \in (-\varepsilon, 0)$. These 'positive' and 'negative' curves bifurcate with the u component tangential to $\pm v_n$, respectively.

The second main result of [13] is a global one, showing that much more information is available in the radial setting than what is given by Theorem 3.2. To state it precisely, we need to introduce some new notation. For each $n \in \mathbb{N}$ we let

$$\begin{aligned} \mathcal{S}_n^{\pm} = \{u \in C^1[0,1] &: u'(0) = u(1) = 0, \\ &u \text{ has exactly } n-1 \text{ simple zeros in } (0,1), \text{ and } \mp u'(1) > 0\}. \end{aligned} \quad (4.2)$$

Theorem 4.2 ([13, Theorem 2])**.** *For each $n \in \mathbb{N}$, let $\mathcal{C}_n$ be given by Theorem 2.2. Under the assumptions of Theorem 4.1, the following properties hold.*

(i) *$\overline{\mathcal{C}_n} = \mathcal{C}_n^+ \cup \{(\mu_{n,p}, 0)\} \cup \mathcal{C}_n^-$, where $\mathcal{C}_n^{\pm}$ are connected, $\mathcal{C}_n^{\pm} \subset \mathbb{R} \times \mathcal{S}_n^{\pm}$, and therefore $\mathcal{C}_n^+ \cap \mathcal{C}_n^- = \emptyset$.*

(ii) *Both connected pieces $\mathcal{C}_n^+$ and $\mathcal{C}_n^-$ are unbounded and do not contain trivial solutions $(\mu, 0)$.*

Remark 4.3. (a) We see from (i) that, in the radial case, the positive and negative components bifurcating from $\mu_{1,p}$ truly consist of positive and negative solutions, respectively.

(b) In view of Theorem 3.2, the assertion that $\mathcal{C}_1^+$ and $\mathcal{C}_1^-$ are both unbounded follows from (i). But Theorem 3.2 was not available to the authors of [13] since [15] was published in 2008 and [13] in 2002.

Proof. Starting with the local informations of Theorem 4.1, the proof of Theorem 4.2 given in [13] follows by standard topological arguments, which we shall not repeat here. The key property is the conservation of the nodal structure of $\pm v_n$

along the sets $\mathcal{C}_n^{\pm}$. This prevents them from meeting the line of trivial solutions at a point $(\mu,0)\neq(\mu_{n,p},0)$, hence implying that they are unbounded. ☐

5. Yours truly

In [14] we consider a slightly different form of (2.5), namely

$$\begin{gathered} -(r^{N-1}\phi_p(u'))' = \lambda r^{N-1} f(r,u), \quad 0<r<1, \\ u'(0)=u(1)=0, \end{gathered} \tag{5.1}$$

where we suppose that $f\in C^1([0,1]\times\mathbb{R})$ satisfies $f(r,0)=0$ for all $r\in[0,1]$, and that there exist $f_0, f_\infty \in C^0[0,1]$ such that

$$(f1) \quad \lim_{\xi\to 0}\frac{f(r,\xi)}{\phi_p(\xi)} = f_0(r)>0 \quad \text{and} \quad (f2) \quad \lim_{|\xi|\to\infty}\frac{f(r,\xi)}{\phi_p(\xi)} = f_\infty(r)>0,$$

uniformly for $r\in[0,1]$. We will make further hypotheses on f, all of which can easily be compared to those of the previous sections by letting

$$h(r,\xi,\lambda)=\lambda[f(r,\xi)-f_0(r)\phi_p(\xi)], \quad (r,\xi,\lambda)\in[0,1]\times\mathbb{R}^2.$$

In particular, two technical assumptions are made in [14] (see hypotheses (H4′) and (H5′) there), which prescribe the behaviour of $\frac{\partial f}{\partial \xi}$ consistently with (f1) and (f2), and ensure that the hypotheses of Theorem 4.1 are satisfied. Note, however, that the present setting is more general than that of Sections 2 and 4, due to the weight f_0, corresponding to the coefficient B in equation (3.1) of Section 3.

Thanks to (f1) and (f2), we are able to control the asymptotic behaviour of the solutions (λ,u), both as $u\to 0$ and $|u|\to\infty$. In this respect, the following two $(p-1)$-homogeneous problems play an important role:

$$\begin{gathered} -(r^{N-1}\phi_p(v'))' = \lambda r^{N-1} f_{0/\infty}(r)\phi_p(v), \quad 0<r<1, \\ v'(0)=v(1)=0. \end{gathered} \tag{$\mathrm{E}_{0/\infty}$}$$

Since the weights f_0 and f_∞ are both positive in $[0,1]$, the structure of the eigenvalues and eigenfunctions of $(\mathrm{E}_{0/\infty})$ is quite analogous to that of (2.6) — see [19, Section 5]. In particular, there exists a (first) eigenvalue $\lambda_{0/\infty}>0$, which is simple, with a corresponding eigenfunction $v_{0/\infty}\in C^1[0,1]$ that can be chosen so that

$$v_{0/\infty}>0 \text{ in } [0,1) \quad \text{and} \quad v_{0/\infty}'<0 \text{ in } (0,1]. \tag{5.2}$$

In the one-dimensional case, $N=1$, we are also able to deal with $f_\infty\equiv 0$, and λ_∞ is then defined to be $+\infty$.

Apart from the precise asymptotic behaviour as $|u|\to\infty$ prescribed by (f2), our main structural hypotheses are the following:

- $(f3)$ $f(r,\xi)\xi>0$, for $r\in[0,1]$ and $\xi\neq 0$;
- $(f4)$ for $r\in[0,1]$ fixed, $\xi\mapsto\dfrac{f(r,\xi)}{\phi_p(\xi)}$ is $\begin{cases}\text{increasing for } \xi\le 0,\\ \text{decreasing for } \xi\ge 0.\end{cases}$

Remark 5.1. (a) It follows from $(f3)$ that any non-trivial solution (λ,u) of (5.1) satisfies either $u>0$ in $[0,1)$ and $u'<0$ in $(0,1]$, or $u<0$ in $[0,1)$ and $u'>0$ in $(0,1]$ (see Section 4 of [14]).

(b) By $(f3)$ and $(f4)$, we have $0<f_\infty(r)\le f_0(r)$ for all $r\in[0,1]$. Note that a little bit more than $(f4)$ is actually required in [14]; the hypothesis (H3′) there implies that $\xi\mapsto\frac{f(r,\xi)}{\phi_p(\xi)}$ is strictly monotonous in an open (r,ξ)-set, which in turns

implies that $f_\infty(r) \neq f_0(r)$ for r in some open interval. It then follows from the comparison principle in [19, Section 4] applied to (5.1) and ($E_{0/\infty}$) that:
(i) $\lambda_0 < \lambda_\infty$;
(ii) any non-trivial solution (λ, u) of (5.1) satisfies $\lambda_0 \leq \lambda \leq \lambda_\infty$.

In order to state precisely the main result of [14], we introduce the spaces

$$X_p = \{u \in C^1[0,1] : \phi_p(u') \in C^1[0,1],\ u'(0) = u(1) = 0\} \quad \text{and} \quad Y = C^0[0,1].$$

We equip Y with its usual sup-norm, which we denote by $|\cdot|_0$.

Theorem 5.2 ([14, Theorem 2.4])**.** *Let $p > 2$ and suppose $(f1)$-$(f4)$. There exist $u_\pm \in C^1((\lambda_0, \lambda_\infty), Y)$ such that $u_\pm(\lambda) \in X_p$, $\pm u_\pm(\lambda) > 0$ on $[0,1)$ and, for any given $\lambda \in (\lambda_0, \lambda_\infty)$, $(\lambda, u_\pm(\lambda))$ are the only non-trivial solutions of* (5.1)*. Furthermore, for each $\nu = \pm$, we have*

$$\lim_{\lambda \to \lambda_0} |u_\nu(\lambda)|_0 = 0 \quad \textit{and} \quad \lim_{\lambda \to \lambda_\infty} |u_\nu(\lambda)|_0 = \infty. \tag{5.3}$$

To prove Theorem 5.2, it is convenient to consider the integral form of (5.1),

$$u = S_p(\lambda f(u)), \quad (\lambda, u) \in \mathbb{R} \times Y, \tag{5.4}$$

where $S_p : C^0[0,1] \to C^1[0,1]$ is the inverse of (minus) the radial p-Laplacian, explicitly given by

$$S_p(h)(r) = \int_r^1 \phi_{p'}\Big(\int_0^s \Big(\frac{t}{s}\Big)^{N-1} h(t)\,dt\Big)\,ds, \quad h \in C^0[0,1]. \tag{5.5}$$

This operator is continuous, bounded and compact. Before we can explain the proof of Theorem 5.2, we need a result about the differentiability of S_p, which depends on the value of $p > 1$. This relies on the related work [5], and we borrow the following notation from there:

$$B_p := \begin{cases} C^1[0,1], & 1 < p \leq 2, \\ W^{1,1}(0,1), & p > 2. \end{cases} \tag{5.6}$$

Theorem 5.3 ([14, Theorem 3.5])**.**
(i) *Suppose $1 < p < 2$. Then $S_p : C^0[0,1] \to B_p$ is C^1, and for all $h, \bar h \in C^0[0,1]$,*

$$DS_p(h)\bar h(s) = \frac{1}{p-1} \int_r^1 |u(h)'(s)|^{2-p} \int_0^s \Big(\frac{t}{s}\Big)^{N-1} \bar h(t)\,dt\,ds, \tag{5.7}$$

where $u(h) = S_p(h)$. Furthermore, $v = DS_p(h)\bar h \iff$

$$v \in B_p \quad \textit{and} \quad \begin{cases} -(p-1)(r^{N-1}|u(h)'(r)|^{p-2} v'(r))' = r^{N-1}\bar h(r), \\ v'(0) = v(1) = 0. \end{cases} \tag{5.8}$$

(ii) *Suppose $p > 2$ and let $h_0 \in C^0[0,1]$ be such that $u(h_0)'(r) = 0 \Rightarrow h_0(r) \neq 0$. Then there exists a neighbourhood V_0 of h_0 in $C^0[0,1]$ such that the mapping $h \mapsto |u(h)'|^{2-p} : V_0 \to L^1(0,1)$ is continuous, $S_p : V_0 \to B_p$ is C^1, and DS_p satisfies* (5.7) *and* (5.8)*, for all $h \in V_0$, $\bar h \in C^0[0,1]$.*

Proof. The proof follows closely that of Theorem 3.4 in Binding and Rynne [5]. In view of the definition of S_p in (5.5), the main difficulty is that, for $1 < p' < 2$, the Nemistkii mapping $u \mapsto \phi_{p'}(u)$ does not map $C^1[0,1]$ into itself — this is due to the lack of differentiability of $\phi_{p'}(s)$ at $s = 0$. Nevertheless, if $g \in C^1[0,1]$ has only simple zeros, then $\phi_{p'}$ maps a neighbourhood of g in $C^1[0,1]$ continuously into $L^1(0,1)$. This result [5, Lemma 2.1] is the key ingredient to the proof of Theorem 5.3. □

Remark 5.4. A similar result was stated in [13, Theorem 5] but there seems to be a mistake in the proof presented there. We do not understand the application of the mean-value theorem on p. 34 of [13], precisely because of the lack of differentiability of the function $\phi_{p'}$. Indeed, we have $1 < p' < 2$ since it is supposed that $p > 2$ for other differentiability reasons — see the proof of Lemma 5.5 below.

Proof of Theorem 5.2. The proof of Theorem 5.2 is in two steps:

(1) local bifurcation from $(\lambda_0, 0)$ in $\mathbb{R} \times Y$;
(2) global continuation and asymptotic analysis.

Step 1. This is essentially given by the $n = 1$ case in Theorem 4.1, although we consider a slightly more general setting here, where bifurcation occurs from the first eigenvalue of the weighted problem (E_0). However, our proof follows closely the arguments in [13].

We normalize the eigenvector v_0 of (E_0) so that $\int_0^1 r^{N-1} f_0 |v_0|^p \, dr = 1$ and, similarly to (4.1), we define the subspace[7]

$$Z = \Big\{ z \in Y : \int_0^1 r^{N-1} f_0 |v_0|^{p-2} v_0 z \, dr = 0 \Big\}.$$

Note that

$$Y = \operatorname{span}\{v_0\} \oplus Z. \tag{5.9}$$

The local bifurcation from $(\lambda_0, 0)$ now follows by applying the implicit function theorem as stated in [7, Appendix A] to the function $G : \mathbb{R}^2 \times Z \to Y$ defined by

$$G(s, \lambda, z) = \begin{cases} v_0 + z - S_p(\lambda f(sv_0 + sz)/\phi_p(s)), & s \neq 0, \\ v_0 + z - S_p(\lambda f_0 \phi_p(v_0 + z)), & s = 0. \end{cases}$$

Lemma 5.5 ([14, Lemma 5.1])**.** *There exist $\varepsilon > 0$, a neighbourhood U of $(\lambda_0, 0)$ in $\mathbb{R} \times Z$ and a continuous mapping $s \mapsto (\lambda(s), z(s)) : (-\varepsilon, \varepsilon) \to U$ such that $(\lambda(0), z(0)) = (\lambda_0, 0)$ and*

$$\{(s, \lambda, z) \in (-\varepsilon, \varepsilon) \times U : G(s, \lambda, z) = 0\} = \{(s, \lambda(s), z(s)) : s \in (-\varepsilon, \varepsilon)\}. \tag{5.10}$$

Proof. Let us first remark that we need $p \geq 2$ here for the Nemitskii mapping $z \mapsto \phi_p(v_0 + z)$ to be differentiable. We are thus in case (ii) of Theorem 5.3. Now, (E_0) is equivalent to

$$v_0 = S_p(\lambda_0 f_0 \phi_p(v_0)),$$

and we have $\lambda_0 f_0(0) \phi_p(v_0(0)) > 0$, where $r = 0$ is the only zero of v_0' by (5.2). Therefore, Theorem 5.3 implies that S_p is C^1 in a neighbourhood of $\lambda_0 f_0 \phi_p(v_0)$ in Y. This enables one to verify the regularity properties required by the implicit function theorem [7, Theorem A]. To apply this theorem, one still needs to check the usual non-degeneracy condition, namely that the linear mapping $D_{(\lambda,z)} G(0, \lambda_0, 0) : \mathbb{R} \times Z \to Y$ be an isomorphism. In view of (5.9), an inspection of the Fréchet derivative $D_{(\lambda,z)} G(0, \lambda_0, 0)$ shows that this condition is equivalent to the invariance of the subspace Z under the mapping

$$\bar{z} \mapsto L\bar{z} := \lambda_0 (p-1) DS_p(\lambda_0 f_0 \phi_p(v_0)) f_0 |v_0|^{p-2} \bar{z}.$$

[7] Our integral formulation of (5.1) automatically takes care of the boundary conditions, so we do not incorporate them in the definition of Z, unlike (4.1).

Using the properties of the derivative DS_p, the relation $L\bar{z} = z$ can be expressed as

$$\begin{gathered} -(r^{N-1}|v_0'|^{p-2}z')' = \lambda_0 r^{N-1} f_0 |v_0|^{p-2}\bar{z}, \quad 0 < r < 1, \\ z'(0) = z(1) = 0. \end{gathered} \tag{5.11}$$

Then, multiplying both sides of the differential equation in (5.11) by v_0 and integrating by parts shows that $\bar{z} \in Z \implies z \in Z$, completing the proof of Lemma 5.5. $\square$

Remark 5.6. Thanks to Theorem 5.3, we were thus able to fill in the gap in the proof of Theorem 4.1, in the case $n = 1$. Since the cases $n \geq 2$ are treated similarly, we believe that the conclusions of Theorem 4.1 are true for all $n \in \mathbb{N}$.

Step 2. Let us denote by $\mathcal{S}^{\pm} \subset \mathbb{R} \times Y$ the sets of positive and negative solutions of (5.4), respectively. We define a function $F : [0, \infty) \times Y \to Y$ by

$$F(\lambda, u) := u - S_p(\lambda f(u)), \quad (\lambda, u) \in [0, \infty) \times Y,$$

so that (5.1) is now equivalent to $F(\lambda, u) = 0$. It follows from Theorem 5.3 and Remark 5.1 (a) that F is C^1 in a neighbourhood of any $(\lambda, u) \in \mathcal{S}^{\pm}$, with

$$D_u F(\lambda, u)v = v - \lambda DS_p(\lambda f(u))\partial_2 f(u)v, \quad v \in Y.$$

Furthermore, using the monotonicity property $(f4)$, standard ODE arguments show that, for any $(\lambda, u) \in \mathcal{S}^{\pm}$, $D_u F(\lambda, u) : Y \to Y$ is an isomorphism. Hence, through each solution $(\lambda, u) \in \mathcal{S}^{\pm}$ passes a unique local C^1 curve, that can be parametrized by λ. It then follows by compactness arguments that any of these curves can be extended smoothly to the whole interval $(\lambda_0, \lambda_\infty)$, and that the solutions along these curves satisfy

$$\lambda \to \lambda_0 \text{ if and only if } |u|_0 \to 0 \quad \text{and} \quad \lambda \to \lambda_\infty \text{ if and only if } |u|_0 \to \infty.$$

Consequently, the uniqueness statement in Theorem 5.2 follows from the local uniqueness in (5.10). This concludes the proof of Theorem 5.2. $\square$

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