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POSITIVE AND FREE BOUNDARY SOLUTIONS TO SINGULAR NONLINEAR ELLIPTIC PROBLEMS WITH ABSORPTION: AN OVERVIEW AND OPEN PROBLEMS

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Abstract. We give a survey of recent results and open problems concerning existence and multiplicity of positive and/or compact support solutions to some semilinear elliptic equations with singular nonlinear terms of absorption type. This includes the case of discontinuous (at the origin) nonlinearities, which is treated by introducing maximal monotone graphs. Extensions to the p-Laplacian are also considered. The one-dimensional case is studied by using energy methods.

1. Introduction

Nonlinear elliptic equations arise in many different places in both pure and applied mathematics. In particular, the so-called reaction-diffusion equations (and then systems) have been studied in the last forty years as mathematical models for applications: chemical reactions, combustion, mathematical biology (mostly Lotka-Volterra systems), nerve impulses (Fitzhugh-Nagumo), genetics, and so on. In most of these reaction-diffusion models its formulation makes a distinction between the positive and negative parts of the reaction term. So they are formulated in terms of the simple equation

$$\begin{gathered} -\Delta u = f(x,u) - g(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, Δ is the Laplacian modeling linear diffusion and $f(x,u)$ and $g(x,u)$ are possibly nonlinear and non-negative functions modeling, respectively the exo-thermic (or sourcing) and endo-thermic (or absorption) reaction term, with the usual notation for models in chemical reactions. In most of the cases the unknown corrresponds to a "physical" magnitude which is not meaningful for negative values: so that, in the following we shall be only interested in non-negative solutions $u(x) \geq 0$ on Ω. Usually $f(x,0), g(x,0) \geq 0$, however, the case of f, g not "clearly" defined at $u = 0$, for instance either $f(x,0)$ or/and $g(x,0)$ going to $+\infty$ when $u > 0$ goes to 0 is still interesting in several applications

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(non-Newtonian fluids, chemical reactions, nonlinear heat equations) and has been widely studied from two important papers by Crandall, Rabinowitz and Tartar [15] and Stuart [49] around 1977. We also mention now (but we shall present more details later) cases in which f or/and g are discontinuous (or even multivalued) at $u = 0$ but their directional limit, as $u > 0$ goes to 0, is finite: it corresponds, for instance, to the so called "obstacle problem", the zero order reaction problem, and so on.

Maybe the simplest case of the above class of problems (1.1) corresponds to the formulation

$$\begin{gathered} -\Delta u = f(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

when $f(x) \geq 0$ in Ω is a prescribed function.

For problem (1.2) it is well known (see, e.g. [9]) that

(i) there at most one solution $u \geq 0$.

Moreover by the Strong Maximum Principle (see, e.g. [45]) it is well known that

(ii:a) $u > 0$, and

(ii:b) $\partial u/\partial n < 0$ on $\partial\Omega$, where n is the outside normal derivative.

Properties (i) and (ii) remain valid for a large class of nonlinear choices of $f(x, u)$ and $g(x, u)$. If for instance we assume $f(x, u) = f(x)$ and $g(x, u) = u^\beta$ for some $\beta \geq 1$ then the properties remain true even if $f \in L^1(\Omega)$ and even under the more general (and optimal if $f \in L^1_{loc}(\Omega)$) class of functions $f \in L^1(\Omega, d)$, where $d(x)$ is the distance to the boundary $\partial\Omega$ (see [12] and [21]). A different case in which the simple properties (i) and (ii) remain true is for the singular problem

$$\begin{gathered} -\Delta u = \frac{h(x)}{u^\alpha} \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where $0 < \alpha$ and $h(x) > 0$ is smooth in $\overline{\Omega}$.

Since the results on (1.3) are more scattered in the literature it seems interesting to make some more detailed comments on them, also since they will be of some relevance in other sections of this paper. Since it is clear that solutions cannot be $C^2(\overline{\Omega})$, the best regularity one could expect is $C^{1,k}(\overline{\Omega})$ for some $0 < k < 1$. Then it seems natural to look for classical solutions in the space $C^2(\Omega) \cap C(\overline{\Omega})$. It is still reasonable to consider weak solutions in the Sobolev space $H^1_0(\Omega)$. Actually, a result due to Lazer and McKenna [41] says that a classical solution is in $H^1_0(\Omega)$ if and only if $0 < \alpha < 3$. The first existence, uniqueness and regularity results for the model problem were proved in [15] and [49], where the regularity results were not always optimal. General regularity results for the model problem were obtained by Gui and Hua Lin [33]. These results can be summarized as follows (although some more general results could be also mentioned as well): for any $\alpha > 0$ there exists a unique solution $u > 0$ in $C^2(\Omega) \cap C(\overline{\Omega})$ of problem (1.3) for $h \equiv 1$. Moreover (a) If $0 < \alpha < 1$, then $u \in C^{1,1-\alpha}(\overline{\Omega})$, (b) if $\alpha > 1$, then $u \in C^{2/(\alpha+1)}(\overline{\Omega})$ and (c) If $\alpha = 1$, then $u \in C^\delta(\overline{\Omega})$ for any $\delta \in (0, 1)$ but $u \notin C^1(\overline{\Omega})$.

For problem (1.3) it is easily seen that once that $u \geq 0$, then necessarily $u > 0$ (see below). This follows from the extension to singular problems of the Strong Maximum Principle (see [37, 50]). The case in which $h(x)$ is also a singular function of the distance to the boundary has been also treated in the literature (see, e.g. [36, 31, 28] and their references).

Coming back to the general introduction of this paper, we can say now that our main interest in the rest of the paper concerns the consideration of problems (1.1) for which properties (i) or/and (ii) are not satisfied and then one may have

(i*) more than one positive solution $u > 0$ or/and

(ii*:a) solutions vanishing in a positively measured subset of Ω, and/or

(ii*:b) solutions $u > 0$ but such that $\partial u/\partial n = 0$ on part of $\partial\Omega$.

In a large sense we can call *free boundary solutions* those which satisfy (ii*:a) or (ii*:b). One typical example concerns the case of non-negative solutions having compact support contained in Ω; i. e., solutions $u \geq 0$ which are zero on subdomains of positive measure in Ω, the boundaries of these domains are called *free boundaries*. For some general references on this subject see, e.g., the books [18] and [1], and the references in [20]. Among the many cases of " non-monotone problems" (in the spirit of Chapter 3 of [18]) leading to such type of properties we shall pay a special attention to problems involving singular coefficients or/and nonlinear terms (for general aspects on singular problems see [36] and [32]).

We will consider first problem (I):

$$\begin{gathered} -\Delta u + \frac{1}{u^\alpha} = \lambda f(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $0 < \alpha$ (mainly $\alpha < 1$) and $f \geq 0$. Also we will consider problem (II):

$$\begin{gathered} -\Delta u + \frac{1}{u^\alpha} = \lambda u^p \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

with $0 < \alpha$ (mainly $\alpha < 1$) and $0 < p < 1$. The reader could have noticed that we are making use of the same exponent α in different terms of the equations and so obeying to different purposes: the only reason to maintain such small confusion is that we prefer to use a notation for the nonlinear terms as close as possible to the original one used by most of the authors which are mentioned in the text. We point out that in order to get the existence of free boundary solutions the singular term $1/u^\alpha$ must be replaced by the expression $\chi_{\{u>0\}}1/u^\alpha$ where, in general, χ_ω denotes the characteristic function of the set $\omega \subset \Omega$. Notice that if $u > 0$ then both expressions coincide.

Maybe one could think, naïvely, that for $f \equiv 1$, problem (II) "tends" in some sense to problem (I) when $p > 0$ goes to 0. We will come back to this question later. A variant to problems (I) and (II) is problem (III):

$$\begin{gathered} -\Delta u = K(x)(\lambda u^\beta - u^\alpha) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Here $K(x) = 1/d(x)^k$ with $0 < k < 2$ and $0 < \alpha < \beta < 1$. For $K \equiv 1$ we recover problem (II).

Recently, in [26] Díaz and Rakotoson obtained some new results concerning the regularity of very weak solutions to linear elliptic equations, showing in particular that the gradients are in some Lorentz-Sobolev space and extending in this way previous work by H. Brezis and other people (see, e.g. [9, 26] and their references). These results were applied in [27] to semilinear elliptic problems and in [21] (see also [47] and [28]) to the above model singular problem (1.3), obtaining existence and regularity for equations even with very irregular coefficients $h(x)$ above. The same kind of ideas are applied in [22] to problems (I) and (II).

In Section 2 we collect known results for both problems (I) and (II) giving only slight information on the different methods of proof. We emphasize the role of compact support solutions. In Section 3 we deal with problem (III) in the same way. In Section 4 we provide an approach (see [20]) which allows to deal, in the one-dimensional case, with both problems (I) and (II) in a unified way, and is applicable to the quasilinear p-Laplacian as well. Finally, in Section 5 we present a short list of some interesting open problems in the theory. Again, they concern principally the possible multiplicity of positive solutions and of free boundary solutions in general domains in $\mathbb{R}^N$ for $N > 1$.

2. Problems (I) and (II): existence and multiplicity of solutions

Before considering different results in the literature on problem (1.1) and its special formulations mentioned above, let us recall that we use the terminology of *weak solution* if, at least, $u \in W_0^{1,1}(\Omega)$, $f(x,u) - g(x,u) \in L^1(\Omega)$ and for any $\zeta \in W_0^{1,\infty}(\Omega)$ we have

$$\int_\Omega \nabla u(x) \cdot \nabla \zeta(x) dx = \int_\Omega (f(x,u(x)) - g(x,u(x)))\zeta(x) dx.$$

The weaker notion of *very weak solution* of problem (1.1) concerns the case in which we merely require that $u \in L^1(\Omega)$, $f(x,u) - g(x,u) \in L^1(\Omega, d)$ and

$$\int_\Omega u(x)\Delta\zeta(x) dx = \int_\Omega (f(x,u(x)) - g(x,u(x)))\zeta(x) dx,$$

for any $\zeta \in W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega)$. Notice that in the case in which $g(x,u) = \chi_{\{u>0\}} 1/u^\alpha$ we shall understand that

$$-\int_\Omega g(x,u(x))\zeta(x) dx = \int_{\{u>0\}} \frac{\zeta(x)}{u(x)^\alpha} dx,$$

(and analogously if $f(x,u)$ involves a term of this type).

Other notions of solution (as, for instance the so called renormalized or entropy solutions) could be also recalled here but we shall avoid such technical details for the moment. Another important class of solutions concerns the case in which problem (1.1) can be understood in a weak sense related to the Calculus of Variations: we say that u is a *variational solution* of (1.1) if $u \in W_0^{1,2}(\Omega)(\equiv H_0^1(\Omega))$, $F(x,u) - G(x,u) \in L^1(\Omega)$ and u is a stationary point of the functional

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v(x)|^2 dx - \int_\Omega F(x,v(x)) dx + \int_\Omega G(x,v(x)) dx,$$

where

$$F(x,r) = \int_0^r f(x,\sigma) d\sigma \quad \text{and} \quad G(x,r) = \int_0^r g(x,\sigma) d\sigma.$$

Notice that, if for instance, $g(x,u) = \chi_{\{u>0\}} 1/u^\alpha$ with $0 < \alpha < 1$ then

$$G(x,v(x)) = \chi_{\{v>0\}}(x) \frac{(-\alpha+1)}{v(x)^{\alpha-1}} \quad \text{a.e. } x \in \Omega.$$

The adaptation of the above notions to the case in which f or/and g are discontinuous (or even multivalued) at $u = 0$ but their directional limit, as $u > 0$ goes to 0, is finite (e.g., the " obstacle problem" or the zero order reaction problem) will be commented later.

Now we pass to mention to some of the results in the literature:

Existence for problem (I). It was proved by Díaz, Morel and Oswald [24] that for $\alpha \in (0,1)$ there exists $\lambda^* > 0$ such that for any $\lambda > \lambda^*$ there exists a maximal positive solution $u_\lambda > 0$ in the Sobolev space $H_0^1(\Omega)$. This result was obtained by using a method of sub and supersolutions adapted to this case. It is easy to show that there are no solutions for $\lambda > 0$ small enough, but it remains the problem of the possible existence of more solutions, either positive or with compact support. This result was extended later for general f's, actually for f's behaving like $d(x)^\beta$ for $0 < \beta < 1$ (also for $\alpha \in (0,1)$), where $d(x)$ is the distance to the boundary $\partial\Omega$ in [38] (see also [46] for some related results concerning the parabolic associated problem).

It was shown in [13] that $\alpha \in (0,1)$ is a necessary condition in order to have existence of a weak solution u (i.e., such that $\frac{1}{u^\alpha} \in L^1(\Omega)$). Estimates on the boundary behavior of the minimal weak solution in the one-dimensional case were obtained in [20] (see (4.3) below). Recently we improve these results in [22] in the sense that when $\alpha \in [1,2)$ there exists a very weak solution (i.e., such that $\frac{1}{u^\alpha} \in L^1(\Omega, d)$ although $\frac{1}{u^\alpha} \notin L^1(\Omega)$). Moreover the estimates (4.3) are preserved as in the one-dimensional case. We also show in [22] that no very weak solution exists if $\alpha \geq 2$.

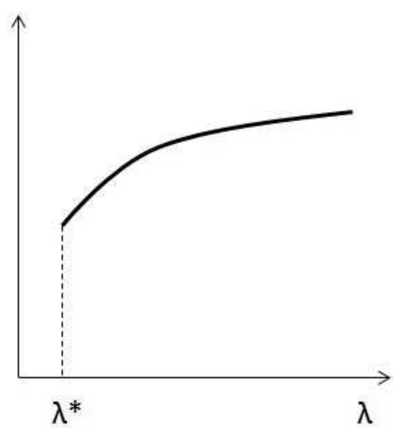

Figure 1. Bifurcation diagram when there is a unique positive solution

Multiplicity for problem (I). Choi, Lazer and McKenna [13] treated the very simple case $f \equiv 1$, $\Omega = (-1,1)$. They obtained, by using ODE methods a maybe surprising result: there are values of $\lambda > 0$ such that there is exactly one positive solution for $\alpha = 1/2$ and at least 2 for $0 < \alpha < 1/3$. It is pretty obvious that it is also a partial result. Some time later a complete answer was given by Horváth and Simon [39] showing that for some λ's there is exactly one solution for $1/2 \leq \alpha < 1$ and exactly 2 for $0 < \alpha < 1/2$.

Existence for problem (II). We have the following results.

(a) Shi and Yao [48] proved that there exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*$ there exists a maximal solution $u_\lambda > 0$, which is obtained this time by approximating the singular problem by a regular one to which well-known techniques (e. g., sub and supersolutions) can be applied to get a unique solution to the approximate problem and then pass to the limit to obtain a solution of the original singular problem. Once again, there is no solution for small λ.

(b) Dávila and Montenegro [16] obtained some time later a related result with the formulation involving the term $\chi_{\{u>0\}} 1/u^\alpha$ (instead of merely $1/u^\alpha$), this time by using variational methods for the associated functional in $H_0^1(\Omega)$. They prove that for any $\lambda > 0$ there exists a maximal solution $u_\lambda \geq 0$ and there exists a $\lambda^* > 0$ satisfying

(i) If $\lambda > \lambda^*$, then $u_\lambda > 0$.
(ii) For $\lambda = \lambda^*$, $u_\lambda > 0$ and $u \in C^{\frac{1-\alpha}{1+\alpha}}(\overline{\Omega})$.
(iii) If $0 < \lambda < \lambda^*$, then $u_\lambda = 0$ on $A \subset \Omega$, where $|A| > 0$.

It is possible to show that in case (iii) $A = \Omega$ if Ω is the ball $B_R(0)$.

(c) Finally, the same result of i) was obtained in a more general framework and by using a completely different technique in [38]: by a change of variable, the problem was reduced to one such that the Implicit Function Theorem (see [37], [36]) could be applied, providing in this way a branch of positive solutions by continuation arguments.

Multiplicity for problem (II). (a) Ouyang, Shi and Yao [44] studied the radial case $\Omega = B_r(0)$ for $N \geq 1$. They only consider radial solutions but they notice that, since the famous Gidas-Ni-Nirenberg theorem is not applicable, existence of non-radial solutions cannot be excluded. By employing the usual ODE methods they show the existence of two values $0 < \lambda^* < \lambda^{**}$ such that for $\lambda^* < \lambda < \lambda^{**}$ there are two radial solutions u_1 and u_2 verifying $u_1 < u_2$ and for $\lambda > \lambda^{**}$ there is only the solution u_2. Moreover, there is a turning point for $\lambda = \lambda^*$ and the normal derivative is < 0 along both branches but it turns out that $\partial u_{1,\lambda}/\partial n|_{\lambda=\lambda^{**}} = 0$. Otherwise stated, the solution branch "lives" in the interior of the positive cone all the time but the lower branch "hurts" the boundary for $\lambda = \lambda^{**}$. Here we have a complete description of radial solutions, and we may ask why the branch "stops" at the value $\lambda = \lambda^{**}$.

(b) In [19] Díaz and Hernández treated the one-dimensional problem

$$\begin{gathered} -u'' + u^\alpha = \lambda u^\beta \quad \text{in } \Omega = (-1,1), \\ u(\pm 1) = 0, \end{gathered} \tag{2.1}$$

where the condition $0 < \alpha < \beta < 1$ is satisfied. One can see that this nonlinearity is not singular but it is not locally Lipschitz close to the origin, which is the main necessary condition related with existence of free boundaries. See the book [18] and the recent paper [20] for more details on these matters. They obtain the same result of part a) concerning positive solutions, but also an additional result answering one of the above questions. Starting from the positive solution having zero boundary derivative in the lower branch it is possible to produce infinite continua of compact support solutions which, since they have zero boundary derivative, are solutions of the equation on the whole real line. A complete description of these continua is given in [19].

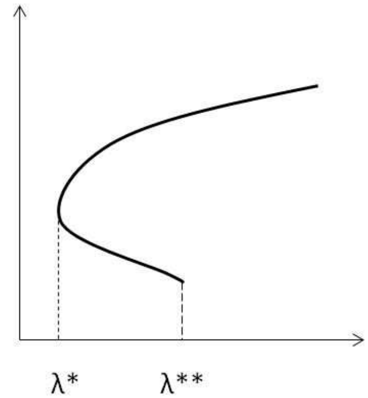

FIGURE 2. Bifurcation diagram including only positive solutions

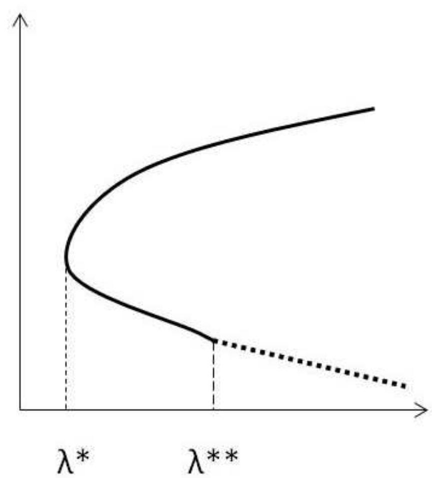

FIGURE 3. Bifurcation diagram including positive and compact supported solutions (dotted line)

Concerning stability of solutions considered as stationary solutions of the associated parabolic problem, one could think that maximal positive solutions are actually (linearly) stable. For this matter see [7, 16, 38, 46]). Periodic-parabolic problems with the same non-singular nonlinear terms were studied in [34] (concerning free boundary periodic solutions see, e.g., [4], its references, and [29]). We mention here that the stability (even the continuous dependence) of free boundary solutions (in the sense of having also the stability of the associated free boundaries) is a quite complicated task where many counterexemples may arise (see, e.g., the recent paper [23] and its references).

3. Existence of positive and compact support solutions to Problem (III)

Now we study the problem

$$
\begin{gathered}
-\Delta u = K(x)(\lambda u^\beta - u^\alpha) \quad \text{in } \Omega,\\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

with $K(x) = 1/d(x)^k$ with $0 < k < 2$ and $0 < \alpha < \beta < 1$, hence the problem is not singular in u. This problem was studied by Haitao in [35], where he proved the following results

(i) If $k - \alpha < 1$, there exists λ^* such that for all $\lambda > \lambda^*$ there is a solution $u_\lambda > 0$ to (III) and there is no solution for λ small;

(ii) If $1 < k - \alpha < 2$ there exists at least a solution with compact support for all $\lambda > \lambda^{**}$ for some $\lambda^{**} > 0$.

The results in (i) were proved by using sub and supersolutions and then those in part ii) by combining variational arguments for the associated functional in $H_0^1(\Omega)$ with a compact support principle. Notice that for $K \equiv 1$ we have the problem (II). This dichotomy can be understood also in the spirit of the "balance between the data and the domain" which is also needed to have free boundary solutions, in a bounded domain, in the monotone case (see [18, Subsection 1.2b]). We also mention that free boundary solutions of singular higher order equations or systems can be studied with the help of energy methods (see [1] and its references).

It seems relevant to mention here the case in which the boundary conditions are not homogeneous as, for instance, problem $P(f, g)$:

$$
\begin{gathered}
-\Delta u = f(x, u) - g(x, u) \quad \text{in } \Omega\\
u = 1 \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.1}
$$

Notice that by making $w = 1 - u$ we get to the formulation

$$
\begin{gathered}
-\Delta w = -f(x, 1 - w) + g(x, 1 - w) \quad \text{in } \Omega\\
w = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.2}
$$

In the special case of $f(x, u) \equiv 0$ it is easy to see that $0 \le u \le 1$ on $\overline{\Omega}$, so that we can write problem (3.2) in the form

$$
\begin{gathered}
-\Delta w + g^*(x, w) = f^*(x) \quad \text{in } \Omega\\
w = 0 \quad \text{on } \partial\Omega
\end{gathered} \tag{3.3}
$$

with $g^*(x, w) = g(x, 1) - g(x, 1 - w)$ and $f^*(x) = g(x, 1)$. For instance, in the case of α-order endothermal chemical reactions, $g(x, u) = \mu u^\alpha$ and if $\alpha \in (0, 1)$, then we transform a concave non-Lipschitz function at $u = 0$ into a convex non-Lipschitz function at $w = 1$. In fact, the free boundary (the boundary of the so called "dead core") is transformed into the free boundary given as the boundary of the level set $\{w = 1\}$. That was extensively used in the book [18] in order to derive many different qualitative properties of solutions. We also send the reader to [18] for the detailed references of many papers in the literature dealing with $\alpha-$order reactions (Bandle, Sperb and Stagold, Díaz and Hernandez, Friedman and Phillips, Caffarelli-Friedman, etc.). Many references dealing with the case in which the order of the reaction is assumed to be negative can be found also there (see, in Subsection 2.3, the mention to the papers by Alt and Phillips, Alt and Caffarelli, Giaquinta-Giusti, Brauner and Nicolaenko, Misiti and Guyot, etc.).

We prefer to drop of our presentation the case in which Ω is unbounded, as it is the case of $\Omega = \mathbb{R}^N$. The list of references could be otherwise very long (see, for instance, the many references of the book [18] and [14] to mention only two of them).

As extensively mentioned in [18], the diffusion operator can be replaced, for many different studies, by a general linear second order elliptic operator (including , or not, some first order transport terms), as well as by some quasilinear second order operators, possibly degenerated as the p-Laplacian operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $p > 1$. For higher order operators some suitable energy methods replace the role of the comparison principle as a fundamental tool to study free boundary solutions (see [1]). We also mention here that for many different purposes the results valid for the zero-order nonlinear terms $f(x,u) - g(x,u)$ can be extended to the case in which there is a explicit dependence on the gradient $f(x,u,\nabla u) - g(x,u,\nabla u)$. The detailed mention here to the extensive literature on this kind of quasilinear equations would make this survey much less shorter than intended, so we limit ourselves to mention here the series of papers [3] [2] and [8]. For multiplicity and free boundary solutions for a quasilinear problem related to the classical *brachistochrone curve* and *fluid bridges* see [25] and its references. As it is well-known, such possible first order terms arise for the equation of an unknown transformation $v = \psi(u)$ where u verifies a problem in which the first order terms are absent (as, for instance problem (1.1)). For an interesting study relating the possible singularities of a quasilinear equation with free boundary solutions of a problem of (1.1) see [11]. Concerning free boundary solutions for quasilinear equations of Hamilton-Jacobi type, without zero order terms see, e.g. [5, 45, 1, 6] and their references. A recent paper concerning free boundary solution for Monge-Ampère equations is [17].

An interesting extension of problem (III) to both the singular case and the p-Laplacian was recently obtained by Giacomoni, Maagli and Sauvy [30]. They consider the problem

$$\begin{gathered} -\Delta_p u = K(x)(\lambda u^\beta - u^\alpha) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

under assumptions $-1 < \alpha < \beta < p-1$, $0 < k < p$ and look for weak solutions in the space $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Actually, they prove an interesting regularity result, that solutions are in both cases in the space $C^{1,\delta}(\overline{\Omega})$, for some $0 < \delta < 1$.

Concerning existence, there is still no solution for small $\lambda > 0$ and

(i) If $k - \alpha < 1$, there exists λ^* such that for all $\lambda > \lambda^*$ there exists a positive solution $u > 0$;

(ii) If $k - \alpha > 1$, it is proved that all solutions have compact support (Propositon 5.3 in [30]). Then sufficient conditions are given for existence for $\lambda > \lambda^{**}$; i.e., for some $\lambda^{**} > 0$, in Theorem 2.2, either $1 + \alpha \geq \beta$ and $1 + \alpha \leq k < 2 + \alpha$ or $1 + \alpha > \beta$ and $k \in [1 + \alpha, 1 + (p-1)(1+\alpha)/(p - \beta + \alpha))$.

Existence in part (i) is proved by sub and supersolutions. In part (ii) variational arguments are combined with suitable truncations and compact support considerations.

4. Problems (I) and (II) "unified" one-dimensional case

A unified treatment of problems (I) and (II) was provided by Díaz, Hernández and Mancebo in [20], where they try to deal simultaneously with problems (I) and

(II), extend the results of Díaz-Hernández [19] to the singular case and generalize the above to the case of the $p-$Laplacian in the one-dimensional case. We consider now the one-dimensional quasilinear boundary value problem

$$
\begin{gathered}
-(|u'|^{p-2}u')' = f(u) \quad \text{in } \Omega = (-1,1), \\
u(\pm 1) = 0,
\end{gathered}
$$

where f satisfies

$$
\begin{gathered}
f(0+) := \limsup_{s\to 0} f(s) \leq 0 \quad (-\infty \text{ is allowed}), \\
\text{There exists a } r_f > 0 \text{ such that } f(r_f) = 0 \text{ and} \\
f(s) < 0 \text{ if } s \in (0, r_f) \text{ and } f(s) > 0 \text{ if } s \in (r_f, +\infty), \\
F(u) = \int_0^u f(s)ds > -\infty \quad \text{for any } u > 0, \\
\limsup_{s\to 0}(pF(s) - sf(s)) = 0, \\
\text{There exists a } r_F > r_f \text{ such that } F(r_F) = 0 \text{ and} \\
F(s) < 0 \text{ if } s \in (0, r_F) \text{ and } F(s) > 0 \text{ if } s \in (r_F, +\infty).
\end{gathered}
$$

A particular case is our preceding example $f(u) = \lambda u^\beta - u^\alpha$ with $-1 < \alpha < \beta < p-1$ ($\alpha = 0$ and $\beta = 0$ are included, see below). If $p = 2$ the condition reads $-1 < \alpha < \beta < 1$.

If we define as above $F(u) = \int_0^u f(s)ds$ there are two different cases, namely:

$$
\int_0^\delta \frac{dr}{(-F(r))^{(p+1)/p}} = +\infty, \quad \text{for some } \delta > 0 \text{ small}, \tag{4.1}
$$

or

$$
\int_0^\delta \frac{dr}{(-F(r))^{(p+1)/p}} < +\infty, \quad \text{for some } \delta > 0 \text{ small}. \tag{4.2}
$$

In particular $0 < \alpha < \beta < 1$ implies (4.1). More generally (4.1) holds if and only if $-1/(p+1) \leq \alpha$ (if $p = 2$, $\alpha \leq -1/3$, a exponent in Choi-Lazer-McKenna [13]) and also (4.2) holds if and only if $-1/p+1 > \alpha > -1$.

Under assumption (4.2) we only obtain a partial result concerning multiplicity. To be more precise, there exists $\delta > 0$ such that there is exactly one solution on the interval $(-1, -1+\delta]$ and there are exactly two solutions on $(-1/p+1-\delta, -1/p+1)$. It is reasonable to guess that there should be a unique α^* such that there is only one solution on $(-1, \alpha^*]$ and two on $(\alpha^*, -1/p+1)$. We were unable to prove that this happens in general, but this is the case if the condition $\beta = 1 + 2\alpha$ is satisfied. In this case we have $\alpha^* = -1/(1 + p/2)$. In particular, for $p = 2$ we get $\alpha^* = 1/2$, another Choi-Lazer-McKenna exponent.

The following estimates on the boundary behavior of the *minimal weak solution* u_1 were obtained in [20],

$$
\underline{k}|\pm 1 - x|^{\frac{p}{p-(1+\alpha)}} \leq u_1(x) \leq \overline{k}|\pm 1 - x|^{\frac{p}{p-(1+\alpha)}}, \quad \text{for any } x \in (-1,1), \tag{4.3}
$$

for some positive constants $\underline{k}$ and $\overline{k}$.

We mention above the relationship which may be established between problems (I) and (II) in the sense that "it seems" that if $p > 0$ goes to zero, then problem (II) "tends" in some sense to problem (I) if $f \equiv 1$. However, one should notice that the limit when $p > 0$ goes to 0 of the nonlinear function u^p is not the constant function equals to 1 but the discontinuous function g such that $g(0) = 0$ and $g(s) = 1$ if $s > 0$. Then we have chosen to treat the problem corresponding to the associated maximal monotone graph obtained by "filling the gap" at 0 by putting $g(0) = [0, 1]$. This allows to consider the associated functional having this maximal monotone graph as its subdifferential and calculate as for the cases studied before. We arrive to formulations of the type of the problem $P^H_{\lambda,\beta}$:

$$\begin{gathered} -(|u'|^{p-2}u')' + H(u) \ni \lambda u^\beta \quad \text{in } \Omega = (-1, 1), \\ u(\pm 1) = 0 \end{gathered} \tag{4.4}$$

which "formally" coincides with problem (2.1) with $\alpha = 0$. Here $0 < \beta < p - 1$ and $H(u)$ denotes the maximal monotone graph of $\mathbb{R}^2$ associated to the Heaviside function

$$H(r) = \begin{cases} 1 & \text{if } r > 0, \\ [0, 1] & \text{if } r = 0, \\ 0 & \text{if } r < 0. \end{cases}$$

The case of the multivalued equation $-(|u'|^{p-2}u')' + u^\alpha \ni \lambda H(u)$ was also considered in [20]. We recall that the *obstacle problem* corresponds to the maximal monotone graph of $\mathbb{R}^2$ given by

$$H(r) = \begin{cases} 0 & \text{if } r > 0, \\ (-\infty, 0] & \text{if } r = 0, \\ \phi & \text{if } r < 0. \end{cases}$$

In the general case of a maximal monotone graph of $\mathbb{R}^2$, $H(u)$, the notion of *non-negative strong solution* of problem $P^H_{\lambda,\beta}$ is introduced by requiring $u \in W^{1,p}_0(\Omega)$, $u \geq 0$ on Ω, $u^\beta \in L^1(\Omega)$, $(|u'|^{p-2}u')' \in L^1(\Omega)$, and $-(|u'|^{p-2}u')'(x) + h(x) = \lambda u(x)^\beta$ for a.e. $x \in \Omega$, for some $h \in L^1(\Omega)$ such that $h(x) \in H(u(x))$ for a.e. $x \in \Omega$ (see [20] and its references).

5. Some open problems

We have mentioned several open questions in the preceding one-dimensional problem, and in Section 2. We now collect some other open problems concerning mostly the case of a general domain (not a ball) for $N > 1$.

(a) Recently Ilyasov and Egorov [40] have studied the problem

$$\begin{gathered} -\Delta u + u^\alpha = \lambda u^\beta \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

with $0 < \alpha < \beta < 1$ as before. Now Ω is a star-shaped domain and the condition $N > 2(1 + \alpha)(1 + \beta)/(1 - \alpha)(1 - \beta)$ is also satisfied. Under these assumptions they prove, by using a combination of variational and continuation arguments, together with Pohozaev identity, that for some λ's

(i) there exists a solution $w_\lambda > 0$ such that $\partial w_\lambda/\partial n \neq 0$ on $\partial\Omega$,

(ii) there exists a solution $u_\lambda \geq 0$ such that $\partial u_\lambda/\partial n = 0$ on $\partial\Omega$.

One could think that in the first case we actually have $\partial w_\lambda/\partial n \leq 0$ on $\partial\Omega$ and in the second $u_\lambda > 0$.

We mention that cases in which the solution is $u > 0$ in Ω but with $\partial u/\partial n = 0$ on a part of $\partial\Omega$ can be understood as solutions with overdetermined boundary conditions (see [10] concerning a formulation close to problem (I)).

(b) A related, but different approach, has been used recently by Montenegro [42] for problem (II), under the same assumptions on α and p. He considers for $\epsilon > 0$ the approximated smooth problem (with $0 < q < p$)

$$
\begin{gathered}
-\Delta u + \frac{u^q}{(u+\epsilon)^{q+\alpha}} = \lambda u^p \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

and the associated functional on the Sobolev space $H_0^1(\Omega)$. Then he proves that this functional has a minimum $u_\epsilon \geq 0$ and also a second critical point $v_\epsilon \geq 0$ (obtained by using the Mountain Pass Theorem) and that when ϵ goes to 0 they tend to (again different) solutions $u \geq 0$ and $v \geq 0$. One can guess that $u > 0$ and that maybe v has compact support, but this is not proved in the paper. In a more recent paper [43] the authors employ a perturbation of domain technique and show that for λ large there is a unique (in some class of functions) positive solution for p "small enough".

(c) Concerning the results for problem (III) obtained by Haitao [35] and the generalization to the p-Laplacian in [30], it would be interesting to know if there is more than one positive solution in the first case $k - \alpha < 1$ and if, also in this case, there are compact support solutions. Information on the structure of the set of compact support solutions is interesting as well.

(d) Multiplicity for different diffusion operators (not only a general linear second order operator or the p-Laplacian operator, but other nonlinear operators of Hamilton-Jacobi type, Monge-Ampère, bi-Laplacian, nonlinear systems, Schroedinger systems, etc.)

(e) Presence of measures at the equation (not only as a sourcing data but also as coefficients of zero order terms, ...)

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