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REAL ANALYTIC SOLUTIONS FOR THE WILLMORE FLOW

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ABSTRACT. In this article, we present a regularity result for the Willmore flow. This is obtained by using a truncated translation technique in conjunction with the implicit function theorem.

1. Introduction

The Willmore flow consists in looking for an oriented, closed, compact moving hypersurface $\Gamma(t)$ immersed in $\mathbb{R}^3$ evolving subject to the law

$$\begin{cases} V(t) = -\Delta_{\Gamma(t)} H_{\Gamma(t)} - 2H_{\Gamma(t)}(H^2_{\Gamma(t)} - K_{\Gamma(t)}), \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{1.1}$$

Here $V(t)$ denotes the velocity in the normal direction of $\Gamma(t)$ at time t. $\Delta_{\Gamma(t)}$ and $H_{\Gamma(t)}$ stand for the Laplace-Beltrami operator and the normalized mean curvature of $\Gamma(t)$, respectively. Finally, $K_{\Gamma(t)}$ denotes the Gaussian curvature.

The equilibria of (1.1) appear as the critical points of the Willmore functional, or sometimes called the Willmore energy. For a smooth immersion $f : \Gamma \to \mathbb{R}^3$ of a closed oriented two-dimensional manifold Γ, the Willmore functional is defined as

$$W(f) = \int_{f(\Gamma)} H^2_{f(\Gamma)}\, d\sigma, \tag{1.2}$$

where $d\sigma$ is the area element on $f(\Gamma)$ with respect to the Euclidean metric in $\mathbb{R}^3$. The critical surfaces of this functional, called the Willmore surfaces, satisfy the equation

$$\Delta_{f(\Gamma)} H_{f(\Gamma)} + 2H^3_{f(\Gamma)} - 2H_{f(\Gamma)} K_{f(\Gamma)} = 0. \tag{1.3}$$

The reader may consult [30, Section 7.4] for a brief historical account and a proof of this variational formula. The proof therein is derived by computing the critical points of all normal variations of the hypersurface $f(\Gamma)$.

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A generalization of the Willmore functional (1.2) in higher dimensions is studied by Chen [5]. He extends (1.2) for smooth immersions $f : \Gamma \to \mathbb{R}^{m+1}$ of the m-dimensional closed oriented manifold Γ into $\mathbb{R}^{m+1}$:

$$W(f) = \int_{f(\Gamma)} H^m_{f(\Gamma)}\, d\sigma$$

with $d\sigma$ standing for the volume element with respect to the Euclidean metric in $\mathbb{R}^{m+1}$. The critical points of this functional are now of the form

$$\Delta_{f(\Gamma)} H^{m-1}_{f(\Gamma)} + m(m-1) H^{m+1}_{f(\Gamma)} - H^{m-1}_{f(\Gamma)} R_{f(\Gamma)} = 0\,.$$

Here $R_{f(\Gamma)}$ denotes the scalar curvature. We may observe that $R_{f(\Gamma)} = 2K_{f(\Gamma)}$ when $m = 2$, so this Euler-Lagrange equation agrees with (1.3) in the two-dimensional case. However, this generalization has the drawback that the corresponding Willmore functional is no longer conformally invariant except when $m = 2$.

The Willmore problem has been studied by many authors, among them Thomsen, Blaschke, Willmore, Chen, Weiner, Li, Yau, Bryant, Kusner, Simon, Mayer, Simonett, Bauer, Kuwert, Schätzle, Pinkall, Sterling, Schmidt, Marques, and Neves; see [2, 3, 4, 5, 10, 12, 13, 14, 15, 16, 18, 19, 20, 21, 23, 26, 27, 28, 29, 30]. It is well-known that the Willmore functional is bounded below by 4π with equality only for the round sphere. Then the famous Willmore conjecture due to Willmore asserts that for any immersed 2-dimensional torus into $\mathbb{R}^3$ we have $W(f) \geq 2\pi^2$, and it suggests that the 2-dimensional Clifford torus achieves the minimum of the Willmore functional amongst all immersed tori in $\mathbb{R}^3$. In 1982, Li and Yau [16] showed that any immersion with $W(f) < 8\pi$ must in fact be an embedding. In other words, it will suffice to estimate $W(f)$ for embeddings. A classification of all Willmore immersions $f : \mathbb{S}^2 \to \mathbb{R}^3$ is obtained by Bryant [4]. The possible values of

$$W(f) = \int_{f(\mathbb{S}^2)} H^2_{f(\mathbb{S}^2)}\, d\sigma$$

are $4n\pi$ with $n = 1$, or $n \geq 4$ and n even, or $n \geq 9$ and n odd. Existence and regularity for embedded tori in the Willmore conjecture has been proven by Simon [26], and later this result is generalized by Bauer, Kuwert [2] for an extension of the conjecture by Kusner [12] to higher genus cases. An existence, uniqueness and regularity result on the Willmore flow is presented by Simonett [27]. It is proven therein that the Willmore flow admits a unique smooth solution. Moreover, this solution exists globally when it is initially close enough to spheres in the $C^{2+\alpha}$-topology and is exponentially attracted by spheres. In [18], Mayer and Simonett proved that the Willmore flow can drive embedded surfaces to a self-intersection in a finite time interval. Moreover, numerical simulations in [19] indicate that the Willmore flow can develop true singularities (topological changes) in finite time. Kuwert and Schätzle [13] show that the smooth solutions are global as long as the initial Willmore energy is sufficiently small. Later, the same authors improve this result in [15] by finding an explicit optimal bound for the restriction on the initial energy; that is, if the smooth immersion $f_0 : \Gamma \to \mathbb{R}^3$ satisfies $W(f_0) \leq 8\pi$, then the solution with initial data f_0 exists smoothly for all time and converges to a round sphere. Recently, in a breakthrough paper, Marques and Neves [10] prove the Willmore conjecture for surfaces of arbitrary genus $g \geq 1$; i.e., $W(f) \geq 2\pi^2$ for all embedded Γ with genus $g \geq 1$, and the equality holds if and only if Γ is conformal to the Clifford torus.

Assumptions. Throughout this paper, we assume that (M, g) is a compact, closed, embedded, oriented, real analytic hypersurface in $\mathbb{R}^3$ endowed with the Euclidean metric g with the exception of Section 3, wherein we remove the restriction on the dimension of M. The notation $(\cdot|\cdot)$ always stands for the standard inner product in $\mathbb{R}^3$. We may find for M a *normalized atlas* $\mathfrak{K} := (\mathsf{O}_\kappa, \varphi_\kappa)_{\kappa\in\mathfrak{K}}$. An atlas $\mathfrak{K}$ is called *normalized* if $\varphi_\kappa(\mathsf{O}_\kappa) = \mathbb{B}^2$ for all $\kappa \in \mathfrak{K}$. Here $\mathbb{B}^2$ is the open unit ball centered at the origin in $\mathbb{R}^2$. Put $\psi_\kappa = \varphi_\kappa^{-1}$.

A family $(\pi_\kappa)_{\kappa\in\mathfrak{K}}$ is called a *localization system subordinate to* $\mathfrak{K}$ if:

- (L1) $\pi_\kappa \in \mathcal{D}(\mathsf{O}_\kappa, [0, 1])$ and $(\pi_\kappa^2)_{\kappa\in\mathfrak{K}}$ is a partition of unity subordinate to $\mathfrak{K}$.
- (L2) Any π_κ and π_η satisfying $\operatorname{supp}(\pi_\kappa) \cap \operatorname{supp}(\pi_\eta) \neq \emptyset$ have their supports located within the same local chart.

For any manifold satisfying the above assumptions, there exists a localization system. See [1, Lemma 3.2] for a proof. Condition (L2) is not an additional assumption, because of the compactness of M.

Notation. Throughout this paper, $\mathbb{N}_0$ stands for the set of natural numbers including 0. For any interval I, $\mathring{I}$ denotes the interior of I, and $\dot{I} := I \setminus \{0\}$.

For a fix $0 < \alpha < 1$. Put $E_0 := h^\alpha(\mathsf{M})$, $E_1 := h^{4+\alpha}(\mathsf{M})$. Please refer to the remark below Theorem 1.1 for the precise definition of the spaces $h^s(\mathsf{M})$. For notational brevity, we simply write $\mathfrak{F}(\mathcal{O}, \mathbb{R})$ and $\mathfrak{F}(\mathsf{M}, \mathbb{R})$ as $\mathfrak{F}(\mathcal{O})$ and $\mathfrak{F}(\mathsf{M})$, where $\mathcal{O}$ is any open subset of $\mathbb{R}^2$ and $\mathfrak{F}$ stands for any of the function spaces in this paper.

Let $\gamma \in (0, 1]$. In the sequel, we denote $(E_0, E_1)_\gamma$ by E_γ, where $(\cdot, \cdot)_\gamma$ is the continuous interpolation method. See [17, Definition 1.2.2] for a definition. In particular, we set $(E_0, E_1)_1 := E_1$.

For some fixed interval $I = [0, T]$ and some Banach space E, we define

$$BUC_{1-\gamma}(I, E) := \{u \in C(\dot{I}, E); [t \mapsto t^{1-\gamma} u] \in BUC(\dot{I}, E), \lim_{t\to 0^+} t^{1-\gamma}\|u\| = 0\},$$

$$\|u\|_{C_{1-\gamma}} := \sup_{t\in\dot{I}} t^{1-\gamma}\|u(t)\|_E,$$

$$BUC^1_{1-\gamma}(I, E) := \{u \in C^1(\dot{I}, E) : u, \dot{u} \in BUC_{1-\gamma}(I, E)\}.$$

In particular, we put

$$BUC_0(I, E) := BUC(I, E), \quad BUC^1_0(I, E) := BUC^1(I, E).$$

In addition, if $I = [0, T)$ is a half open interval, then

$$C_{1-\gamma}(I, E) := \{v \in C(\dot{I}, E) : v \in BUC_{1-\gamma}([0, t], E),\ t < T\},$$

$$C^1_{1-\gamma}(I, E) := \{v \in C^1(\dot{I}, E) : v, \dot{v} \in C_{1-\gamma}(I, E)\}.$$

We equip these two spaces with the natural Fréchet topology induced by the topology of $BUC_{1-\gamma}([0, t], E)$ and $BUC^1_{1-\gamma}([0, t], E)$, respectively.

Also we set

$$\mathbb{E}_0(I) := C(I, E_0), \quad \mathbb{E}_1(I) := C(I, E_1) \cap C^1(I, E_0).$$

In this article, we will show that the Willmore flow (1.1) admits a real analytic solution jointly in time and space. Our motivation for a real analytic solution is mainly stimulated by the following facts: a compact closed real analytic manifold cannot have a “flat part”, and real analyticity in time implies that the hypersurface should move permanently in the interval of existence.

Theorem 1.1. *Let $0 < \alpha < 1$. Suppose that Γ_0 is a compact closed embedded oriented hypersurface in $\mathbb{R}^3$ belonging to the class $h^{2+\alpha}$. Then the Willmore flow* (1.1) *has a unique local solution $\Gamma = \{\Gamma(t) : t \in [0,T)\}$ for some $T > 0$. Moreover,*

$$\mathcal{M} := \cup_{t\in(0,T)}(\{t\} \times \Gamma(t))$$

is a real analytic submanifold in $\mathbb{R}^4$. In particular, each manifold $\Gamma(t)$ is real analytic for $t \in (0,T)$.

For any open subset $\mathcal{O} \subset \mathbb{R}^2$, the little Hölder space $h^s(\mathcal{O})$ of order $s > 0$ with $s \notin \mathbb{N}$ is the closure of $BUC^\infty(\mathcal{O})$ in $BUC^s(\mathcal{O})$. Here $BUC^s(\mathcal{O})$ is the Banach space of all bounded and uniformly Hölder continuous functions. The little Hölder space $h^s(\mathsf{M})$ on M is defined in terms of the atlas $\mathfrak{K}$; that is, a function u belongs to $h^s(\mathsf{M})$ if and only if $\psi_\kappa^* \pi_\kappa u \in h^s(\mathbb{R}^2)$, for each $\kappa \in \mathfrak{K}$.

2. Parameterization over a reference manifold

In equation (1.1), if we fix an embedded initial hypersurface Γ_0 belonging to the class $h^{2+\alpha}$, then by the discussion in [22, Section 4] we can find a real analytic compact closed embedded oriented hypersurface M, a function $\rho_0 \in h^{2+\alpha}(\mathsf{M})$ and a parameterization

$$\Psi_{\rho_0} : \mathsf{M} \to \mathbb{R}^3, \quad \Psi_{\rho_0}(p) := p + \rho_0(p)\nu_{\mathsf{M}}(p)$$

such that $\Gamma_0 = \mathrm{im}(\Psi_{\rho_0})$. Here $\nu_{\mathsf{M}}(p)$ denotes the unit normal with respect to a chosen orientation of M at p, and $\rho_0 : \mathsf{M} \to (-a,a)$ is a real-valued function on M, where a is a sufficiently small positive number depending on the inner and outer ball condition of M. The reader may consult [22, Section 4.1] for the precise bound of a. Thus Γ_0 lies in the a-tubular neighborhood of M. In fact, it will suffice to assume Γ_0 to be a C^2-manifold for the existence of such a parameterization and a real analytic reference manifold. See [22, Section 4] for a detailed proof.

Analogously, if $\Gamma(t)$ is C^1-close enough to M, then we can find a function $\rho : [0,T) \times \mathsf{M} \to (-a,a)$ for some $T > 0$ and a parameterization

$$\Psi_\rho : [0,T) \times \mathsf{M} \to \mathbb{R}^3, \quad \Psi_\rho(t,p) := p + \rho(t,p)\nu_{\mathsf{M}}(p)$$

such that $\Gamma(t) = \mathrm{im}(\Psi_\rho(t,\cdot))$ for every $t \in [0,T)$. It is worthwhile to mention that Ψ_ρ admits an extension on $\mathbb{R}^3$, called Hanzawa transform, which is first introduced by Hanzawa in [11].

For any fixed t, I do not distinguish between $\rho(t,\cdot)$ and $\rho(t,\psi_\kappa(\cdot))$ in each local coordinate $(\mathsf{O}_\kappa, \varphi_\kappa)$ and abbreviate $\Psi_\rho(t,\cdot)$ to be $\Psi_\rho := \Psi_\rho(t,\cdot)$. In addition, the hypersurface $\Gamma(t)$ will be simply written as Γ_ρ as long as the choice of t is of no importance in the context, or ρ is independent of t.

We put

$$\mho := \{\rho \in h^{2+\alpha}(\mathsf{M}) : \|\rho\|_\infty^{\mathsf{M}} < a\}.$$

Here $\|\rho\|_\infty^{\mathsf{M}} := \sup_{p\in\mathsf{M}} |\rho(p)|$. For any $\rho \in \mho$, $\mathrm{im}(\Psi_\rho)$ constitutes a $h^{2+\alpha}$-hypersurface Γ_ρ. In this case, Ψ_ρ defines a $h^{2+\alpha}$-diffeomorphism from M onto Γ_ρ.

Here and in the following, it is understood that the Einstein summation convention is employed and all the summations run from 1 to 2 for all repeated indices.

In [22], J. Prüss and G. Simonett derive global expressions for many geometric objects of Γ_ρ in terms of the function ρ. I will use some results therein to translate equation (1.1) into a differential equation in ρ. By [22, formula (23), (28)], we have the following explicit expressions for the components of the first fundamental form and the normal vector of Γ_ρ:

$$g_{ij}^\Gamma = g_{ij} - 2\rho l_{ij} + \rho^2 l_i^r l_{jr} + \partial_i\rho\partial_j\rho, \tag{2.1}$$

$$\nu_\Gamma = \beta(\rho)(\nu_{\mathsf{M}} - a(\rho)). \tag{2.2}$$

In (2.1), the l_j^i's are the components of the Weingarten tensor L_{M} of M with respect to g; i.e., $L_{\mathsf{M}} = l_i^j \tau^i \otimes \tau_j$, where $\{\tau_i = \partial_i\}$ forms a basis of $T_p\mathsf{M}$ at $p \in \mathsf{M}$ and $\{\tau^i\}$ is the dual basis to $\{\tau_i\}$; i.e., $(\tau^i|\tau_j) = \delta_j^i$. The extension of L_{M} into $\mathbb{R}^3$, by identifying it to be zero in the normal direction, is denoted by $L_{\mathsf{M}}^{\mathcal{E}}$, namely, $L_{\mathsf{M}}^{\mathcal{E}} = l_i^j\tau^i \otimes \tau_j + 0 \cdot \nu_{\mathsf{M}} \otimes \nu_{\mathsf{M}}$. It is a simple matter to check that

$$\tau_i^\Gamma = (I - \rho L_{\mathsf{M}}^{\mathcal{E}})\tau_i + \nu_{\mathsf{M}}\partial_i\rho \tag{2.3}$$

forms the standard basis of $T_{\Psi_\rho(p)}\Gamma_\rho$. In addition, the l_{ij}'s are the components of the second fundamental form L^{M} of the metric g. Finally, $g_{ij}^\Gamma = (\tau_i^\Gamma|\tau_j^\Gamma)$ are the components of the first fundamental form of the Euclidean metric g_Γ on Γ_ρ. We set $G^\Gamma(\rho) = (g_{ij}^\Gamma)_{ij}$ and $G_\Gamma^{-1}(\rho)$ for its inverse.

In (2.2), the terms $a(\rho)$ and $\beta(\rho)$ read

$$a(\rho) = (I - \rho L_{\mathsf{M}}^{\mathcal{E}})^{-1}\nabla_{\mathsf{M}}\rho, \quad \beta(\rho) = [1 + |a(\rho)|^2]^{-1/2}.$$

Here ∇_{M} is the surface gradient on M.

For sufficiently small $a > 0$, the operator $(I - \rho L_{\mathsf{M}}^{\mathcal{E}})$ is invertible. One can check that

$$I - \rho L_{\mathsf{M}}^{\mathcal{E}} = (\delta_i^j - \rho l_i^j)\tau^i \otimes \tau_j + \nu_{\mathsf{M}} \otimes \nu_{\mathsf{M}}.$$

Thus

$$(I - \rho L_{\mathsf{M}}^{\mathcal{E}})^{-1} = r_i^j(\rho)\tau^i \otimes \tau_j + \nu_{\mathsf{M}} \otimes \nu_{\mathsf{M}}, \tag{2.4}$$

where $R_\rho = (r_i^j(\rho))_{ij} = [(\delta_i^j - \rho l_i^j)_{ij}]^{-1}$. By Cramer's rule, all the entries of R_ρ possess the expression

$$r_i^j(\rho) = \frac{P_i^j(\rho)}{Q_i^j(\rho)}$$

in every local chart, where P_i^j and Q_i^j are polynomials in ρ with real analytic coefficients and $Q_i^j \neq 0$.

Substituting $(I - \rho L_{\mathsf{M}}^{\mathcal{E}})^{-1}$ by (2.4), we obtain

$$|a(\rho)|^2 = (r_i^j(\rho)\partial_j\rho\tau^i | r_k^l(\rho)\partial_l\rho\tau^k) = g^{ik}r_i^j(\rho)r_k^l(\rho)\partial_j\rho\partial_l\rho.$$

Then

$$\beta(\rho) = [1 + |a(\rho)|^2]^{-1/2} = [1 + g^{ik}r_i^j(\rho)r_k^l(\rho)\partial_j\rho\partial_l\rho]^{-1/2}.$$

Note that in every local chart

$$\beta^2(\rho) = \frac{P^\beta(\rho)}{Q^\beta(\rho, \partial_j\rho)},$$

where $P^\beta(\rho)$ is a polynomial in ρ with real analytic coefficients and $Q^\beta(\rho, \partial_j\rho) \neq 0$ is a polynomial in ρ and its first order derivatives with real analytic coefficients.

The normal velocity can be expressed as

$$V(t) = (\partial_t\Psi_\rho|\nu_\Gamma) = (\rho_t\nu_{\mathsf{M}}|\nu_\Gamma) = \beta(\rho)\rho_t.$$

Therefore, the first line of (1.1) is equivalent to

$$\rho_t = -\frac{1}{\beta(\rho)}[\Psi^*_\rho \Delta_{\Gamma_\rho} H_{\Gamma_\rho} + 2\Psi^*_\rho H_{\Gamma_\rho}(H^2_{\Gamma_\rho} - K_{\Gamma_\rho})].$$

Next we shall calculate the Gaussian curvature K_{Γ_ρ} in terms of ρ. For simplicity, we write K_ρ instead of $\Psi^*_\rho K_{\Gamma_\rho}$. Using that

$$\partial_j \tau_i = \Gamma^k_{ij}\tau_k + l_{ij}\nu_{\mathsf{M}}, \quad \partial_j \tau^i = -\Gamma^i_{jk}\tau^k + l^i_j \nu_{\mathsf{M}},$$

one may readily obtain

$$\partial_j L^{\mathcal{E}}_{\mathsf{M}} = \partial_j l^k_i \tau^i \otimes \tau_k - \Gamma^i_{jl} l^k_i \tau^l \otimes \tau_k + \Gamma^l_{jk} l^k_i \tau^i \otimes \tau_l + l^i_j l^k_i \nu_{\mathsf{M}} \otimes \tau_k + l_{jk} l^k_i \tau^i \otimes \nu_{\mathsf{M}}. \tag{2.5}$$

Denote by $L^\Gamma = (l^\Gamma_{ij})_{ij}$ the second fundamental form of Γ_ρ with respect to g_Γ. Then by (2.2) and (2.3), we can compute its components l^Γ_{ij} as follows:

$$\begin{aligned}
l^\Gamma_{ij} &= -(\tau^\Gamma_i | \partial_j \nu_\Gamma) \\
&= -((I-\rho L^{\mathcal{E}}_{\mathsf{M}})\tau_i + \nu_{\mathsf{M}}\partial_i\rho | \beta(\partial_j \nu_{\mathsf{M}} - \partial_j a(\rho))) - (\tau^\Gamma_i | \frac{\partial_j \beta}{\beta}\nu_\Gamma) \\
&= \beta\{l_{ij} + \rho(L^{\mathcal{E}}_{\mathsf{M}}\tau_i|\partial_j\nu_{\mathsf{M}}) + (\tau_i|\partial_j(\nabla_{\mathsf{M}}\rho)) + ((I-\rho L^{\mathcal{E}}_{\mathsf{M}})\tau_i|\partial_j[(I-\rho L^{\mathcal{E}}_{\mathsf{M}})^{-1}]\nabla_{\mathsf{M}}\rho) \\
&\quad + \partial_i\rho(\nu_{\mathsf{M}}|\partial_j[(I-\rho L^{\mathcal{E}}_{\mathsf{M}})^{-1}]\nabla_{\mathsf{M}}\rho) + \partial_i\rho(\nu_{\mathsf{M}}|(I-\rho L^{\mathcal{E}}_{\mathsf{M}})^{-1}[\partial_j(\nabla_{\mathsf{M}}\rho)])\} \\
&= \beta\{l_{ij} + \rho l_{ik}(\tau^k|\partial_j\nu_{\mathsf{M}}) + (\tau_i|\partial_j(\nabla_{\mathsf{M}}\rho)) + (\tau_i|\partial_j(\rho L^{\mathcal{E}}_{\mathsf{M}})(I-\rho L^{\mathcal{E}}_{\mathsf{M}})^{-1}\nabla_{\mathsf{M}}\rho) \\
&\quad + \partial_i\rho(\nu_{\mathsf{M}}|\partial_j(\rho L^{\mathcal{E}}_{\mathsf{M}})(I-\rho L^{\mathcal{E}}_{\mathsf{M}})^{-1}\nabla_{\mathsf{M}}\rho) + \partial_i\rho(\nu_{\mathsf{M}}|\partial_j(\nabla_{\mathsf{M}}\rho))\} \\
&= \beta[l_{ij} - l_{ik}l^k_j\rho + \partial_{ij}\rho - \Gamma^k_{ij}\partial_k\rho + r^l_k(\rho)(\partial_j l^k_i + \Gamma^k_{jh}l^h_i - \Gamma^h_{ij}l^k_h)\rho\partial_l\rho \\
&\quad + r^l_k(\rho)l^k_i\partial_j\rho\partial_l\rho + r^l_k(\rho)l^h_j l^k_h\rho\partial_i\rho\partial_l\rho + l^k_j\partial_i\rho\partial_k\rho].
\end{aligned}$$

Here we have used (2.5) and the following facts:

- $(\nu_{\mathsf{M}}|\partial_j\nu_{\mathsf{M}}) = 0$.
- $(\tau^\Gamma_i|\nu_\Gamma) = 0$.
- $\partial_j\nu_{\mathsf{M}} = -l_{ij}\tau^i$.
- $(I-\rho L^{\mathcal{E}}_{\mathsf{M}})^{-1}\nu_{\mathsf{M}} = \nu_{\mathsf{M}}$.
- $\partial_j a(\rho) = (I-\rho L^{\mathcal{E}}_{\mathsf{M}})^{-1}\partial_j(\nabla_{\mathsf{M}}\rho) + \partial_j[(I-\rho L^{\mathcal{E}}_{\mathsf{M}})^{-1}]\nabla_{\mathsf{M}}\rho$.
- $\partial_j[(I-\rho L^{\mathcal{E}}_{\mathsf{M}})^{-1}] = (I-\rho L^{\mathcal{E}}_{\mathsf{M}})^{-1}\partial_j(\rho L^{\mathcal{E}}_{\mathsf{M}})(I-\rho L^{\mathcal{E}}_{\mathsf{M}})^{-1}$.

Therefore, $\det(L^\Gamma)$ can be expressed in every local chart as

$$\det(L^\Gamma) = \beta^2(\rho)\frac{P^\Gamma(\rho,\partial_j\rho,\partial_{ij}\rho)}{Q^\Gamma(\rho)}.$$

Here $P^\Gamma(\rho,\partial_j\rho,\partial_{ij}\rho)$ is a polynomial in ρ and its derivatives up to second order with real analytic coefficients. Moreover, $Q^\Gamma(\rho)$ is a polynomial in ρ with real analytic coefficients. In particular, we have $Q^\Gamma \neq 0$.

In view of the above computations, within every local chart $K_\rho = \det[G^{-1}_\Gamma(\rho)L^\Gamma]$ can be expressed locally as

$$K_\rho = \beta^2(\rho)\frac{P^\Gamma(\rho,\partial_j\rho,\partial_{ij}\rho)}{\det(G^\Gamma(\rho))Q^\Gamma(\rho)}. \tag{2.6}$$

As a straightforward conclusion of the above computation, we obtain an explicit expression for $H_\rho := \Psi_\rho^* H_{\Gamma_\rho}$:

$$\begin{aligned}2H_\rho &= g_\Gamma^{ij} l_{ij}^\Gamma \\ &= \beta(\rho) g_\Gamma^{ij}[l_{ij} - l_{ik}l_j^k\rho + \partial_{ij}\rho - \Gamma_{ij}^k\partial_k\rho + r_k^l(\rho)l_i^k\partial_j\rho\partial_l\rho \\ &\quad + r_k^l(\rho)(\partial_j l_i^k + \Gamma_{jh}^k l_i^h - \Gamma_{ij}^h l_h^k)\rho\partial_l\rho + r_k^l(\rho)l_j^h l_h^k\rho\partial_i\rho\partial_l\rho + l_j^k\partial_i\rho\partial_k\rho].\end{aligned} \tag{2.7}$$

The reader may also find a different global expression for H_ρ in [22, formula (32)]. We can decompose H_ρ into $H_\rho = P_1(\rho)\rho + F_1(\rho)$:

$$F_1(\rho) = \frac{\beta(\rho)}{2} g_\Gamma^{ij}(l_{ij} - l_{ik}l_j^k\rho) = \frac{\beta(\rho)}{2}\operatorname{Tr}[G_\Gamma^{-1}(\rho)(L^{\mathsf{M}} - \rho L^{\mathsf{M}} L_{\mathsf{M}})],$$

where $\operatorname{Tr}(\cdot)$ denotes the trace operator, and

$$\begin{aligned}P_1(\rho) = \frac{\beta(\rho)}{2}\Big\{ & g_\Gamma^{ij}\partial_{ij} + g_\Gamma^{ij}(l_j^k\partial_i\rho - \Gamma_{ij}^k)\partial_k \\ & + g_\Gamma^{ij}[r_k^l(\rho)l_i^k\partial_j\rho + r_k^l(\rho)(\partial_j l_i^k + \Gamma_{jh}^k l_i^h - \Gamma_{ij}^h l_h^k)\rho + r_k^l(\rho)l_j^h l_h^k\rho\partial_i\rho]\partial_l\Big\}\end{aligned}$$

in every local chart. Note that $\operatorname{Tr}[G_\Gamma^{-1}(\rho)L^{\mathsf{M}}]$ changes like H_{M} under transition maps and thus is invariant. Analogously, we can check that F_1 is a well-defined global operator. Hence so is $P_1(\rho)$.

In addition, it is a well-known fact that $\Psi_\rho^*\Delta_{\Gamma_\rho} = \Delta_\rho\Psi_\rho^*$, where Δ_{Γ_ρ} and Δ_ρ are the Laplace-Beltrami operators on (Γ_ρ, g_Γ) and $(\mathsf{M}, \sigma(\rho))$, respectively. Here $\sigma(\rho) := \Psi_\rho^* g_\Gamma$ stands for the pull-back metric of g_Γ on M by Ψ_ρ. Then in every local chart, the Laplace-Beltrami operator Δ_ρ can be expressed as

$$\Delta_\rho = \sigma^{jk}(\rho)(\partial_j\partial_k - \gamma_{jk}^i(\rho)\partial_i). \tag{2.8}$$

Here $\sigma^{jk}(\rho)$ are the components of the induced metric $\sigma^*(\rho)$ of $\sigma(\rho)$ on the cotangent bundle. Note that $\sigma^{jk}(\rho)$ involves the derivatives of ρ merely up to order one. $\gamma_{jk}^i(\rho)$ are the corresponding Christoffel symbols of $\sigma(\rho)$, which contain the derivatives of ρ up to second order.

There exists a global operator $R(\rho) \in \mathcal{L}(h^{3+\alpha}(\mathsf{M}), E_0)$ such that $R(\cdot)$ is well defined on $\mho$ and

$$R(\rho)\rho = \frac{1}{2\beta(\rho)}\Delta_\rho[\beta(\rho)\operatorname{Tr}(G_\Gamma^{-1}(\rho)L^{\mathsf{M}})] - \frac{\rho}{2\beta(\rho)}\Delta_\rho[\beta(\rho)\operatorname{Tr}(G_\Gamma^{-1}(\rho)L^{\mathsf{M}}L_{\mathsf{M}})].$$

We set

$$\begin{aligned}P(\rho) &:= \frac{1}{\beta(\rho)}\Delta_\rho P_1(\rho) + R(\rho), \quad \rho\in\mho, \\ F(\rho) &:= -\frac{1}{\beta(\rho)}\Delta_\rho F_1(\rho) + R(\rho)\rho - \frac{2}{\beta(\rho)}H_\rho(H_\rho^2 - K_\rho), \quad \rho\in\mho\cap h^{3+\alpha}(\mathsf{M}).\end{aligned}$$

Note that third order derivatives of ρ do not appear in $F(\rho)$. Hence it is actually well-defined on $\mho$. Based on the above discussion, these two maps enjoy the following smoothness properties:

$$P\in C^\omega(\mho, \mathcal{L}(E_1, E_0)), \quad F\in C^\omega(\mho, E_0).$$

Here ω is the symbol for real analyticity. Please refer to [24, Appendix] for a proof.

Definition 2.1. Let $l \in \mathbb{N}_0$. A linear operator $\mathcal{A} : \mathcal{D}(\mathsf{M}) \to \mathbb{R}(\mathsf{M})$ is called a linear differential operator of order l on M if in every local chart $(\mathsf{O}_\kappa, \varphi_\kappa)$, there exists some linear differential operator

$$\mathcal{A}_\kappa = \sum_{|\alpha| \le l} a^\kappa_\alpha \partial^\alpha$$

with $a^\kappa_\alpha \in \mathbb{R}^{\mathbb{B}^2}$ defined on $\mathbb{B}^2$ such that for any $u \in \mathcal{D}(\mathsf{M})$ it holds

$$\psi^*_\kappa(\mathcal{A}u) = \mathcal{A}_\kappa(\psi^*_\kappa u)$$

Moreover, at least one of the $\mathcal{A}_\kappa$'s is of order l. In particular, when $l = 0$, $\mathcal{A}u = au$ for some $a \in \mathbb{R}^{\mathsf{M}}$.

By the above definition, $P(\rho)$ is a fourth order linear differential operator on M for each $\rho \in \mho$. In every local chart $(\mathsf{O}_\kappa, \varphi_\kappa)$, the principal part of the local expression of $P(\rho)$ can be written as

$$P^\pi_\kappa(\rho) := \frac{1}{2}\sigma^{kl}(\rho) g^{ij}_\Gamma \partial_{ijkl}.$$

Given $\xi \in T^*\mathsf{M}$, we estimate the symbol of $P^\pi_\kappa(\rho)$ as follows.

$$P^\pi_\kappa(\rho)(\xi) = \frac{1}{2}\sigma^*(\rho)(\xi,\xi) g^*_\Gamma(\xi,\xi) \ge c|\xi|^4$$

for some $c > 0$, and g^*_Γ denotes the induced metric of g_Γ on the cotangent bundle of M. Hence, $P(\rho)$ is a normally elliptic fourth order operator acting on functions over M for each $\rho \in \mho$. By [24, Theorem 3.4], $P(\rho) \in \mathcal{H}(E_1, E_0)$, namely, $-P(\rho)$ generates an analytic semigroup on E_0 with $D(-P(\rho)) = E_1$, $\rho \in \mho$.

Now the Willmore flow (1.1) can be rewritten as

$$\begin{aligned} \rho_t + P(\rho)\rho &= F(\rho), \\ \rho(0) &= \rho_0, \end{aligned} \tag{2.9}$$

where $\rho_0 \in \mho$. See [7, 8, 27] for related work.

Applying [6, Theorem 4.1], the existence and regularity result in [27] can be restated as follows.

Theorem 2.2 ([27, Theorem 1.1])**.** *Suppose that $\rho_0 \in \mho$. Then equation (2.9) has a unique solution ρ in the interval of maximal existence $J(\rho_0) := [0, T(\rho_0))$ for some $T(\rho_0) > 0$ such that*

$$\rho \in C^1_{\frac{1}{2}}(J(\rho_0), E_0) \cap C_{\frac{1}{2}}(J(\rho_0), E_1) \cap C(J(\rho_0), \mho) \cap C^{\frac{1}{2}-\beta_0}(J(\rho_0), E_{\beta_0})$$

for any $\beta_0 \in [0, \frac{1}{2}]$. Moreover, each hypersurface $\Gamma(t)$ is of class C^∞ for $t \in \dot{J}(\rho_0)$.

3. Parameter-dependent diffeomorphisms

The main purpose of the last two sections is to show that the classical solution obtained in Theorem 2.2 is in fact real analytic jointly in time and space. To this end, I will construct a family of parameter-dependent diffeomorphisms acting on functions over M. Because the construction applies to manifolds of arbitrary dimensions, in this section we assume that M is a m-dimensional manifold with the properties imposed in Section 1.

For a given point $p \in \mathsf{M}$, we choose a normalized atlas $\mathfrak{K}$ for M such that $\varphi_1(p) = 0 \in \mathbb{R}^m$. Choose several open subsets B_i in $\mathbb{B}^m$, the open unit ball centered at the origin in $\mathbb{R}^m$, in such a manner that

- $B_i := \mathbb{B}^m(0, i\varepsilon_0)$, for $i = 1, 2, 3$ and some $\varepsilon_0 > 0$.
- $B_3 \subset\subset B_4 \subset\subset \mathbb{B}^m$.

Next, We further select two cut-off functions on $\mathbb{B}^m$:

- $\chi \in \mathcal{D}(B_2, [0,1])$ such that $\chi|_{\overline{B}_1} \equiv 1$. We write $\chi_\kappa = \varphi_\kappa^* \chi$.
- $\zeta \in \mathcal{D}(B_4, [0,1])$ such that $\zeta|_{\overline{B}_3} \equiv 1$. We write $\zeta_\kappa = \varphi_\kappa^* \zeta$.

We define a re-scaled translation on $\mathbb{B}^m$ for any $\mu \in \mathbb{B}(0, r) \subset \mathbb{R}^m$ with r sufficiently small:

$$\theta_\mu(x) := x + \chi(x)\mu, \quad x \in \mathbb{B}^m.$$

This localization technique in Euclidean spaces is first introduced in [9] by Escher, Prüss and Simonett to establish regularity for solutions to parabolic and elliptic equations.

Given a function $v \in L_{1,loc}(\mathbb{B}^m)$, its pull-back and push-forward induced by θ_μ are defined as

$$\theta_\mu^* v := v \circ \theta_\mu, \quad \theta_*^\mu v := v \circ \theta_\mu^{-1}.$$

The diffeomorphism θ_μ induces a transformation Θ_μ on M by

$$\Theta_\mu(q) = \begin{cases} \psi_1(\theta_\mu(\varphi_1(q))) & q \in \mathsf{O}_1, \\ q & q \notin \mathsf{O}_1. \end{cases}$$

It can be shown that $\Theta_\mu \in \mathrm{Diff}^\infty(\mathsf{M})$ for $\mu \in \mathbb{B}(0, r)$ with sufficiently small $r > 0$. See [25] for details. For any $u \in L_{1,\mathrm{loc}}(\mathsf{M})$, we can define its pull-back and push-forward induced by Θ_μ analogously as

$$\Theta_\mu^* u := u \circ \Theta_\mu, \quad \Theta_*^\mu u := u \circ \Theta_\mu^{-1}.$$

We may find an explicit global expression for the transformation Θ_μ^* on M,

$$\Theta_\mu^* u = \varphi_1^* \theta_\mu^* \psi_1^* (\zeta_1 u) + (1 - \zeta_1) u.$$

Here and in the following it is understood that a partially defined and compactly supported function is automatically extended over the whole base manifold by identifying it to be zero outside its original domain. Likewise, we can express Θ_*^μ as

$$\Theta_*^\mu = \varphi_1^* \theta_*^\mu \psi_1^* (\zeta_1 u) + (1 - \zeta_1) u.$$

Let $I = [0, T]$, $T > 0$. Assuming that $J \subset \mathring{I}$ is an open interval and $t_0 \in J$ is a fixed point, we choose ε_0 so small that $\mathbb{B}(t_0, 3\varepsilon_0) \subset J$. Next we pick another auxiliary function

$$\xi \in \mathcal{D}(\mathbb{B}(t_0, 2\varepsilon_0), [0,1]) \quad \text{with } \xi|_{\mathbb{B}(t_0, \varepsilon_0)} \equiv 1.$$

The above construction now engenders a parameter-dependent transformation in terms of the time variable:

$$\varrho_\lambda(t) := t + \xi(t)\lambda, \quad \text{for any } t \in I \text{ and } \lambda \in \mathbb{R}.$$

Now we define a family of parameter-dependent transformations on $I \times \mathsf{M}$. Given a function $u : I \times \mathsf{M} \to \mathbb{R}$, we set

$$u_{\lambda,\mu}(t, \cdot) := \Theta_{\lambda,\mu}^* u(t, \cdot) := T_\mu(t) \varrho_\lambda^* u(t, \cdot),$$

where $T_\mu(t) = \Theta_{\xi(t)\mu}^*$ and $(\lambda, \mu) \in \mathbb{B}(0, r)$. It is important to note that $u_{\lambda,\mu}(0, \cdot) = u(0, \cdot)$ for any $(\lambda, \mu) \in \mathbb{B}(0, r)$ and any function u.

The importance of this family of parameter-dependent diffeomorphisms lies in the following theorems. Their proofs as well as additional properties of this technique can be found in [25].

Theorem 3.1. *Let $k \in \mathbb{N}_0 \cup \{\infty, \omega\}$. Suppose that $u \in C(I \times \mathsf{M})$. Then we have that $u \in C^k(\mathring{I} \times \mathsf{M})$ if and only if for any $(t_0, p) \in \mathring{I} \times \mathsf{M}$, there exists $r = r(t_0, p) > 0$ and a corresponding family of parameter-dependent diffeomorphisms $\{\Theta^*_{\lambda,\mu} : (\lambda, \mu) \in \mathbb{B}(0, r)\}$ such that*

$$[(\lambda, \mu) \mapsto \Theta^*_{\lambda,\mu} u] \in C^k(\mathbb{B}(0, r), C(I \times \mathsf{M})).$$

Proposition 3.2. *Suppose that $u \in \mathbb{E}_1(I)$. Then $u_{\lambda,\mu} \in \mathbb{E}_1(I)$, and*

$$\partial_t[u_{\lambda,\mu}] = (1 + \xi'\lambda)\Theta^*_{\lambda,\mu} u_t + B_{\lambda,\mu}(u_{\lambda,\mu}),$$

where

$$[(\lambda, \mu) \mapsto B_{\lambda,\mu}] \in C^\omega(\mathbb{B}(0, r), C(I, \mathcal{L}(E_1, E_0))).$$

Furthermore, $B_{\lambda,0} = 0$.

Proposition 3.3. *Let $s \in (0, t)$ and $l \in \mathbb{N}_0$. Suppose that $\mathcal{A}$ is a linear differential operator of order l on M satisfying $a^\kappa_\alpha \in BC^t(\mathbb{B}^m)$ and $a^1_\alpha \in BC^t(\mathbb{B}^m) \cap C^\omega(\mathsf{O})$ for some open subset O such that $B_3 \subset\subset \mathsf{O} \subset\subset \mathbb{B}^m$. Then*

$$[\mu \mapsto T_\mu \mathcal{A} T_\mu^{-1}] \in C^\omega(\mathbb{B}(0, r), C(I, \mathcal{L}(h^{s+l}(\mathsf{M}), h^s(\mathsf{M})))).$$

Proposition 3.4. *Let $s > 0$. Suppose that $u \in C^\omega(\psi_1(\mathsf{O})) \cap h^s(\mathsf{M})$, where O is defined in Proposition 3.3. Then*

$$[\mu \mapsto T_\mu u] \in C^\omega(\mathbb{B}(0, r), C(I, h^s(\mathsf{M}))).$$

4. Real analyticity

By setting $G(\rho) := P(\rho)\rho - F(\rho)$, we may rewrite (2.9) as

$$\begin{gathered} \rho_t + G(\rho) = 0, \\ \rho(0) = \rho_0. \end{gathered} \tag{4.1}$$

Theorem 4.1. *Let $0 < \alpha < 1$. Suppose that $\rho_0 \in \mho$. Then (4.1) has a unique local solution ρ in the interval of maximal existence $J(\rho_0)$ such that*

$$\rho \in C^\omega(\dot{J}(\rho_0) \times \mathsf{M}).$$

Proof. The key steps of the proof are indicated here, while the details can be found in [25].

For any $(t_0, p) \in \dot{J}(\rho_0) \times \mathsf{M}$ and sufficiently small $r > 0$, a family of parameter-dependent diffeomorphisms $\Theta^*_{\lambda,\mu}$ can be defined for $(\lambda, \mu) \in \mathbb{B}(0, r)$. Henceforth, we always use the notation ρ exclusively for the solution to (2.9) and hence to (4.1). Set $u := \rho_{\lambda,\mu}$. Then as a consequence of Proposition 3.2, u satisfies the equation

$$\begin{aligned} u_t = \partial_t[\rho_{\lambda,\mu}] &= (1 + \xi'\lambda)\Theta^*_{\lambda,\mu}\rho_t + B_{\lambda,\mu}(u) \\ &= -(1 + \xi'\lambda)\Theta^*_{\lambda,\mu} G(\rho) + B_{\lambda,\mu}(u) \\ &= -(1 + \xi'\lambda) T_\mu G(\varrho^*_\lambda \rho) + B_{\lambda,\mu}(u) \\ &= -(1 + \xi'\lambda) T_\mu G(T_\mu^{-1} u) + B_{\lambda,\mu}(u) := -H_{\lambda,\mu}(u). \end{aligned}$$

Select $I : [\varepsilon, T] \subset\subset \dot{J}(\rho_0)$ such that $t_0 \in \mathring{I}$ and $\mathbb{B}(t_0, 3\varepsilon_0) \subset\subset \mathring{I}$. Then we define $\mathbb{E}_0(I)$ and $\mathbb{E}_1(I)$ as in Section 1 by moving the initial point from 0 to ε. Set

$$\mathbb{E}^a_1(I) := \{v \in \mathbb{E}_1(I) : \|v\|_\infty < a\},$$

where $\|v\|_\infty := \sup_{(t,q) \in I \times \mathsf{M}} |v(t, q)|$.

For $\mathcal{A} \in \mathcal{H}(E_1, E_0)$, we say that $(\mathbb{E}_0(I), \mathbb{E}_1(I))$ is a pair of maximal regularity of $\mathcal{A}$, if

$$(\frac{d}{dt} + \mathcal{A}, \gamma_\varepsilon) \in \mathrm{Isom}(\mathbb{E}_1(I), \mathbb{E}_0(I) \times E_1),$$

where γ_ε is the evaluation map at ε; i.e., $\gamma_\varepsilon(u) = u(\varepsilon)$. Next we define

$$\Phi : \mathbb{E}_1^a(I) \times \mathbb{B}(0, r) \to \mathbb{E}_0(I) \times E_1 \quad \text{as } \Phi(v, (\lambda, \mu)) \mapsto \begin{pmatrix} v_t + H_{\lambda,\mu}(v) \\ \gamma_\varepsilon(v) - \rho(\varepsilon) \end{pmatrix}.$$

Note that $\Phi(\rho_{\lambda,\mu}, (\lambda, \mu)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for any $(\lambda, \mu) \in \mathbb{B}(0, r)$.

(i) Our first goal is to prove that $\Phi \in C^\omega(\mathbb{E}_1^a(I) \times \mathbb{B}(0, r), \mathbb{E}_0(I) \times E_1)$. By Proposition 3.2, $B_{\lambda,\mu} \in C^\omega(\mathbb{B}(0, r), C(I, \mathcal{L}(E_1, E_0)))$. We define a bilinear and continuous map

$$f : C(I, \mathcal{L}(E_1, E_0)) \times \mathbb{E}_1(I) \to \mathbb{E}_0(I), \quad (T(t), u(t)) \mapsto T(t)(u(t)).$$

Hence $[(v, (\lambda, \mu)) \mapsto f(B_{\lambda,\mu}, v) = B_{\lambda,\mu}(v)] \in C^\omega(\mathbb{E}_1^a(I) \times \mathbb{B}(0, r), \mathbb{E}_0(I))$.

On the other hand, let $\pi = \sum_{\eta \in \mathcal{C}(1)} \pi_\eta^2$, where

$$\mathcal{C}(1) := \{\eta \in \mathfrak{K} : \mathrm{supp}(\pi_\eta) \cap \mathrm{supp}(\pi_1) \neq \emptyset\}.$$

We decompose G into

$$G = \pi G + \sum_{\eta \notin \mathcal{C}(1)} \pi_\eta^2 G.$$

According to our construction of Θ_μ^* and of the localization system, we may assume that $\pi|_{\mathsf{O}} \equiv 1$, where O is defined in Proposition 3.3 with $m = 2$. See [1, Lemma 3.2] for details.

Taking into account (2.6), (2.7) and (2.8), in every local chart $(\mathsf{O}_\kappa, \varphi_\kappa)$ and for any $v \in \mathbb{E}_1^a(I)$, $G(v)$ can be expressed as

$$\frac{\beta^{2n}(v) P^G(v, \dots, \partial_{ijkl} v)}{\det(G^\Gamma(v))^{s_1} \det([\sigma(v)])^{s_2} Q^G(v)},$$

where $n, s_1, s_2 \in \mathbb{N}$. $[\sigma(v)]$ is the matrix representation of the metric $\sigma(v)$. Here $\sigma(v)$ is defined in a similar manner to $\sigma(\rho)$ with ρ replaced by v. Analogously, $G^\Gamma(v)$ is defined in a similar way to $G^\Gamma(\rho)$. Meanwhile, P^G is a polynomial in v and its derivatives up to fourth order with real analytic coefficients, and Q^G is a polynomial in v with real analytic coefficients. In particular, $\det([\sigma(v)])$ only involves first order derivatives of v.

Therefore, $\pi G(v)$ can be decomposed globally into

$$\frac{\mathcal{P}_0 + \mathcal{P}_1^1 v \dots \mathcal{P}_{k_1}^1 v + \dots + \mathcal{P}_1^r v \dots \mathcal{P}_{k_r}^r v}{\mathcal{Q}_0 + \mathcal{Q}_1^1 v \dots \mathcal{Q}_{l_1}^1 v + \dots + \mathcal{Q}_1^s v \dots \mathcal{Q}_{l_s}^s v},$$

where $\mathcal{P}_0$, $\mathcal{Q}_0 \in C^\infty(\mathsf{M}) \cap C^\omega(\psi_1(\mathsf{O}))$. The $\mathcal{P}_j^i$'s are linear differential operators on M up to fourth order, and the $\mathcal{Q}_j^i$'s are linear differential operators of order at most one on M. Their coefficients in every local chart satisfy that $a_\alpha^\kappa \in BC^\infty(\mathbb{B}^2)$ and $a_\alpha^1 \in BC^\infty(\mathbb{B}^2) \cap C^\omega(\mathsf{O})$. By Proposition 3.4, we deduce that

$$[\mu \mapsto (T_\mu \mathcal{P}_0, T_\mu \mathcal{Q}_0)] \in C^\omega(\mathbb{B}(0, r), C(I, E_1) \times C(I, E_1)).$$

Analogously, it follows from Proposition 3.3 that

$$[\mu \mapsto T_\mu \mathcal{P}_j^i T_\mu^{-1}] \in C^\omega(\mathbb{B}(0, r), C(I, \mathcal{L}(E_1, E_0)))$$

and

$$[\mu \mapsto T_\mu \mathcal{Q}^i_j T_\mu^{-1}] \in C^\omega(\mathbb{B}(0,r), C(I, \mathcal{L}(E_1, h^{3+\alpha}(\mathsf{M})))).$$

Combining the above discussion with point-wise multiplication theorems on Riemannian manifolds, we infer that

$$[(v,\mu) \mapsto T_\mu(\pi G)T_\mu^{-1} v] \in C^\omega(\mathbb{E}_1^a(I) \times \mathbb{B}(0,r), \mathbb{E}_0(I)).$$

Applying these arguments repeatedly to the other terms $\pi_\eta^2 G$, we conclude that

$$\Phi \in C^\omega(\mathbb{E}_1^a(I) \times \mathbb{B}(0,r), \mathbb{E}_0(I) \times E_1).$$

(ii) Next we look at the Fréchet derivative of Φ in the first component:

$$D_1\Phi(v,(\lambda,\mu))w = \begin{pmatrix} w_t + (1+\xi'\lambda)T_\mu DG(T_\mu^{-1}v)T_\mu^{-1}w - B_{\lambda,\mu}(w) \\ \gamma_\varepsilon w \end{pmatrix}.$$

Thus

$$D_1\Phi(\rho,(0,0))w = \begin{pmatrix} w_t + DG(\rho)w \\ \gamma_\varepsilon w \end{pmatrix}.$$

Observe that $DG(\rho)$ is a fourth order linear differential operator whose coefficients satisfy $a^\kappa_\alpha \in E_0$. The principal part of $DG(\rho)$ in every local chart coincides with that of $P(\rho)$; that is, $P^\pi_\kappa(\rho)$. By the discussion in Section 2, we know that $DG(\rho(t,\cdot))$ is a normally elliptic operator for every fixed $t \geq 0$. As a consequence of [24, Theorem 3.6], it follows that $(\mathbb{E}_0(I), \mathbb{E}_1(I))$ is a pair of maximal regularity for $DG(\rho(t,\cdot))$.

We set $\mathcal{A}(t) = DG(\rho(t,\cdot))$. It follows that

$$(\frac{d}{dt} + \mathcal{A}(s), \gamma_\varepsilon) \in \mathrm{Isom}(\mathbb{E}_1(I), \mathbb{E}_0(I) \times E_1), \quad \text{for every } s \in I.$$

By [6, Lemma 2.8(a)], we have

$$(\frac{d}{dt} + \mathcal{A}(\cdot), \gamma_\varepsilon) \in \mathrm{Isom}(\mathbb{E}_1(I), \mathbb{E}_0(I) \times E_1).$$

Now we apply the implicit function theorem. It follows right away that there exists an open neighborhood, say $\mathbb{B}(0,r_0) \subset \mathbb{B}(0,r)$, such that

$$[(\lambda,\mu) \mapsto \rho_{\lambda,\mu}] \in C^\omega(\mathbb{B}(0,r_0), \mathbb{E}_1(I)).$$

As a consequence of Theorem 3.1, we deduce that $\rho \in C^\omega(\dot{J}(\rho_0) \times \mathsf{M})$. This completes the proof. □

Proof of Theorem 1.1. For each $(t_0,q) \in \mathcal{M} = \cup_{t\in\dot{J}(\rho_0)}(\{t\} \times \Gamma(t))$, there exists a $p \in \mathsf{M}$ such that $\Psi_\rho(t_0,p) = q$. Here $\Gamma(t) = \mathrm{im}(\Psi_\rho(t,\cdot))$. Theorem 4.1 states that there exists a local patch $(\mathsf{O}_\kappa, \varphi_\kappa)$ such that $p \in \mathsf{O}_\kappa$ and $\rho \circ \psi_\kappa$ is real analytic in $\dot{J}(\rho_0) \times \mathbb{B}^2$. Therefore, we conclude that

$$[(t,x) \mapsto (t, \psi_\kappa(x) + \rho(t,\psi_\kappa(x))\nu_{\mathsf{M}}(\psi_\kappa(x))] \in C^\omega(\dot{J}(\rho_0) \times \mathbb{B}^2, \mathcal{M}).$$

This proves the assertion of Theorem 1.1. □

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