Ninth MSU-UAB Conference on Differential Equations and Computational Simulations.
Electronic Journal of Differential Equations, Conference 20 (2013), pp. 53–63.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

TOTAL VARIATION STABILITY AND SECOND-ORDER ACCURACY AT EXTREMA

RITESH KUMAR DUBEY

ABSTRACT. It is well known that high order total variation diminishing (TVD) schemes for hyperbolic conservation laws degenerate to first-order accuracy, even at smooth extrema; hence they suffer from clipping error. In this work, TVD bounds on representative three-point second-order accurate schemes are given for the scalar case, which show that it is possible to obtain second order TVD approximation at points of extrema as well as in steep gradient regions. These bounds can be used to improve existing high order TVD schemes and to reduce clipping error. In a 1D scalar test cases, an existing limiters based high order TVD scheme is applied, along with these second-order schemes using their TVD bounds to show improvement in the numerical results at extrema and steep gradient regions.

1. INTRODUCTION

The concept of non-linear stability condition total variation diminishing (TVD) was first introduced by Harten [2] and later by Sanders [11]. TVD condition is the weakest possible condition for monotonicity preservation which ensures stability of scheme for both monotone and non-monotone solutions. In other words, it guarantees that maxima or minima will not increase or decrease respectively. Many modern shock capturing schemes are devised with TVD property in Harten's sense and implemented successfully by scientific community over the past 30 years. Examples of such high order TVD schemes are, flux limiter based schemes [14, 12, 15, 4], slope limiters based schemes [1] and relaxation schemes [3]. Despite of huge success, these schemes are criticized because they *degenerate to first order accuracy at smooth extrema of solution* even for one-dimensional scalar conservation laws [7, 8]. In [11], Sanders defined the total variation by measuring the variation of the reconstructed polynomials rather than the traditional way of measuring the variation of the grid values as in [2]. He also gave a uniformly third order accurate finite volume scheme in [11]. Recently, following TVD definition given by Sanders, Zhang and Shu constructed a finite volume TVD schemes which are up to sixth order accurate in the L_1 norm but only in 1D case [16]. Hence it can be concluded that uniformly high

---

2000 *Mathematics Subject Classification.* 35L65, 65M06, 65M12.
Key words and phrases. Second order accurate schemes; total variation diminishing schemes; smoothness parameter; hyperbolic equations.

Published October 31, 2013.
Supported by DST India grant SR/FTP/MS-015/2011.

order approximation can be achieved for TVD schemes defined in Sanders sense but for TVD schemes defined in Harten's sense such uniform accuracy is impossible due to their degeneracy at extrema.

We consider uniformly second order accurate centered and upwind schemes like Lax-Wendroff and Beam-Warming type schemes. Numerically, it can be seen that these three points schemes fail to produce total variation stable solution though total variation bounded (TVB) solution can be obtained. It shows that these schemes are not TVD in general. Till now no theoretical discussion or proof is reported on the TVD or TVB properties of these schemes [5]. We investigate these uniformly second order schemes to achieve second order TVD approximation (in Harten's sense) at points of extrema and steep gradient region of solution. This results into explicit bounds on the solution region where these schemes remains total variation stable. The bounds are given in terms of smoothness parameter which is a function of consecutive gradient ratios. These obtained bounds can play significant role to construct or improve existing high order schemes which can give second order approximation for solution region with extreme point and steep gradient. In section 2, we analyze for total variation (TV) stability of the representative second order accurate central and upwind biased schemes. In section 3 we give numerical results to show the improvement in a well known limiters based scheme when applied with three point second order schemes. Conclusion and future work is discussed in the last section.

2. Total variation stability bounds

We consider the scalar conservation law,

$$\begin{aligned} u(x,t)_t + f(u(x,t))_x = 0, \\ u(x,0) = u_0(x) \end{aligned} \tag{2.1}$$

where $u(x,t)$ is a conserved variable and $f(u)$ is the non linear flux function. The characteristics speed associated with (2.1) is defined by $a(u) = f'(u) = \frac{\partial f(u)}{\partial u}$. For discritization we divide the spatial space into N equispaced cells $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $i = 0, 1, \dots N$ of length Δx and temporal space into M equispaced intervals $[t^n, t^{n+1}]$, $n = 0, 1, \dots, M$ of length Δt. The quantity $x_{i\pm\frac{1}{2}}$ is called cell interface and t^n denotes the n^{th} time level respectively. A conservative numerical approximation for above equation is defined by

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \lambda\big(\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}}\big) \tag{2.2}$$

where $\lambda = \frac{\Delta t}{\Delta x}$, $\mathcal{F}_{i+\frac{1}{2}}$ is time-integral average of flux function at cell interface and $\bar{u}_i^n$ is spatial cell-integral average defined as

$$\bar{u}_i^n \approx \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t^n)dx, \quad \mathcal{F}_{i+\frac{1}{2}} \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u(x_{i+\frac{1}{2}},t))dt. \tag{2.3}$$

The choice of numerical flux function $\mathcal{F}_{i\pm\frac{1}{2}}$ governs the spatial performance like accuracy, dissipation, numerical oscillations or shock capturing feature of resulting conservative scheme. Further, numerically characteristics speed is approximated at cell interface $x_{i+\frac{1}{2}}$ as

$$a_{i+\frac{1}{2}} = \begin{cases} \frac{F_{i+1}-F_i}{\bar{u}_{i+1}-\bar{u}_i} & \text{if } \bar{u}_{i+1} \neq \bar{u}_i, \\ f'(\bar{u}_i) & \text{if } \bar{u}_{i+1} = \bar{u}_i, \end{cases} \tag{2.4}$$

where $F_i = f(\bar{u}_i)$ and the superscript for time level n is dropped. The following well known TVD criteria given by Harten in [2] is used for the main results.

Lemma 2.1. *Conditions $\alpha_{i+\frac{1}{2}} \geq 0$, $\beta_{i-\frac{1}{2}} \geq 0$ and $\alpha_{i+\frac{1}{2}} + \beta_{i+\frac{1}{2}} \leq 1$ are sufficient for a conservative scheme in Incremental form (I-form)*

$$\bar{u}_i^{n+1} = \bar{u}_i^n + \alpha_{i+\frac{1}{2}}\Delta_+\bar{u}_i - \beta_{i-\frac{1}{2}}\Delta_-\bar{u}_i \tag{2.5}$$

to be TVD, where $\Delta_+\bar{u}_i = \Delta_-\bar{u}_{i+1} = \bar{u}_{i+1} - \bar{u}_i$.

Theorem 2.2. *For non-linear scalar conservation law* (2.1)*, Lax-Wendroff scheme is TVD under CFL like condition $0 < \lambda \max_u |f'(u)| < 1$, if $r_i \in (-\infty, -1] \cup [1/3, \infty)$, where smoothness parameter is defined as*

$$r_i = \begin{cases} \frac{(1-\lambda a_{i-1/2})\Delta_- F_i}{(1-\lambda a_{i+1/2})\Delta_+ F_i} & a_{i+\frac{1}{2}} > 0 \\ \frac{(1+\lambda a_{i+1/2})\Delta_+ F_i}{(1+\lambda a_{i-1/2})\Delta_- F_i} & a_{i+\frac{1}{2}} < 0 \end{cases}$$

Proof. Consider the numerical flux function of Lax-Wendroff scheme

$$F_{i+\frac{1}{2}}^{n,LxW} = \frac{1}{2}\left(F_{i+1} + F_i\right) - \frac{\lambda\, a_{i+\frac{1}{2}}}{2}\Delta_+\bar{u}_i. \tag{2.6}$$

Case $a(u) > 0$**:** The conservative approximation using (2.6) can be written as

$$\bar{u}_i^{n+1} = \bar{u}_i - \Big[\frac{\lambda a_{i+\frac{1}{2}}}{2}\big(1 - \lambda a_{i+\frac{1}{2}}\big)\Delta_+\bar{u}_i + \lambda a_{i-\frac{1}{2}}\Delta_-\bar{u}_i - \frac{\lambda a_{i-\frac{1}{2}}}{2}\big(1 - \lambda a_{i-\frac{1}{2}}\big)\Delta_-\bar{u}_i\Big]. \tag{2.7}$$

which can be written in the following Incremental form,

$$\bar{u}_i^{n+1} = \bar{u}_i - \Big[\frac{\lambda a_{i+\frac{1}{2}}}{2}\big(1 - \lambda a_{i+\frac{1}{2}}\big)\frac{\Delta_+\bar{u}_i}{\Delta_-\bar{u}_i} + \lambda a_{i-\frac{1}{2}} - \frac{\lambda a_{i-\frac{1}{2}}}{2}\big(1 - \lambda a_{i-\frac{1}{2}}\big)\Big]\Delta_-\bar{u}_i. \tag{2.8}$$

From Lemma 2.1, (2.8) will be TVD if,

$$0 \leq \Big[\frac{\lambda a_{i+\frac{1}{2}}}{2}\big(1 - \lambda a_{i+\frac{1}{2}}\big)\frac{\Delta_+\bar{u}_i}{\Delta_-\bar{u}_i} + \lambda a_{i-\frac{1}{2}} - \frac{\lambda a_{i-\frac{1}{2}}}{2}\big(1 - \lambda a_{i-\frac{1}{2}}\big)\Big] \leq 1. \tag{2.9}$$

Under CFL condition,

$$0 < \lambda a_{i+\frac{1}{2}} < 1 \Rightarrow (1 - \lambda a_{i+\frac{1}{2}}) > 0, \quad \forall i. \tag{2.10}$$

Inequality (2.9) can be rewritten as

$$\frac{-2}{1 - \lambda a_{i-\frac{1}{2}}} \leq \frac{a_{i+\frac{1}{2}}(1 - \lambda a_{i+\frac{1}{2}})}{a_{i-\frac{1}{2}}(1 - \lambda a_{i-\frac{1}{2}})}\frac{\Delta_+\bar{u}_i}{\Delta_-\bar{u}_i} - 1 \leq \frac{2}{\lambda a_{i-\frac{1}{2}}}. \tag{2.11}$$

Note that under (2.10), $\sup\{\frac{-2}{1-\lambda\, a_{i-\frac{1}{2}}}\} = -2$ and $\inf\{\frac{2}{\lambda\, a_{i-\frac{1}{2}}}\} = 2$. Hence the above inequality is satisfied if

$$-1 \leq \frac{a_{i+\frac{1}{2}}(1 - \lambda a_{i+\frac{1}{2}})}{a_{i-\frac{1}{2}}(1 - \lambda a_{i-\frac{1}{2}})}\frac{\Delta_+\bar{u}_i}{\Delta_-\bar{u}_i} \leq 3,$$

which implies $r_i \in (-\infty, -1] \cup [\frac{1}{3}, \infty)$, where

$$r_i = \frac{a_{i-\frac{1}{2}}(1-\lambda a_{i-\frac{1}{2}})\Delta_- \bar{u}_i}{a_{i+\frac{1}{2}}(1-\lambda a_{i+\frac{1}{2}})\Delta_+ \bar{u}_i} = \frac{(1-\lambda a_{i-\frac{1}{2}})\Delta_- \bar{F}_i}{(1-\lambda a_{i+\frac{1}{2}})\Delta_+ \bar{F}_i}$$

Case $a(u) < 0$**:** The resulting approximation can be written as

$$\bar{u}_i^{n+1} = \bar{u}_i + \Big[\frac{\lambda a_{i+\frac{1}{2}}}{2}\big(1+\lambda a_{i+\frac{1}{2}}\big) - \lambda a_{i-\frac{1}{2}} - \frac{\lambda a_{i-\frac{1}{2}}}{2}\big(1+\lambda a_{i-\frac{1}{2}}\big)\frac{\Delta_- \bar{u}_i}{\Delta_+ \bar{u}_i}\Big]\Delta_+ \bar{u}_i. \quad (2.12)$$

Using Lemma 2.1, it can be shown that (2.12) will be TVD, if

$$\lambda a_{i+\frac{1}{2}} \le \frac{\lambda a_{i+\frac{1}{2}}}{2}\big(1+\lambda a_{i+\frac{1}{2}}\big) - \frac{\lambda a_{i-\frac{1}{2}}}{2}\big(1+\lambda a_{i-\frac{1}{2}}\big)\frac{\Delta_- \bar{u}_i}{\Delta_+ \bar{u}_i} \le 1 + \lambda a_{i+\frac{1}{2}}. \quad (2.13)$$

Under the CFL condition for $a(u) < 0$,

$$-1 < \lambda a_{i+\frac{1}{2}} < 0, \forall i. \quad (2.14)$$

Inequality (2.13) results to

$$\frac{2}{1+\lambda a_{i+\frac{1}{2}}} \ge 1 - \frac{a_{i-\frac{1}{2}}(1+\lambda a_{i-\frac{1}{2}})\Delta_- \bar{u}_i}{a_{i+\frac{1}{2}}(1+\lambda a_{i+\frac{1}{2}})\Delta_+ \bar{u}_i} \ge \frac{2}{\lambda a_{i+\frac{1}{2}}}. \quad (2.15)$$

Note that under (2.14) $\inf\{\frac{2}{1+\lambda a_{i+\frac{1}{2}}}\} = 2$ and $\sup\{\frac{2}{\lambda a_{i+\frac{1}{2}}}\} = -2$. Hence (2.15) is satisfied, if

$$-1 \le \frac{a_{i-\frac{1}{2}}(1+\lambda a_{i-\frac{1}{2}})\Delta_- \bar{u}_i}{a_{i+\frac{1}{2}}(1+\lambda a_{i+\frac{1}{2}})\Delta_+ \bar{u}_i} \le 3.$$

On inversion, $r_i \in (-\infty, -1] \cup [\frac{1}{3}, \infty)$, where

$$r_i = \frac{a_{i+\frac{1}{2}}(1+\lambda a_{i+\frac{1}{2}})\Delta_+ \bar{u}_i}{a_{i-\frac{1}{2}}(1+\lambda a_{i-\frac{1}{2}})\Delta_- \bar{u}_i} = \frac{(1+\lambda a_{i+\frac{1}{2}})\Delta_+ F_i}{(1+\lambda a_{i-\frac{1}{2}})\Delta_- F_i}$$

□

Theorem 2.3. *For non-linear scalar conservation law* (2.1)*, the Beam-Warming scheme is TVD under CFL like condition* $0 < \lambda \max_u |f'(u)| < 1$*, if* $r_{i+\sigma_{i+1/2}} \in [-1, 3]$*. Smoothness parameter* r *is defined as*

$$r_{i+\sigma_{i+1/2}} = \frac{\big(1+\sigma_{i+\frac{1}{2}}\lambda a_{i+\frac{3}{2}\sigma_{i+\frac{1}{2}}}\big)}{\big(1+\sigma_{i-\frac{1}{2}}\lambda a_{i+\frac{1}{2}\sigma_{i-\frac{1}{2}}}\big)}\theta(F_{i+\sigma_{i+1/2}}),$$

where

$$\sigma_{i+\frac{1}{2}} = \sigma(a_{i+1/2}) = \begin{cases} +1 & a_{i+1/2} > 0, \\ -1 & a_{i+1/2} < 0. \end{cases} \quad (2.16)$$

and

$$\theta(F_i) = \begin{cases} \frac{\Delta_- F_i}{\Delta_+ F_i} & a_{i+\frac{1}{2}} > 0, \\ \frac{\Delta_+ F_i}{\Delta_- F_i} & a_{i+\frac{1}{2}} < 0. \end{cases} \quad (2.17)$$

Proof. **Case $a(u) > 0$:** The numerical flux of Beam-Warming scheme is given by

$$F^{n,BW}_{i+\frac{1}{2}} = F_i + \frac{a_{i-\frac{1}{2}}}{2}\big(1-\lambda\, a_{i-\frac{1}{2}}\big)\Delta_-\bar{u}_i. \tag{2.18}$$

The resulting conservative I-form can be written as

$$\bar{u}^{n+1}_i = \bar{u}_i - \Big[\lambda a_{i-\frac{1}{2}} + \frac{\lambda a_{i-\frac{1}{2}}}{2}\big(1-\lambda a_{i-\frac{1}{2}}\big) - \frac{\lambda a_{i-\frac{3}{2}}}{2}\big(1-\lambda a_{i-\frac{3}{2}}\big)\frac{\Delta_-\bar{u}_{i-1}}{\Delta_-\bar{u}_i}\Big]\Delta_-\bar{u}_i. \tag{2.19}$$

A condition for (2.19) to be TVD is

$$0 \le \lambda a_{i-\frac{1}{2}} + \frac{\lambda\, a_{i-\frac{1}{2}}}{2}\big(1-\lambda\, a_{i-\frac{1}{2}}\big) - \frac{\lambda\, a_{i-\frac{3}{2}}}{2}\big(1-\lambda\, a_{i-\frac{3}{2}}\big)\frac{\Delta_-\bar{u}_{i-1}}{\Delta_-\bar{u}_i} \le 1. \tag{2.20}$$

Under theCFL condition (2.10), (2.20) reduce to

$$\frac{-2}{1-\lambda a_{i-\frac{1}{2}}} \le 1 - \frac{a_{i-\frac{3}{2}}(1-\lambda\, a_{i-\frac{3}{2}})\Delta_-\bar{u}_{i-1}}{a_{i-\frac{1}{2}}(1-\lambda\, a_{i-\frac{1}{2}})\Delta_-\bar{u}_i} \le \frac{2}{\lambda a_{i-\frac{1}{2}}}. \tag{2.21}$$

As $\sup\{\frac{-2}{1-\lambda a_{i-\frac{1}{2}}}\} = -2$ and $\inf\{\frac{2}{\lambda a_{i-\frac{1}{2}}}\} = 2$ under (2.10), (2.21) reduce to

$$-1 \le r_{i-1} \le 3 \tag{2.22}$$

where

$$r_{i-1} = \frac{a_{i-\frac{3}{2}}(1-\lambda\, a_{i-\frac{3}{2}})\Delta_-\bar{u}_{i-1}}{a_{i-\frac{1}{2}}(1-\lambda\, a_{i-\frac{1}{2}})\Delta_-\bar{u}_i} = \frac{(1-\lambda\, a_{i-\frac{3}{2}})\Delta_-F_{i-1}}{(1-\lambda\, a_{i-\frac{1}{2}})\Delta_+F_{i-1}} \tag{2.23}$$

Case $a(u) < 0$: The Beam-Warming flux is given by

$$F^{n,BW}_{i+\frac{1}{2}} = F_{i+1} - \frac{a_{i+\frac{3}{2}}}{2}\big(1+\lambda\, a_{i+\frac{3}{2}}\big)\Delta_+\bar{u}_{i+1}. \tag{2.24}$$

The resulting conservative I-form is

$$\bar{u}^{n+1}_i = \bar{u}_i + \Big[\frac{\lambda\, a_{i+\frac{3}{2}}}{2}(1+\lambda\, a_{i+\frac{3}{2}})\frac{\Delta_+\bar{u}_{i+1}}{\Delta_+\bar{u}_i} - \lambda\, a_{i+\frac{1}{2}} - \frac{\lambda\, a_{i+\frac{1}{2}}}{2}(1+\lambda\, a_{i+\frac{1}{2}})\Big]\Delta_+\bar{u}_i. \tag{2.25}$$

A condition for (2.25) to be TVD is

$$0 \le \frac{\lambda\, a_{i+\frac{3}{2}}}{2}\Big(1+\lambda\, a_{i+\frac{3}{2}}\Big)\frac{\Delta_+\bar{u}_{i+1}}{\Delta_+\bar{u}_i} - \lambda\, a_{i+\frac{1}{2}} - \frac{\lambda\, a_{i+\frac{1}{2}}}{2}\Big(1+\lambda\, a_{i+\frac{1}{2}}\Big) \le 1. \tag{2.26}$$

Note under CFL condition (2.14) $-1 < \lambda\, a_{i+\frac{1}{2}} < 0$, $\forall i$ and $0 < 1 + \lambda\, a_{i+\frac{1}{2}} \le 1$ hence (2.26) reduced to

$$\frac{2}{1+\lambda\, a_{i+\frac{1}{2}}} \ge \frac{a_{i+\frac{3}{2}}(1+\lambda\, a_{i+\frac{3}{2}})\Delta_+\bar{u}_{i+1}}{a_{i+\frac{1}{2}}(1+\lambda\, a_{i+\frac{1}{2}})\Delta_+\bar{u}_i} - 1 \ge \frac{2}{\lambda a_{i+\frac{1}{2}}}. \tag{2.27}$$

Inequality (2.27) can be satisfied if

$$-1 \le r_{i+1} \le 3, \tag{2.28}$$

where

$$r_{i+1} = \frac{a_{i+\frac{3}{2}}(1+\lambda\, a_{i+\frac{3}{2}})\Delta_+\bar{u}_{i+1}}{a_{i+\frac{1}{2}}(1+\lambda\, a_{i+\frac{1}{2}})\Delta_+\bar{u}_i} = \frac{(1+\lambda\, a_{i+\frac{3}{2}})\Delta_+F_{i+1}}{(1+\lambda\, a_{i+\frac{1}{2}})\Delta_-F_{i+1}} \tag{2.29}$$

□

Similar to above it is easy to prove the following result which show that second order upwind scheme share the same TVD bounds as Beam-Warming upwind scheme.

Theorem 2.4. *For non-linear scalar conservation law* (2.1)*, second order upwind scheme is TVD under CFL like condition* $\lambda \max_u |f'(u)| \leq \frac{1}{2}$*, if* $r_{i+\sigma_{i+1/2}} \in [-1, 3]$*. Smoothness parameter* r *is defined as*

$$r_{i+\sigma_{i+1/2}} = \theta(F_{i+\sigma_{i+1/2}}),$$

where σ *and* θ *is given by* (2.16) *and* (2.17) *respectively.*

3. Numerical results

In this section numerical results are given to show the improvement in approximation of smooth solution by existing high order TVD schemes. We take well known Lax-Wendroff type TVD flux limited method **(LxWflm)** [12, 9] with second order diffusive Minmod, third order Superbee and VanLeer limiters [10, 14]. These limiters are defined in terms of smoothness parameter r as

- Minmod: $\phi(r) = \min\{\max(0,\ br), 1\}, b \in [1, 2]$
- Superbee: $\phi(r) = \max\{0, \min(2r, 1), \min(r, 2)\}$
- VanLeer: $\phi(r) = \frac{r+|r|}{1+|r|}$

More details on these methods can be found in [13, 5]. Note that all flux limited method degenerate to first order at extrema and it is impossible to have second order accuracy with them in steep gradient region where $r \to 0^+$ [6]. This sudden drop in accuracy of high order TVD method cause a flatten approximation for the smooth solution profile widely know as clipping error. In order to show the improvement in such region, We use a hybrid method defined as: *Use Lax-Wendroff and Beam Warming schemes in such degeneracy region if permitted by their TVD bounds otherwise use LxWflm.* Numerical results obtained with such hybrid approach are shown by **ModLxWflm**. Other possible hybrid approach can also be defined.

3.1. Example 1.

We solve the linear transport equation $u_t + u_x = 0$ with periodic boundary condition along with following initial data.

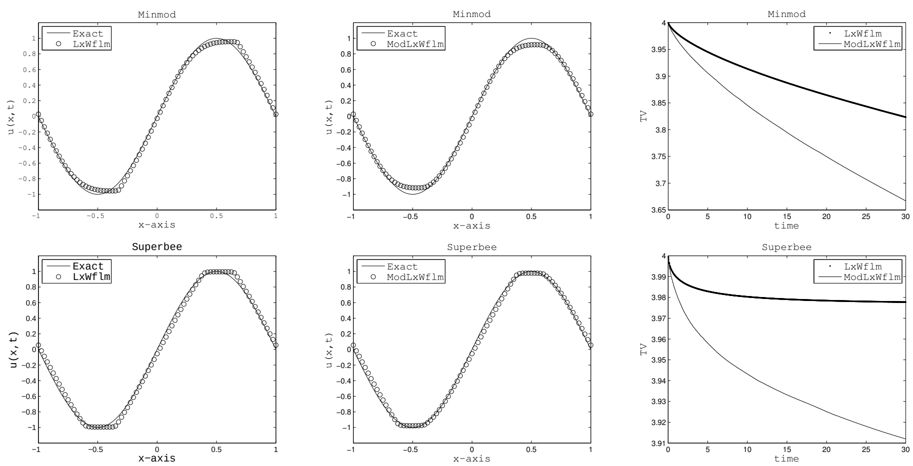
3.1.1. *Smooth Initial data.*

We take two smooth initial conditions to show the improvement in approximating smooth solution by Lax-Wendorff type flux limited scheme while applied with uniformly second order accurate Lax-Wendroff and Beam-Warming schemes as defined above using the TVD bounds of Theorem 2.2 and 2.3 respectively.

IC 1: $u(x, 0) = \sin(\pi x)$, $x \in [-1,\ 1]$. The solution remains smooth and approximation with LxW type flux limited method (LxWflm) using Minmod or Superbee limiter produces solution with corners or flatten profile respectively due to clipping error Figure 1(a). On the other hand results by hybrid method ModLxWflm yield a smoother approximation with reduced clipping Figure 1(b). Total variation of the computed solution by both the approach is also shown in Figure 1(c). Note that ModLxWflm not only has reduced clipping error but also has a better total variation diminishing property compared to LxWflm.

In Table 1, using different norms error convergence rate is shown for both LxWflm and ModLxWflm method. Convergence rate is shown at time $t = 4$ and $t = 30$ to see the short and long time behavior of approximation error. Error table shows a consistent improvement in the convergence rate especially at $t = 30$. It can also be seen from the Table 1 that due to clipping problem convergence rate of

LxWflm in all norm behave erratically (rows $N = 20, 40, 80$) whereas convergence rate of ModLxWflm remain consistent in all norm.



(a) (b) (c)

FIGURE 1. Numerical results for $\frac{\Delta t}{\Delta x} = 0.8$, $N = 80$, $T = 30$

	T=4						T=30					
	Minmod			Superbee			Minmod			Superbee		
N	L_1	L_2	L_∞	L_1	L_2	L_∞	L_1	L_2	L_∞	L_1	L_2	L_∞
LxWflm												
10	—	—	—	—	—	—	—	—	—	—	—	—
20	1.13	1.70	2.43	1.09	1.76	2.14	1.61	2.05	2.38	1.80	2.34	3.00
40	1.65	2.10	2.00	1.40	1.72	1.72	1.21	1.68	2.31	0.80	1.33	1.26
80	1.84	2.15	2.01	1.81	2.13	2.24	1.60	2.07	1.92	0.91	1.33	1.48
160	1.94	2.20	2.28	1.92	2.20	2.40	1.80	2.16	2.11	1.74	2.10	2.21
320	1.97	2.20	2.36	1.98	2.21	2.20	1.90	2.19	2.27	1.86	2.16	2.32
ModLxWflm												
10	—	—	—	—	—	—	—	—	—	—	—	—
20	2.90	2.32	2.43	1.84	2.26	2.64	1.16	1.63	2.20	1.74	2.26	2.70
40	1.87	2.27	2.27	1.75	2.21	2.24	1.75	2.26	2.49	1.80	2.14	2.48
80	1.93	2.20	2.26	1.95	2.27	2.07	1.87	2.28	2.31	1.69	2.23	2.16
160	1.96	2.18	2.28	2.00	2.28	2.49	1.93	2.24	2.30	1.98	2.36	2.20
320	1.96	2.17	2.31	2.03	2.29	2.14	1.96	2.21	2.32	2.01	2.35	2.32

TABLE 1. Convergence rate for linear case with initial condition $u_0(x) = \sin(\pi x)$ in different norms with the mesh refinement for $C = 0.9$

IC 2: $u(x, 0) = \sin^4(\pi x)$, $x \in [0, 1]$. This test case is taken from [16]. Initial data has a smooth peak and strict increasing or decreasing monotone solution regions towards the bottom where $r \to 0+$ or $r >> 1$ respectively. Numerical results are shown in Figure 2(a) with LxWflm method. Similar to first test in this case too, cornered or flatten approximation for the smooth peak can be seen. Also cornered approximation to high gradient left bottom region can be easily observed. Result in Figure 2(b), obtained by ModLxWflm show a smoother approximation for the

	LxWflm						ModLxWflm					
	Minmod			Superbee			Minmod			Superbee		
N	L_1	L_2	L_∞	L_1	L_2	L_∞	L_1	L_2	L_∞	L_1	L_2	L_∞
10	—	—	—	—	—	—	—	—	—	—	—	—
20	1.43	1.77	2.28	1.23	1.83	2.45	1.47	1.82	2.22	1.77	2.10	2.31
40	1.26	1.82	2.28	1.04	1.54	2.02	1.75	2.16	2.30	1.54	2.04	2.54
80	1.75	2.14	2.07	1.48	1.85	1.53	1.84	2.27	2.30	1.71	2.23	2.14
160	1.87	2.20	2.01	1.84	2.17	2.29	1.92	2.25	2.30	1.92	2.32	2.21
320	1.94	2.23	2.29	1.92	2.21	2.17	1.97	2.23	2.30	1.99	2.35	2.33

TABLE 2. Convergence rate for linear case with initial condition $u_0(x) = \sin^4(\pi x)$ for data $\frac{\Delta t}{\Delta x} = 0.8$ at $T = 2$

smooth peak and bottom region.

In Table 2 convergence rate are given using L_1, L_2 and L_∞ error norms. Results show that due to improved approximation of smooth extrema and high gradient region ModLxWflm show better and consistent convergence rate in all norms as compared to LxWflm.

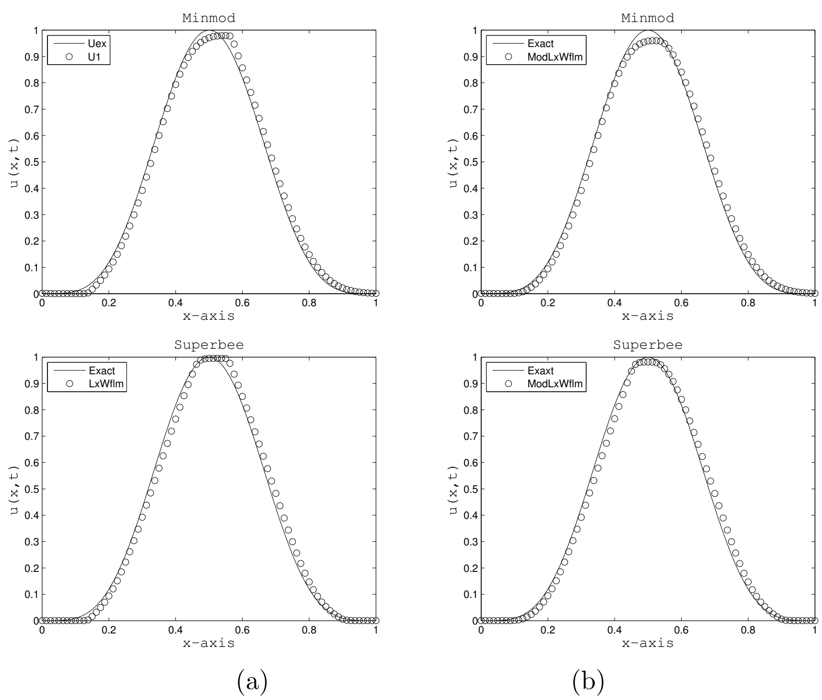
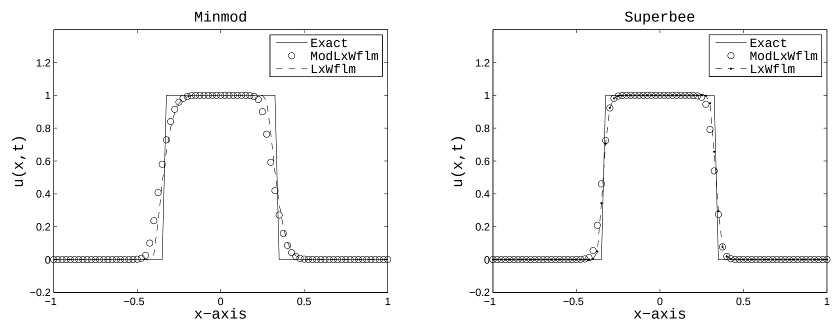


FIGURE 2. Numerical results are given for $\frac{\Delta t}{\Delta x} = 0.9$, $N = 80$, $T = 10$

3.1.2. *Discontinuous solution.* In this test the initial data given by

$$u(x,0) = \begin{cases} 1, & |x| \le 1/3, \\ 0, & \text{else}, \end{cases}$$

$x \in [-1, 1]$ which has two discontinuities present in it. Numerical results given in Figure 3 show that LxWflm give crisp resolution to discontinuous solution profile whereas hybrid approach ModLxWflm give acceptable little diffusive approximation at some corners. This little dissipative behavior can be easily understood by noting that LxWflm inherent the clipping effect which cause cornered approximation even for smooth solution therefore LxWflm produces a solution with crisp resolution for discontinuity. On the other hand hybrid method ModLxWflm has reduced clipping error hence give a nice smoother resolution for corners of discontinuity.



(a) (b)

FIGURE 3. Numerical results are given for $\frac{\Delta t}{\Delta x} = 0.8$, $N = 80$, $T = 4.0$

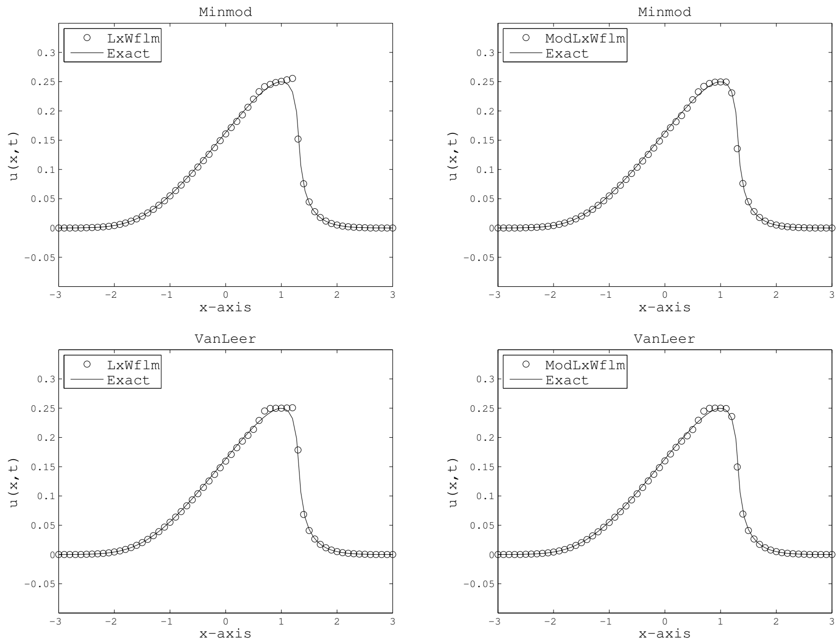
3.2. **Example 2: Non-linear scalar.** We solve the Burgers equation $u_t + (\frac{u^2}{2})_x = 0$, $-a \le x \le b$ with periodic boundary conditions. The time step Δt is chosen by relation $\Delta t = \frac{C\Delta x}{\max_u |u|}$, $0 < C < 1$. Two initial conditions $u(x, 0) = u_0(x)$ are taken as

(1) $u_0(x) = \frac{1}{4}\exp(-x^2)$, $x \in [-3, 3]$. In this case, solution remain smooth till $t = 4.66$. In Figure 4(a) solution obtained by LxWflm with Minmod and VanLeer limiters are shown. Approximate solutions by LxWflm fail to capture smooth profile. Solution obtained using VanLeer limiter exhibit flatness whereas solution by Minmod limiter show a spike in smooth solution with extrema. Numerical result in Figure 4(b) by hybrid approach ModLxWflm gives improved smoother approximation to exact solution. In Table 3 convergence rate of ModLxWflm is shown using L_1 and L_∞ error norm at time $T = 1.0$ while solution remain smooth. In L_1/L_∞ norm ModLxWflm show consistent second/higher order convergence rate.

$$u_0(x, 0) = \begin{cases} 1, & |x| \le 1/3, \\ -1, & \text{else.} \end{cases}$$

$x \in [-1, 1]$. In this test, left jump at $x = -1/3$ in initial data create a sonic expansion fan where as right jump at $x = 1/3$ results into stationary shock [5]. In this test case LxW flux limited method with Harten's Entropy fix is applied using VanLeer Limiter. Results given in Figure 5 show that the left expansion fan is better captured by ModLxWflm with less entropy glitch compared to LxWflm.

Conclusion and Future work. Three-point second-order schemes are investigated in terms of smoothness parameter for their TV stability bounds. These bounds show that it is possible to have second-order accuracy at extrema and steep gradient regions where r is negative. A hybrid approach is used to show



(a) (b)

FIGURE 4. Numerical results are given for $C = 0.6$, $N = 60$, $T = 4.0$

N	L_1	Rate	L_∞	Rate
20	0.03894062	—	0.00646461	—
40	0.02136945	1.82	0.00201133	3.2
80	0.01090251	1.96	0.00055271	3.6
160	0.00554425	1.96	0.00014610	3.8
320	0.00282380	1.96	0.00003719	3.9

TABLE 3. Convergence rate of ModLxWflm for Burgers equation with initial condition $u_0(x) = \frac{\exp(-x^2)}{4}$ at $T = 1.0$ using $C = 0.6$.

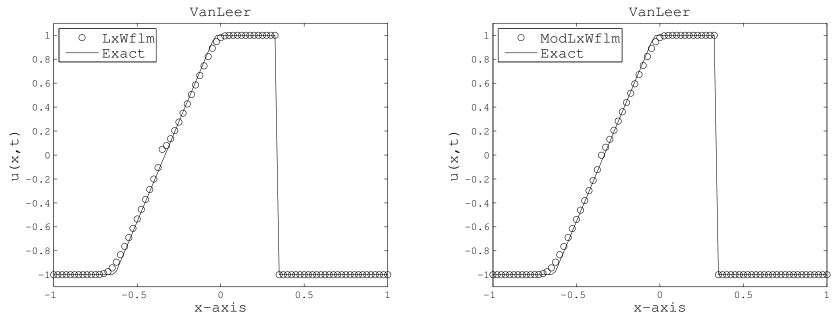
improvement in TVD approximation with well known flux limited method using these bounds. In future it would be interesting to device better hybrid methods to improve the accuracy of existing high order TVD schemes at extrema. Also five point schemes will be analyzed to see possibility for improvement when $r \to 0^{\pm}$.

References

[1] Jonathan B. Goodman, Randall J. LeVeque; A geometric approach to high resolution tvd schemes. *SIAM J. Numer. Anal.*, 25:268–284, April 1988.

[2] Ami Harten; High resolution schemes for hyperbolic conservation laws,. *Journal of Computational Physics*, 49(3):357 – 393, 1983.

[3] S. Jin, Z. Xin, Shi Jin, Zhouping Xin; The relaxation schemes for systems of conservation laws in arbitrary space dimensions. *Comm. Pure Appl. Math*, 48:235–277, 1995.



(a) (b)

FIGURE 5. Numerical results are given for $C = 0.8$, $N = 80$, $T = 0.3$

[4] M. K. Kadalbajoo, Ritesh Kumar; A high resolution total variation diminishing scheme for hyperbolic conservation law and related problems. *Applied Mathematics and Computation*, 175(2):1556 – 1573, 2006.

[5] Culbert B. Laney; *Computational gasdynamics*. Cambridge University Press, 1998.

[6] Randall J. LeVeque; *Numerical Methods for Conservation Laws*. Lectures in mathematics ETH Zürich. Birkhäuser Basel, 2nd edition, February 1992.

[7] S. Osher, S. Chakravarthy; High resolution schemes and entropy condition. *SIAM J. Numer. Anal.*, 21:955–984, 1984.

[8] Stanley Osher, Eitan Tadmor; On the convergence of difference approximations to scalar conservation laws. *Mathematics of Computation*, 50(181):pp. 19–51, 1988.

[9] William J. Rider; A comparison of tvd lax-wendroff methods. *Communications in Numerical Methods in Engineering*, 9(2):147–155, 1993.

[10] Philip L. Roe; Characteristic-based schemes for the euler equations. *Annual Review of Fluid Mechanics*, 18:337–365, 1986.

[11] Richard Sanders; A third-order accurate variation nonexpansive difference scheme for single nonlinear conservation laws. *Mathematics of Computation*, 51(184):pp. 535–558, 1988.

[12] P. K. Sweby; High resolution schemes using flux limiters for hyperbolic conservation laws. *Siam Journal on Numerical Analysis*, 21(5):995–1011, 1984.

[13] E.F. Toro; *Riemann Solvers and Numerical Methods for Fluid Dynamics: A Practical Introduction*. Applied mechanics: Researchers and students. Springer-Verlag GmbH, 1999.

[14] Bram van Leer; Towards the ultimate conservative difference scheme. ii. monotonicity and conservation combined in a second-order scheme. *Journal of Computational Physics*, 14(4):361 – 370, 1974.

[15] H. C. Yee; Construction of explicit and implicit symmetric tvd schemes and their applications. *Journal of Computational Physics*, 68(1):151 – 179, 1987.

[16] Xiangxiong Zhang, Chi-Wang Shu; A genuinely high order total variation diminishing scheme for one-dimensional scalar conservation laws. *SIAM J. Numerical Analysis*, 48(2):772–795, 2010.

RITESH KUMAR DUBEY
RESEARCH INSTITUTE, SRM UNIVERSITY, TAMILNADU, INDIA
E-mail address: riteshkd@gmail.com, riteshkumar.d@res.srmuniv.ac.in