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LOCALIZATION PHENOMENA IN A DEGENERATE LOGISTIC EQUATION

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ABSTRACT. We analyze the behavior of positive solutions of elliptic equations with a degenerate logistic nonlinearity and Dirichlet boundary conditions. Our results concern existence and strong localization in the spatial region in which the logistic nonlinearity cancels. This type of nonlinearity has applications in the nonlinear Schrodinger equation and the study of Bose-Einstein condensates. In this context, our analysis explains the fact that the ground state presents a strong localization in the spatial region in which the nonlinearity cancels.

1. INTRODUCTION

In this paper we analyze the behavior of positive solutions of elliptic equations with a degenerate logistic nonlinearity and Dirichlet boundary conditions

$$\begin{gathered} -\Delta u = \lambda u - n(x)u^{\rho} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain, $\rho > 1$, $\lambda \in \mathbb{R}$ and $n(x) \geq 0$ in Ω and $n(x)$ is not identically zero.

We will also assume that $n(x)$ remains strictly positive near the boundary of Ω and therefore

$$K_0 = \{x \in \Omega : n(x) = 0\} \subset \Omega \quad \text{and } K_0 \text{ is a nonempty compact set.} \tag{1.2}$$

Despite a large amount of mathematical literature in this kind of logistic equations, see below, this type of nonlinearity has applications in the nonlinear Schrodinger equation and the study of Bose-Einstein condensates. In this context, assumption (1.2) implies the fact that the *ground state* presents a strong localization in the spatial region K_0, see [19] and references therein.

Throughout this article we shall assume that the compact set K_0 and the function $n(x)$ satisfy the following hypotheses:

(Hn) $n(x)$ is a Hölder continuous function and

$$n(x) \geq C\big(d_0(x)\big)^{\gamma} \quad \text{for some } \gamma > 0.$$

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where $d_0(x) := \mathrm{dist}(x, K_0)$, and

(HK) $K_0 = K_1 \cup K_2 \subset \Omega$, where K_1 and K_2 are compact sets and $K_1 = \overline{\Omega}_0$ is the closure of a regular connected open set $\Omega_0 \neq \emptyset$, K_2 has zero Lebesgue measure.

In some cases (HK) will be strengthened to

(HK') K_0 satisfies (HK) and K_2 is a closed regular d-dimensional manifold, with $d \leq N-1$.

When the set K_0 is empty, that is, if $n(x)$ is strictly bounded away from zero, problem (1.1) is classical and well understood, see e.g. [20] and references therein. Also, when K_0 is "smooth" in the sense that in (HK) we have $K_0 = K_1 = \overline{\Omega}_0$ where Ω_0 is a smooth open set, and $K_2 = \emptyset$, this problem has also been studied in [17, 7, 8, 9, 10, 15] and further developments in [11, 12, 16]. Therefore here we focus on the effect on the solutions of the presence of the part with empty interior K_2.

As a general notation, we will denote by $\lambda_1(U)$ the first eigenvalue of the Laplace operator with Dirichlet boundary conditions in the open and smooth set U.

As will be shown below, by standard estimates on (1.1), if the parameter λ is below the value $\lambda_1(\Omega)$, the unique non negative solution is $u \equiv 0$. Moreover, as λ crosses the value $\lambda_1(\Omega)$, a bifurcation phenomena takes place and a unique positive solution emanates from the trivial one. This solution can be continued in λ up until it reaches some critical value, λ_c. By monotonicity properties of the first eigenvalue (with respect to the domains and to the potentials), it is an easy task to realize that the critical value λ_c is equal to $\lambda_1(\Omega_0)$, see Lemma 2.1, part (i). Note that this is precisely the same situation as when K_0 is "smooth", i.e. $K_2 = \emptyset$. When K_0 is empty, the picture is also as above, with $\lambda_c = \infty$.

Our goal is then to give a detailed description of the behavior of this branch of solutions for $\lambda \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$ and specially as $\lambda \to \lambda_1(\Omega_0)$. First we show that the solutions blow up in compact sets of Ω_0 (see Lemma 3.1 below). Also, we will show that the solutions are uniformly bounded in compact sets of $\Omega \setminus K_0$ (see Proposition 3.3 below). Hence, it remains to analyze the behavior of solutions in K_2, which is not so clear at all. In K_2 we have two competing mechanisms: on one hand the fact that $n(x) \equiv 0$ in K_2 "pushes" the solution towards $+\infty$ while the fact that K_2 is not "fat" enough means that this effect may not have enough room to force the solution to go to infinity.

We will distinguish two situations for which we will be able to show that the solutions remain bounded in K_2. In case $K_2 \cap K_1 = \emptyset$, then any solution will be bounded in K_2, actually it will be so in a neighborhood of K_2. In the case $K_2 \cap K_1 \neq \emptyset$, it will turn out that a balance between the geometry of K_2 and the strength of the logistic term, given by the exponent ρ and the behavior of the function $n(x)$ near K_2, will determine the behavior of the solution. As a matter of fact we will be able to prove the following result.

Theorem 1.1. *Assume K_0 satisfies* (HK) *and $n(x)$ satisfies* (Hn). *Then for any $\lambda \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$ there exists a unique positive solution of* (1.1), *φ_λ, and we have*

$$\lim_{\lambda \to \lambda_1(\Omega_0)} \varphi_\lambda(x) = \infty, \quad \text{for all } x \in \Omega_0, \tag{1.3}$$

and the limit is uniform in compact sets of Ω_0. Moreover, we have the following two cases:

(i) *If $K_1 \cap K_2 = \emptyset$, then there exists a $\delta > 0$ and $M > 0$ such that*
$$|\varphi_\lambda(x)| \le M, \quad \forall x,\, d(x, K_2) \le \delta, \quad \forall\, \lambda \in (\lambda_1(\Omega), \lambda_1(\Omega_0)).$$
(ii) *If $K_1 \cap K_2 \neq \emptyset$ and K_0 satisfies* (HK') *and*
$$\gamma + 2 < (\rho - 1)(N - d), \tag{1.4}$$
then φ_λ remains uniformly bounded on compact sets of $\Omega \setminus K_1$. In particular it remains bounded at each point of $K_2 \setminus K_1$.

The proof of this result relies on the following argument. If we denote by u a nonnegative solution of (1.1), then we obtain first an upper bound of u, independent of λ, in compact sets of $\Omega \setminus K_0$. If $\bar{B}(x_0, a) \subset \Omega \setminus K_0$, where $n(x) \ge n_0$ in this ball, we may compare the solution u with radial solutions of singular Dirichlet problems, posed in $B(x_0, a)$, going to infinity at the boundary, see [9, 14, 18]. By radial symmetry, the minimum of the singular solution is attained at the center of the ball (that is in x_0), and can be estimated in terms of n_0, a, ρ and the dimension N. Translating this result to our problem, we can move those balls for points in $\Omega \setminus K_0$ next to the boundary of K_0, and state some rate for the upper bounds in terms of some inverse power of the distance to the boundary of K_0. This estimates provide a rate at which the solution may diverge to infinity as we approach K_0. See Lemma 3.2, Proposition 3.3 and Lemma 3.5.

Once this estimate is obtained we may consider a point $z \in K_2 \setminus K_1$ and consider for instance a small ball $B(z, \delta)$, where in principle the solution u may become unbounded as λ increases. Nevertheless, the rate obtained with the argument above may imply that the solution u restricted to the sphere $S(z, \delta) = \{|x - z| = \delta\}$ is in $L^r(S(z, \delta))$ for some $r \ge 1$, with a norm independent of λ. Hence, u will be a solution of an elliptic problem in $B(z, \delta)$ with an L^r trace at the boundary. Elliptic regularity will imply that the solution u is bounded, independent of λ, in compact sets of $B(z, \delta)$ and in particular in a neighborhood of $z \in K_2$. Therefore, we may obtain conditions on ρ, the dimensions N and d and the rate γ at which $n(x)$ approaches to zero, see (Hn), which may guarantee that the solution is bounded in $K_2 \setminus K_1$, see (1.4).

This article is organized as follows. In Section 2 we have collected some relevant results on the stationary solutions of logistic degenerated equations. All those results are essentially well know in case $K_2 = \emptyset$ and we now cover the case when $K_2 \neq \emptyset$. In Section 3 we state our main results.

2. Existence of the positive equilibria

Our main result in this Section states that for any $\lambda \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$, there exists a unique classical positive solution of (1.1) and their L^∞-norms approach infinity as $\lambda \to \lambda_1(\Omega_0)$, see Theorem 2.3. As mentioned before, this result is already know in the particular case when $n(x)$ is a smooth function, $K_2 = \emptyset$, and $K_0 = K_1 = \overline{\Omega_0}$, an open set with regular boundary, see [17, 8, 7].

We first state the following preliminary result. Assuming that for a fixed value of the parameter $\lambda = \lambda_0$, there exists a positive stationary solution of (1.1), then λ_0 must lie inside a precise open bounded interval. Moreover, for this λ_0, there is a small δ_0 such that for each $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, there exists a unique positive solution, which is smooth and increasing in the parameter. More precisely, we have the following lemma.

Lemma 2.1. *Assume $n(x)$ is Hölder continuous and K_0 satisfies* (HK). *Assume that φ_0 is a nontrivial nonegative classical stationary solution of* (1.1) *for $\lambda = \lambda_0$. Then the following holds:*

- (i) $\lambda_0 \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$
- (ii) *For each λ in a neighborhood of λ_0 there exists a unique nonnegative stationary solution of* (1.1)*, φ_λ, close to φ_0 which is moreover a smooth function of λ.*
- (iii) *The equilibria φ_λ is an increasing function of λ.*

Proof. (i) Assume that (λ_0, φ_0) is a non negative nontrivial stationary solution, then

$$\lambda_0 = \lambda_1(-\Delta + n(x)\varphi_0^{\rho-1}, \Omega), \tag{2.1}$$

that is, λ_0 is the first eigenvalue of the operator $-\Delta + n(x)\varphi_0^{\rho-1}$ in Ω, with Dirichlet boundary conditions. This fact, together with the monotonicity of the first eigenvalue with respect to the potential implies that, since $n(x)\varphi_0^{\rho-1} \geq 0$,

$$\lambda_0 > \lambda_1(\Omega).$$

On the other hand, the monotonicity with respect to the domain of this eigenvalue gives

$$\lambda_0 < \lambda_1(-\Delta + n(x)\varphi_0^{\rho-1}, \Omega_0).$$

Also note that $n(x) = 0$ on Ω_0 and so

$$\lambda_1(-\Delta + n(x)\varphi_0^{\rho-1}, \Omega_0) = \lambda_1(-\Delta, \Omega_0) = \lambda_1(\Omega_0),$$

and therefore, part (i) is already proved.

(ii) Since n is C^α Hölder continuous, we consider the map

$$F : (\lambda, u) \to -\Delta u - \lambda u + n(x)u^\rho$$

from $\mathbb{R} \times C_0^{2,\alpha}(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$ where $C_0^{2,\alpha}(\overline{\Omega}) := \{u \in C^{2,\alpha}(\overline{\Omega}) : u = 0, \text{ on } \partial\Omega\}$. Then F is a continuously differentiable map, and we apply the implicit function theorem at $(\lambda, u) = (\lambda_0, \varphi_0)$. By hypothesis φ_0 is a nonnegative stationary solution of (1.1), then $F(\lambda_0, \varphi_0) = 0$.

Moreover, the derivative with respect to u at $(\lambda, u) = (\lambda_0, \varphi_0)$ is

$$D_uF(\lambda_0, \varphi_0) = -\Delta - \lambda_0 + \rho n(x)\varphi_0^{\rho-1}.$$

Since $\rho > 1$ and taking into account the monotonicity of the first eigenvalue with respect to the potential and (2.1) we obtain

$$\lambda_1(-\Delta - \lambda_0 + \rho n(x)\varphi_0^{\rho-1}) > \lambda_1(-\Delta - \lambda_0 + n(x)\varphi_0^{\rho-1}) = 0.$$

This implies that the derivative $D_uF(\lambda_0, \varphi_0)$ is an isomorphism.

So, for each λ in a neighborhood of λ_0 there is a unique solution φ_λ of (1.1) in a neighborhood of φ_0 and the map $\lambda \to \varphi_\lambda$ is continuously differentiable with $\varphi_{\lambda_0} = \varphi_0$, ending this part of the proof.

(iii) Let

$$v := \frac{d\varphi_\lambda}{d\lambda},$$

taking derivatives with respect to λ in (1.1) we obtain

$$\begin{gathered} -\Delta v = \lambda v + \varphi - \rho n(x)\varphi^{\rho-1} v \quad \text{in } \Omega \\ v = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

We shall reason as before. Since $\lambda = \lambda_1(-\Delta + n(x)\varphi_\lambda^{\rho-1})$ and

$$\lambda_1(-\Delta - \lambda + \rho n(x)\varphi_\lambda^{\rho-1}) > \lambda_1(-\Delta - \lambda + n(x)\varphi_\lambda^{\rho-1}) = 0,$$

the maximum principle gives

$$v := \frac{d\varphi_\lambda}{d\lambda} > 0,$$

therefore φ_λ is an increasing function of λ. □

The next result gives some "spectral" property of the set K_0 that will be used below.

Lemma 2.2. *Assume K_0 satisfies* (HK). *If we denote by $U_\delta = \{x \in \Omega : d(x, K_0) < \delta\}$, then*

$$\lambda_1(U_\delta) \nearrow \lambda_1(\Omega_0), \qquad \text{as } \delta \to 0. \tag{2.2}$$

Proof. Observe that the family U_δ is decreasing in δ and we have $\Omega_0 \subset K_0 \subset U_\delta$. Therefore, $\lambda_1(U_\delta)$ is an increasing sequence in δ with $\lambda_1(U_\delta) < \lambda_1(\Omega_0)$. Nevertheless, U_δ does not converge in the Haussdorf distances to Ω_0 so the convergence stated in (2.2) is not obvious at all.

Notice first that if $K_1 \cap K_2 = \emptyset$ then for $\delta < \frac{1}{2} d(K_1, K_2)$, we have $U_\delta = U_\delta^1 \cup U_\delta^2$, where $U_\delta^i = \{x \in \Omega : d(x, K_i) < \delta\}$ for $i = 1, 2$ and $U_\delta^1 \cap U_\delta^2 = \emptyset$. This implies that $\lambda_1(U_\delta) = \min\{\lambda_1(U_\delta^1), \lambda_1(U_\delta^2)\}$. But since $|K_2| = 0$ then $|U_\delta^2| \to 0$ and therefore $\lambda_1(U_\delta^2) \to +\infty$. To see this, we just use Faber-Krahn inequality, see for instance [13]. This implies that $\lambda_1(U_\delta) = \lambda_1(U_\delta^1)$ and since Ω_0 is a smooth open set, then $\lambda_1(U_\delta^2) \to \lambda_1(\Omega_0)$, see [5, 4].

If $K_1 \cap K_2 \neq \emptyset$, then the argument is not so straightforward. Nevertheless, since $|K_2| = 0$, we have that for each fixed ball $B \subset \mathbb{R}^N \setminus \bar{\Omega}_0$ we have $|B \cap U_\delta| \to 0$ as $\delta \to 0$ and this implies, see [3, 6] that $\lambda_1(U_\delta) \to \lambda_1(\Omega_0)$. □

Next, we state the following result. For each parameter inside the interval determined in Lemma 2.1, part (i), there exists a unique positive solution. Moreover, the L^∞ norm of the solutions grows to infinity as the parameter λ approaches $\lambda_1(\Omega_0)$.

Theorem 2.3. *Assume K_0 satisfies* (HK). *Then the following holds:*

- (i) *For any $\lambda \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$ there exists a unique strictly positive classical solution $\varphi_\lambda \in C_0^2(\overline{\Omega})$ of* (1.1).
- (ii) *furthermore, as $\lambda \to \lambda_1(\Omega_0)$, we have*

$$\|\varphi_\lambda\|_{L^\infty(\Omega)} \to \infty. \tag{2.3}$$

Proof. (i) If $\lambda \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$, by sub-supersolutions method, we will prove that there is a bounded solution of (1.1). Specifically, observe that $\underline{u} := \varepsilon\Phi_1$ is a subsolution choosing ε small enough, in particular for any $\varepsilon \leq \big(\frac{\lambda - \lambda_1(\Omega)}{\|n\|_\infty}\big)^{1/(p-1)}$.

On the other hand, from Lemma 2.2, we can choose regular domains Ω_1, Ω_2, with

$$\Omega_0 \subset K_0 \Subset \Omega_1 \Subset \Omega_2 \subset \Omega$$

such that $\lambda < \lambda_1(\Omega_2) < \lambda_1(\Omega_1) < \lambda_1(\Omega_0)$. Set $w \in C^2(\overline{\Omega})$ a function strictly positive such that

$$w(x) := \begin{cases} 1 & \text{for } x \in \Omega \setminus \Omega_2 \\ \Phi_1(\Omega_2) & \text{for } x \in \Omega_1 \end{cases}$$

where $\Phi_1(\Omega_2) > 0$ is the first eigenfunction corresponding to the eigenvalue problem in Ω_2 with Dirichlet boundary conditions.

Then a supersolution can be chose in the following way $\overline{u} := Mw$ for M big enough, [8]. Thus existence of a pair of ordered positive solutions $\varphi_1 \le \varphi_2$, follows from [1].

To prove uniqueness, observe that if $\varphi_1 \le \varphi_2$ are not the same, then we would have

$$\lambda = \lambda_1(-\Delta + n(x)\varphi_2^{\rho-1}) > \lambda_1(-\Delta + n(x)\varphi_1^{\rho-1}) = \lambda,$$

which is absurd.

(ii) From the monotonicity in λ, see Lemma 2.1, there exists the monotone pointwise limit

$$\varphi^*(x) = \lim_{\lambda\to\lambda_1(\Omega_0)} \varphi_\lambda(x).$$

We next prove (2.3). In fact, otherwise, we get $\varphi^* \in L^\infty(\Omega)$ and by elliptic regularity we would have $\|\varphi_\lambda\|_{W^{2,p}(\Omega)} \le C$, for all $\lambda \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$ and any $1 < p < \infty$.

Sobolev's compact imbedding Theorem implies then that at least for a subsequence, $\varphi_\lambda \to \varphi^*$ in $W^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$, for $p > N$ and therefore φ^* is a weak solution of

$$\begin{gathered}-\Delta\varphi^* = \lambda_1(\Omega_0)\varphi^* - n(x)(\varphi^*)^\rho \quad \text{in } \Omega\\ \varphi^* = 0 \quad \text{on } \partial\Omega.\end{gathered}$$

Moreover, φ^* is bounded and therefore, by a bootstrap argument φ^* will be a classical solution of (1.1) with $\lambda = \lambda_1(\Omega_0)$, which contradicts part (i) of Lemma 2.1, which ends the proof. □

Remark 2.4. *It can be shown that φ_λ is globally asymptotically stable for nonnegative nontrivial solutions of* (1.1)*; see* [2].

3. Boundedness and unboundedness of solutions

The questions are now: What happens as $\lambda \to \lambda_1(\Omega_0)$? Where and how solutions become unbounded?

The first that we can say is that the blow-up is a complete blow-up at every point in Ω_0. For the for the proofs of the following results, we refer to [2].

Lemma 3.1. *Assume K_0 satisfies* (HK) *and let $\{\varphi_\lambda\}$ for $\lambda \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$ denote the family of positive solutions of* (1.1)*. Then*

$$\lim_{\lambda\to\lambda_1(\Omega_0)} \varphi_\lambda(x) = \infty, \quad \textit{for all } x \in \Omega_0.$$

To obtain upper bounds on the solutions outside Ω_0 we will use the following Lemma, see [9]. This Lemma analyzes the minimum of a radially symmetric solution of a singular logistic equation with constant coefficients and going to infinity at the boundary, see [14, 18].

Lemma 3.2. *Assume $\rho > 1$ and $\lambda, \beta > 0$ and consider a ball in $\mathbb{R}^N$ of radius $a > 0$ and the following singular Dirichlet problem*

$$\begin{gathered}-\Delta z = \lambda z - \beta z^\rho \quad \textit{in } B(0,a)\\ z = \infty \quad \textit{on } \partial B(0,a).\end{gathered}$$

Then, there exists a unique positive radial solution, $z_a(x)$. Moreover, the solution satisfies

$$\Big(\frac{\lambda}{\beta}\Big)^{1/(\rho-1)} \le z_a(0) = \inf_{B(0,a)} z_a(x) \le \Big(\frac{\lambda(\rho+1)}{2\beta} + \frac{B}{\beta a^2}\Big)^{1/(\rho-1)}$$

for some constant $B = B(\rho, N) > 0$, B independent of λ.

The above Lemma gives a local upper bound.

Proposition 3.3. *Let $x_0 \in \Omega \setminus K_0$ and let $\varphi > 0$ be a stationary solution of* (1.1) *for some $\lambda < \lambda_1(\Omega_0)$. Then there exists $a > 0$ and $M > 0$ independent of λ, such that*

$$0 \le \varphi(x) \le M, \quad \forall x \in B(x_0, a).$$

Proof. Let $x_0 \in \Omega \setminus K_0$ and let $a > 0$ be such that $B(x_0, 3a) \subset \Omega \setminus K_0$. Denote

$$\beta = \inf\{n(x),\ x \in B(x_0, 2a)\} > 0.$$

For each $y \in B(x_0, a)$, consider $z(x)$ the translation to $B(y, a)$ of the function in Lemma 3.2, with $\lambda = \lambda_1(\Omega_0)$. Hence $z(x)$ is a supersolution for $\varphi(x)$ and then

$$\varphi(x) \le z(x), \quad x \in B(y, a).$$

In particular, taking $x = y$, we have

$$\varphi(y) \le \Big(\frac{\lambda_1(\Omega_0)(\rho+1)}{2\beta} + \frac{B}{\beta a^2}\Big)^{1/(\rho-1)}, \quad \forall y \in B(x_0, a),$$

which proves the result with $M = \Big(\frac{\lambda_1(\Omega_0)(\rho+1)}{2\beta} + \frac{B}{\beta a^2}\Big)^{1/(\rho-1)}$. □

Assume now that the two parts K_1 and K_2 of K_0 are disjoint. The following result shows that, for $\lambda \to \lambda_1(\Omega_0)$, all solutions of (1.1) remain bounded in K_2, while they start to grow up in K_1.

Theorem 3.4. *Assume K_0 satisfies* (HK) *and $K_1 \cap K_2 = \emptyset$. Then the following holds*

- (i) *There exists a $\delta > 0$ and $M > 0$ such that*
$$|\varphi_\lambda(x)| \le M, \quad \forall x :\ d(x, K_2) \le \delta, \quad \forall\, \lambda \in (\lambda_1(\Omega), \lambda_1(\Omega_0)).$$
- (ii) *For $\lambda \to \lambda_1(\Omega_0)$ all solution of* (1.1) *are bounded on K_2.*
- (iii) *If $\lambda \to \lambda_1(\Omega_0)$ then the pointwise limit of the solutions of* (1.1) *is unbounded on K_1.*

Now we turn to the case in which K_1 and K_2 are glued together. First using Lemma 3.2 we prove the following universal bounds for solutions of (1.1).

Lemma 3.5. *Assume that $n(x)$ satisfies* (Hn)*. Then there exists a constant A, independent of λ such that for any solution of* (1.1) *we have*

$$0 \le \varphi(x) \le h(x) = \Big(\frac{A}{d_0(x)}\Big)^{\frac{\gamma+2}{\rho-1}}$$

with $d_0(x) = \mathrm{dist}(x, K_0)$.

The following result will be used further below and gives a criteria to check whether a function that is infinity on a smooth compact set of measure zero, is integrable. As shown below, this criteria depends on the dimension of the set and the rate at which the function diverges on it.

Lemma 3.6. *Assume $K \subset \mathbb{R}^N$ is a closed regular $d-$dimensional manifold with $d \leq N-1$, and consider a function defined on a bounded neighborhood Ω of K of the form*

$$f(x) = \big(dist(x,K)\big)^{-\alpha} \quad \text{for } \alpha > 0.$$

If $r \geq 1$ satisfies $r\alpha < N-d$, then $f \in L^r(\Omega)$.

With all these we can state the following result.

Theorem 3.7. *Assume K_0 satisfies* (HK') *and*

$$K_1 \cap K_2 \neq \emptyset.$$

Assume $n(x)$ satisfies (Hn). *Assume also that*

$$\gamma + 2 < (\rho - 1)(N-d).$$

Then, the positive solutions of (1.1) *remain bounded on compact sets of $\Omega \setminus K_1$. In particular they remain bounded at each point of $K_2 \setminus K_1$.*

Remark 3.8. It is an interesting open problem to determine whether we always obtain that the solution of (1.1) are bounded in compact sets of $\Omega \setminus K_1$ or, in the contrary, that we have cases in which φ_λ becomes infinity in K_2 as $\lambda \to \lambda_1(\Omega_0)$.

Remark 3.9. This work is still in progress, and we refer to [2] for details and more general results, including more general configurations for the set K_0 and the analysis of the solutions of the parabolic problem associated to (1.1).

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