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POPULATION MODELS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We study a two point boundary-value problem describing the steady states of a Logistic growth population model with diffusion and constant yield harvesting. In particular, we focus on a model when a certain nonlinear boundary condition is satisfied.

1. Introduction

Consider the Logistic growth population dynamics model with nonlinear boundary conditions:

$$u_t = d\Delta u + au - bu^2 - ch(x) \quad \text{in } \Omega, \tag{1.1}$$

$$d\alpha(x,u)\frac{\partial u}{\partial \eta} + [1-\alpha(x,u)]u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 1$, Δ is the Laplace operator, d is the diffusion coefficient, a, b are positive parameters, $c \geq 0$ is the harvesting parameter, $h(x) : \overline{\Omega} \to \mathbb{R}$ is a C^1 function, $\frac{\partial u}{\partial \eta}$ is the outward normal derivative, and $\alpha(x,u) : \Omega \times \mathbb{R} \to [0,1]$ is a nondecreasing C^1 function.

The parameter $c \geq 0$ represents the level of harvesting, $h(x) \geq 0$ for $x \in \Omega$, $h(x) = 0$ for $x \in \partial\Omega$, and $\|h\|_\infty = 1$. Here $ch(x)$ can be understood as the rate of the harvesting distribution. The nonlinear boundary condition (1.2) has only been recently studied by such authors as [1, 2, 3], among others. Here

$$\alpha(x,u) = \alpha(u) = \frac{u}{u - d\frac{\partial u}{\partial \eta}}$$

represents the fraction of the population that remains on the boundary when reached. For the case when $\alpha(x,u) \equiv 0$, (1.2) becomes the well known Dirichlet boundary condition. If $\alpha(x,u) \equiv 1$ then (1.2) becomes the Neumann boundary condition. Here we will be interested in the study of positive steady state solutions

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of (1.1)–(1.2) when $d = 1$ and

$$\alpha(x,u) = \frac{u}{u+1} \quad \text{on } \partial\Omega.$$

Hence, we consider the model

$$-\Delta u = au - bu^2 - ch(x) =: f(x,u) \quad \text{in } \Omega, \tag{1.3}$$

$$u[\frac{\partial u}{\partial \eta} + 1] = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

We will present the results of the case when $n = 1$, $\Omega = (0,1)$, and $h(x) \equiv 1$. Thus, we study the nonlinear boundary-value problem

$$-u'' = au - bu^2 - c, \quad x \in (0,1), \tag{1.5}$$

$$[-u'(0) + 1]u(0) = 0, \tag{1.6}$$

$$[u'(1) + 1]u(1) = 0. \tag{1.7}$$

It is easy to see that analyzing the positive solutions of (1.5)–(1.7) is equivalent to studying the four boundary-value problems

$$-u'' = au - bu^2 - c, \quad x \in (0,1), \tag{1.8}$$

$$u(0) = 0, \quad u(1) = 0; \tag{1.9}$$

$$-u'' = au - bu^2 - c, \quad x \in (0,1), \tag{1.10}$$

$$u(0) = 0, \quad u'(1) = -1; \tag{1.11}$$

$$-u'' = au - bu^2 - c, \quad x \in (0,1), \tag{1.12}$$

$$u'(0) = 1, \quad u(1) = 0; \tag{1.13}$$

$$-u'' = au - bu^2 - c, \quad x \in (0,1), \tag{1.14}$$

$$u'(0) = 1, \quad u'(1) = -1. \tag{1.15}$$

Hence, the positive solutions of these four BVPs are the positive solutions of (1.5)–(1.7). Notice that if $u(x)$ is a solution of (1.10)–(1.11) then $v(x) := u(1-x)$ is a solution of (1.12)–(1.13). Thus, it suffices to only consider (1.8)–(1.9), (1.10)–(1.11), and (1.14)–(1.15). The structure of positive solutions for (1.8)–(1.9) is known (see [4] and [7]) via the quadrature method introduced by Laetsch in [8]. We develop quadrature methods in Section 2 to completely determine the bifurcation diagram of (1.5)–(1.7). In Section 3 we use Mathematica computations to show that for certain subsets of the parameter space, (1.5)–(1.7) has up to exactly 8 positive solutions. For higher dimensional results, in the case when $\alpha(x,u) = 0$ on $\partial\Omega$ (Dirichlet boundary conditions) see [9], and for the case when $\alpha(x,u) = \frac{u}{u+1}$ on $\partial\Omega$ see recent work in [5].

2. Results via the quadrature method

2.1. **Positive solutions of** (1.8)–(1.9)**.**

In this section we summarize the known results (see [9]) for positive solutions of (1.8)–(1.9). Consider the boundary value problem:

$$-u'' = au - bu^2 - c =: f(u), \quad x \in (0,1), \tag{2.1}$$

$$u(0) = 0, \quad u(1) = 0. \tag{2.2}$$

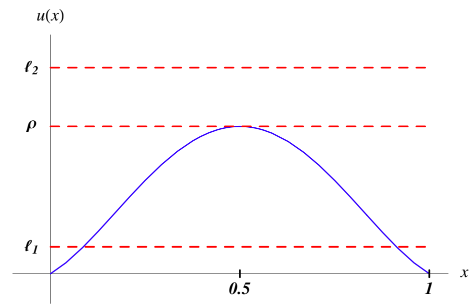

FIGURE 1. Typical solution of (2.1)–(2.2)

It is easy to see that positive solutions of (2.1)–(2.2) must resemble Figure 1 where ℓ_i for $i = 1, 2$ are the positive zeros of $f(u)$. The following theorem details the structure of positive solutions of (2.1)–(2.2) for the case when $b = 1$:

Theorem 2.1 ([4, 9])**.** (1) *If $a < \lambda_1$ then* (2.1)–(2.2) *has no positive solution for any $c \geq 0$.*

(2) *If $\lambda_1 \leq a < \lambda^*$ (some $\lambda^* > \lambda_1$) then there exists a $c_0 > 0$ such that if*
   (a) *$0 \leq c < c_0$ then* (2.1)–(2.2) *has 2 positive solutions.*
   (b) *$c = c_0$ then* (2.1)–(2.2) *has a unique positive solution.*
   (c) *$c > c_0$ then* (2.1)–(2.2) *has no positive solution.*

(3) *If $a > \lambda^*$ then there exist $c_0, \tilde{c} > 0$ such that if*
   (a) *$\tilde{c} < c < c_0$ then* (2.1)–(2.2) *has 2 positive solutions.*
   (b) *$0 \leq c < \tilde{c}$ or $c = c_0$ then* (2.1)–(2.2) *has a unique positive solution.*
   (c) *$c > c_0$ then* (2.1)–(2.2) *has no positive solution.*

Figure 2 illustrates this theorem.

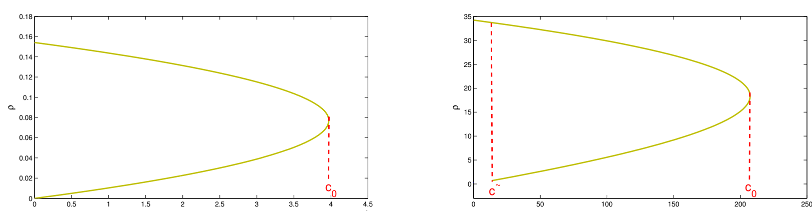

FIGURE 2. $a = 10$, $b = 1$ (left), and $a = 40$, $b = 1$ (right)

2.2. **Positive solutions of** (1.10)–(1.11)**.**

In this subsection, we adapt the quadrature method in [8] to study

$$-u'' = au - bu^2 - c =: f(u), \quad x \in (0,1), \tag{2.3}$$

$$u(0) = 0, \quad u'(1) = -1. \tag{2.4}$$

Now, define $F(u) = \int_0^u f(s)ds$, the primitive of $f(u)$. Since (2.3) is an autonomous differential equation, if $u(x)$ is a positive solution of (2.3) with $u'(x_0) = 0$ for some

$x_0 \in (0,1)$ then $v(x) := u(x_0 - x)$ and $w(x) := u(x_0 + x)$ both satisfy the initial value problem,

$$-z'' = f(z) \tag{2.5}$$

$$z(0) = u(x_0) \tag{2.6}$$

$$z'(0) = 0 \tag{2.7}$$

for all $x \in [0, d)$ where $d = min\{x_0, 1 - x_0\}$. As a result of Picard's existence and uniqueness theorem, $u(x_0 - x) \equiv u(x_0 + x)$. Thus, if we assume that $u(x)$ is a positive solution of (2.3)–(2.4) then it is symmetric around x_0 with $\rho := \|u\|_\infty = u(x_0)$. This implies that $u'(x_0) = 0$, $u'(x) > 0$; $[0, x_0)$, and $u'(x) < 0$; $(x_0, 1]$. Using symmetry about x_0, the boundary conditions (2.4), and the sign of u'' given by $f(u)$ we see that positive solutions of (2.3)–(2.4) must resemble Figure 3, where $\rho = \|u\|_\infty$ and $q = u(1)$. This implies that $\ell_1 < \rho < \ell_2$ and $0 \le q < \rho$ where ℓ_i, $i = 1, 2$ are the zeros of $f(u)$.

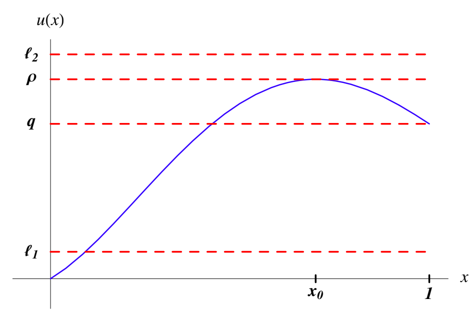

FIGURE 3. Typical solution of (2.3) -(2.4)

Multiplying (2.3) by u' gives

$$-u'u'' = f(u)u' \tag{2.8}$$

Integration of (2.8) with respect to x gives,

$$-\Big(\frac{[u'(x)]^2}{2}\Big) = [F(u(x))] + K. \tag{2.9}$$

Substituting $x = 1$ and $x = x_0$ into (2.9) yields,

$$-K = F(q) + \frac{1}{2} \tag{2.10}$$

$$K = -F(\rho). \tag{2.11}$$

Combining (2.10) and (2.11), we have

$$F(\rho) = F(q) + \frac{1}{2}. \tag{2.12}$$

Substituting (2.11) into (2.9) yields,

$$-\Big(\frac{[u'(x)]^2}{2}\Big) = [F(u(x))] - F(\rho). \tag{2.13}$$

Now, solving for u' in (2.13) gives

$$u'(x) = \sqrt{2}\sqrt{F(\rho) - F(u(x))}, \quad x \in [0, x_0], \tag{2.14}$$

$$u'(x) = -\sqrt{2}\sqrt{F(\rho) - F(u(x))}, \quad x \in [x_0, 1]. \tag{2.15}$$

Integrating (2.14) and (2.15) with respect to x and using a change of variables, we have

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2}x, \quad x \in [0, x_0], \tag{2.16}$$

$$\int_\rho^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = -\sqrt{2}(x - x_0), \quad x \in [x_0, 1]. \tag{2.17}$$

Substitution of $x = x_0$ into (2.16) and $x = 1$ into (2.17) gives

$$\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2}x_0 \tag{2.18}$$

$$\int_\rho^q \frac{ds}{\sqrt{F(\rho) - F(s)}} = -\sqrt{2}(1 - x_0). \tag{2.19}$$

Finally, subtracting (2.19) from (2.18), yields

$$\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_q^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2}, \tag{2.20}$$

or equivalently,

$$2\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \int_0^q \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2}. \tag{2.21}$$

We note that in order for $\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}}$ to be well defined, $F(\rho) > F(s)$ for all $s \in [0, \rho)$. Moreover, the improper integral is convergent if $f(\rho) > 0$. Thus, for such a positive solution to exist, $f(u)$ and $F(u)$ must resemble Figure 4, where μ_1, ℓ_i, and θ_i are the zeros of $f'(u)$, $f(u)$, and $F(u)$ respectively for $i = 1, 2$.

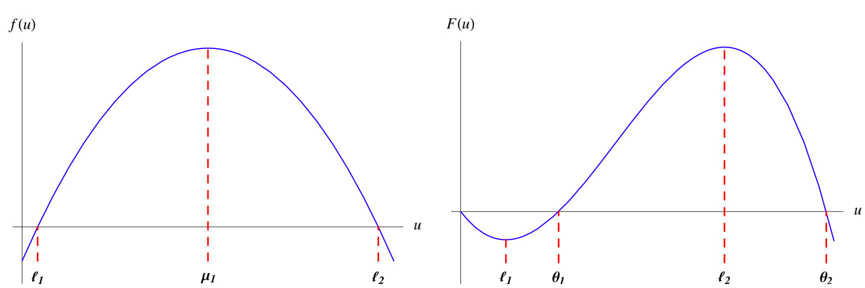

FIGURE 4. Graph of $f(u)$ (left), and of $F(u)$ (right)

From Figure 4, we note that if $\rho \in (\theta_1, \ell_2)$ then both of these conditions hold and the integrals in (2.21) are well defined. From this and letting $c_1 := \frac{3a^2}{16b}$ and $c_2 := \frac{a^2}{4b}$, we can arrive at the following result.

Theorem 2.2. *If $c > c^*(a, b)$ then (2.3)–(2.4) has no positive solution, where $c^*(a, b) = \min\{c_1, c_2\} = \frac{3a^2}{16b}$.*

Further, since $x_0 \in (0,1)$ is fixed for each $\rho > 0$, we need a unique $q < \rho$ corresponding to each ρ-value such that (2.12) is satisfied. Otherwise, uniqueness of solutions to the initial value problem, (2.5)–(2.7), would be violated. Let

$$H(x) := F(x) + \frac{1}{2}.$$

It follows that $H'(x) = -bx^2 + ax - c$, $H(0) = 1/2$, and $H'(0) = -c < 0$. In order for a unique $q < \rho$ to exist such that $H(q) = F(\rho)$, $H(x)$ must have the following structure in Figure 5, where $H'(\ell_2) = 0$. So, for such a unique $q < \rho$ to exist $F(\rho) > 1/2$.

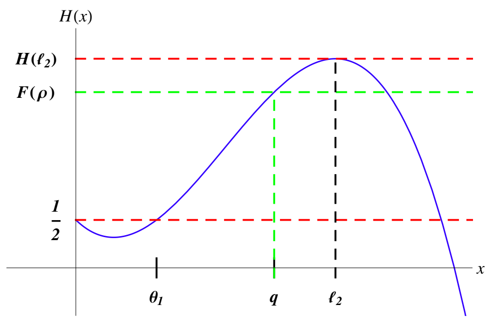

FIGURE 5. Graph of $H(x)$

Since $\rho \in (\theta_1, \ell_2)$, for this to be true we will need $H(\ell_2) > 1/2$. In fact, if

$$F(\ell_2) > \frac{1}{2} \tag{2.22}$$

then clearly for $\rho \in (\theta_1, \ell_2)$ with $\rho \approx \ell_2$ we have $F(\rho) > 1/2$. It is easy to see that (2.22) will be satisfied if (solving using Mathematica)

$$\begin{aligned} c < c_3 := \frac{9a^2}{144b} - \frac{9(a^4 - 96ab^2)}{144b\Big(-a^6 - 240a^3b^2 + 16\big(72b^4 + \sqrt{3}\sqrt{b^2(a^3+12b^2)^3}\big)\Big)^{1/3}} \\ - \frac{9}{144b}\Big(-a^6 - 240a^3b^2 + 16\big(72b^4 + \sqrt{3}\sqrt{b^2(a^3+12b^2)^3}\big)\Big) \end{aligned}$$

and for c_3 to be positive (again using Mathematica)

$$a > a_0 := \sqrt[3]{3b^2}$$

both hold. This leads to the following results.

Theorem 2.3. *If $a \le a_0$ then (2.3)–(2.4) has no positive solution for any $c \ge 0$.*

Theorem 2.4. *If $a > a_0$ then there is a $c^*(a,b) \le min\{c_1, c_2, c_3\}$ such that for $c \ge c^*$ (2.3)–(2.4) has no positive solution.*

We now state and prove the main theorem of this subsection.

Theorem 2.5. *If $a > a_0$ and $c < c^*(a,b)$ then there is a unique $r(a,b,c) \in (\theta_1, \ell_2)$ such that $F(r) = 1/2$ and*

$$G(\rho) := 2\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \int_0^q \frac{ds}{\sqrt{F(\rho) - F(s)}}$$

is well defined for all $\rho \in [r, \ell_2)$ *where* $q < \rho$ *is the unique solution of* $F(\rho) = H(q)$. *Moreover,* (2.3)–(2.4) *has a positive solution,* $u(x)$*, with* $\rho = \|u\|_\infty$ *if and only if* $G(\rho) = \sqrt{2}$ *for some* $\rho \in [r, \ell_2)$.

Proof. Let $a, b > 0$ s.t. $a > a_0$ and $c \in [0, c^*(a, b))$. From the preceding discussion, it follows that if u is a positive solution to (2.3)–(2.4) with $\rho = \|u\|_\infty$ then $G(\rho) = \sqrt{2}$. Next, suppose $G(\rho) = \sqrt{2}$ for some $\rho \in [r, \ell_2)$. Define $u(x) : (0, 1) \to \mathbb{R}$ by

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2}x, \quad x \in [0, x_0], \tag{2.23}$$

$$\int_\rho^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = -\sqrt{2}(x - x_0), \quad x \in [x_0, 1]. \tag{2.24}$$

Now, we show that $u(x)$ is a positive solution to (2.3)–(2.4). It is easy to see that the turning point is given by $x_0 = \frac{1}{\sqrt{2}} \int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}}$. The function, $\int_0^u \frac{ds}{\sqrt{F(\rho)-F(s)}}$, is a differentiable function of u which is strictly increasing from 0 to x_0 as u increases from 0 to ρ. Thus, for each $x \in [0, x_0]$, there is a unique $u(x)$ such that

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2}x \tag{2.25}$$

Moreover, by the Implicit Function theorem, u is differentiable with respect to x. Differentiating (2.25) gives

$$u'(x) = \sqrt{2[F(\rho) - F(u)]}, \quad x \in [0, x_0].$$

Similarly, u is a decreasing function of x for $x \in [x_0, 1]$ which yields,

$$u'(x) = -\sqrt{2[F(\rho) - F(u)]}, \quad x \in [x_0, 1].$$

This implies

$$\frac{-(u')^2}{2} = F(\rho) - F(u(x)).$$

Differentiating again, we have $-u''(x) = f(u(x))$. Thus, $u(x)$ satisfies (2.3). Now, from our assumption, $G(\rho) = \sqrt{2}$, it follows that $u(0) = 0$ and $u(1) = q(\rho)$. Since $F(\rho) = H(q(\rho)) = F(q) + \frac{1}{2}$, we have that $u'(1) = -\sqrt{2[F(\rho) - F(q)]} = -1$. Hence, the boundary conditions (2.4) are both satisfied. □

2.3. **Positive solutions of** (1.14)–(1.15)**.** A similar quadrature method can be adapted to study

$$-u'' = au - bu^2 - c =: f(u), \quad x \in (0, 1), \tag{2.26}$$

$$u'(0) = 1, \quad u'(1) = -1. \tag{2.27}$$

Again, define $F(u) = \int_0^u f(s)ds$, the primitive of $f(u)$. Using a similar argument as before, symmetry about x_0, the boundary conditions (2.26)–(2.27), and the sign of u'' given by $f(u)$ ensure that positive solutions of (2.26)–(2.27) must resemble Figure 6, where $\rho = \|u\|_\infty$ and $q = u(0) = u(1)$. Clearly, $x_0 = 1/2$ in this case.

Through an almost identical approach as the one in Section 2.2, we can prove the following results.

Theorem 2.6. *If* $a \leq a_0$ *then* (2.26)–(2.27) *has no positive solution for any* $c \geq 0$.

Theorem 2.7. *If* $a > a_0$ *then there is a* $c^*(a, b) \leq \min\{c_1, c_2, c_3\}$ *such that for* $c \geq c^*$ (2.26)–(2.27) *has no positive solution.*

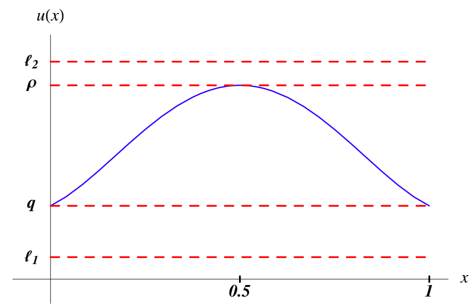

FIGURE 6. Typical solution of (2.3)–(2.4)

We now state the main theorem of this subsection.

Theorem 2.8. *If $a > a_0$ and $c < c^*(a,b)$ then there is a unique $r(a,b,c) \in (\theta_1, \ell_2)$ such that $F(r) = \frac{1}{2}$ and*

$$\widetilde{G}(\rho) := 2\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}} - 2\int_0^q \frac{ds}{\sqrt{F(\rho)-F(s)}}$$

is well defined for all $\rho \in [r, \ell_2)$ where $q < \rho$ is the unique solution of $F(\rho) = H(q)$. Moreover, (2.26)–(2.27) *has a positive solution, $u(x)$, with $\rho = \|u\|_\infty$ if and only if $\widetilde{G}(\rho) = \sqrt{2}$ for some $\rho \in [r, \ell_2)$.*

Remark. See [7] where Ladner et al. adapted the quadrature method to study the case when $\alpha(x,u) = \frac{u}{a}$ on $\partial\Omega$. Also, see [6] where the quadrature method was adapted to study the case with a Strong Allee effect and $\alpha(x,u) = \frac{u}{b}$ on $\partial\Omega$.

3. Computational results

3.1. **Positive solutions of** (1.10)–(1.11) **and** (1.12)–(1.13)**.** We are particularly interested in the case when $b = 1$. From Theorem 2.5, we plot the level sets of

$$G(\rho) - \sqrt{2} = 0 \tag{3.1}$$

for $a > \sqrt[3]{3}$ and $\rho \in [r, \ell_2)$. By implementing a numerical root-finding algorithm in Mathematica we were able to solve equation (3.1). Explicit formulas were used to calculate the unique $r = r(a,b,c)$ and $q = q(\rho)$ values. Note that these computations are expensive due to the natural of the improper integral equations involved. Figures 7 - 9 depict several level sets plotted within $[r, \ell_2) \times [0, c^*]$. In what follows, the green curve represents ρ vs c while the upper and lower branches of the dotted black curve represent ℓ_2 and r, respectively. The green curve's lower branch begins to shrink for $a \geq 10.1388$. This is due to the fact that solutions of (3.1) are outside of $[r, \ell_2)$. The bifurcation diagrams also indicate the following results.

Theorem 3.1. *For $b = 1$, if $a < a_4$ (for $a_4 \approx 5.0407$) then* (1.10)–(1.11) *and* (1.12)–(1.13) *have no positive solution for any $c \geq 0$.*

Theorem 3.2. *If $b = 1$ then $c_0(a) \to c^*(a)$ as $a \to \infty$. Furthermore, $\rho \to \ell_2$ as $a \to \infty$ where $u(x)$ is a positive solution to* (1.10)–(1.11) *or* (1.12)–(1.13) *with $\|u\|_\infty = \rho$.*

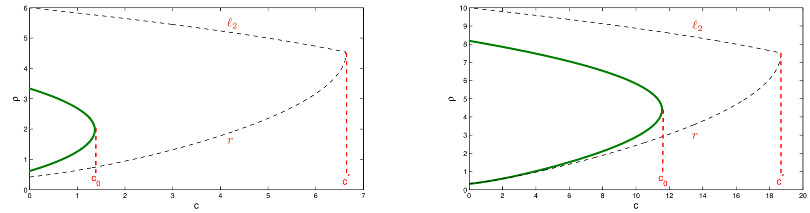

FIGURE 7. $a = 6$, $b = 1$ (left), and $a = 10$, $b = 1$ (right)

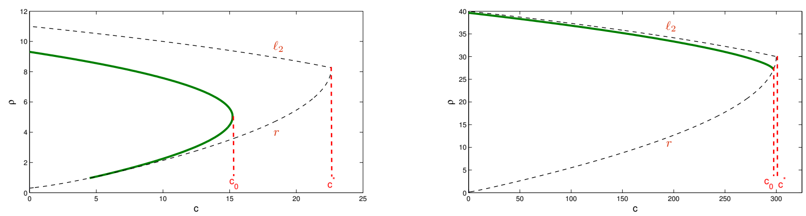

FIGURE 8. $a = 11$, $b = 1$ (left), and $a = 40$, $b = 1$ (right)

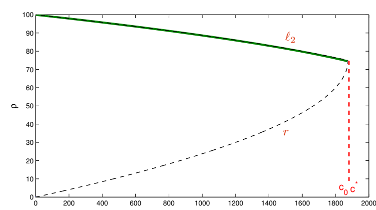

FIGURE 9. $a = 100$, $b = 1$

3.2. **Positive solutions of** (1.14)–(1.15)**.** Again, we are particularly interested in the case when $b = 1$. Recalling Theorem 2.8, we plot the level sets of

$$\widetilde{G}(\rho) - \sqrt{2} = 0 \tag{3.2}$$

Using our numerical root-finding algorithm in Mathematica to solve equation (3.2) and explicit formulas to calculate the unique $r = r(a, b, c)$ and $q = q(\rho)$ values, level sets were plotted within $[r, \ell_2) \times [0, c^*]$. The blue curve breaks into two components somewhere around $a = 4.39$, with the lower component vanishing for $a > 10.1387$. This is due to the fact that the ρ-values, which are solutions of (3.2), are outside of $[r, \ell_2)$. These bifurcation diagrams also indicate the following results.

Theorem 3.3. *For $b = 1$, if $a < a_1$ (for $a_1 \approx 2.8324$) then* (1.14)–(1.15) *has no positive solution for any $c \geq 0$.*

Theorem 3.4. *If $b = 1$ then $c_0(a) \to c^*(a)$ as $a \to \infty$. Furthermore, $\rho \to \ell_2$ as $a \to \infty$ where $u(x)$ is a positive solution to* (1.14)–(1.15) *with $\|u\|_\infty = \rho$.*

3.3. **Structure of Positive solutions to** (1.5)–(1.7)**.** Combining results from the three cases, (1.8)–(1.9), (1.10)–(1.11), and (1.14)–(1.15) while recalling that the

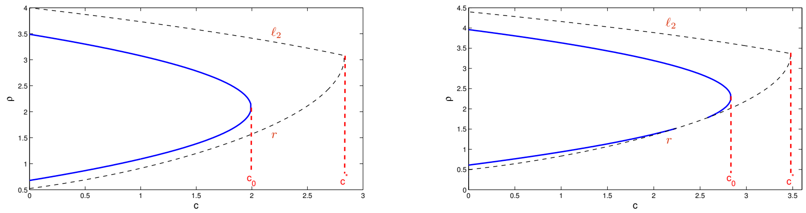

FIGURE 10. $a = 4$, $b = 1$ (left), and $a = 4.4$, $b = 1$ (right)

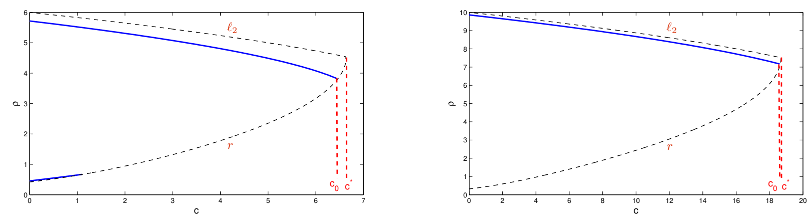

FIGURE 11. $a = 6$, $b = 1$ (left), and $a = 10$, $b = 1$ (right)

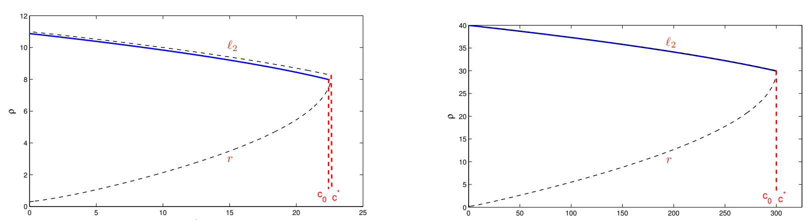

FIGURE 12. $a = 11$, $b = 1$ (left), and $a = 40$, $b = 1$ (right)

(1.10)–(1.11) case represents two symmetric solutions, we are able to completely determine the structure of positive solutions to (1.5)–(1.7). As before, we are primarily interested in the case when $b = 1$. Comparison of nonexistence Theorems 2.1, 2.3, and 2.6 from Section 3 yields the following nonexistence result for (1.5)–(1.7).

Theorem 3.5. *If* $a \le \min[\sqrt[3]{3b^2}, \lambda_1]$ *then* (1.5)–(1.7) *has no positive solution for any* $c \ge 0$*.*

Moreover, our computational results for the case $b = 1$ provide the following nonexistence result.

Theorem 3.6. *For* $b = 1$*, if* $a < a_1$ *(for* $a_1 \approx 2.8324$*) then* (1.5)–(1.7) *has no positive solution for any* $c \ge 0$*.*

Also, our computations indicate the following existence results for $b = 1$. For what follows, (1.8)–(1.9) is depicted in yellow, (1.10)–(1.11) and (1.12)–(1.13) both in green, and (1.14)–(1.15) in blue.

Theorem 3.7. *For $b = 1$, if $a \in [a_1, a_2)$ (for some $a_2 > a_1$) (for $a_2 \approx 4.39$) then there exists a $C_0 > 0$ such that if*

1. *$0 \le c < C_0$ then* (1.5)–(1.7) *has exactly 2 positive solutions.*
2. *$c = C_0$ then* (1.5)–(1.7) *has a unique positive solution.*
3. *$c > C_0$ then* (1.5)–(1.7) *has no positive solution.*

A bifurcation diagram of the case when $b = 1$ and $a = 4$ is shown in Figure 13.

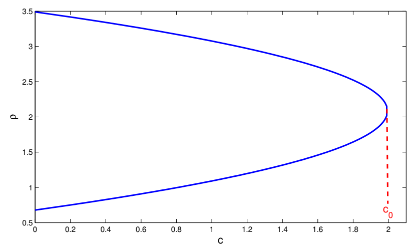

FIGURE 13. ρ vs c for $a = 4$, $b = 1$

Theorem 3.8. *For $b = 1$, if $a \in [a_2, a_3)$ (some $a_3 \in (4.4, 5)$) then there exist $C_i > 0$, $i = 0, 1, 2$, such that if*

1. *$0 \le c \le C_2$ or $C_1 \le c < C_0$ then* (1.5)–(1.7) *has exactly 2 positive solutions.*
2. *$C_2 < c < C_1$ or $c = C_0$ then* (1.5)–(1.7) *has a unique positive solution.*
3. *$c > C_0$ then* (1.5)–(1.7) *has no positive solution.*

Figure 14 illustrates Theorem 3.8.

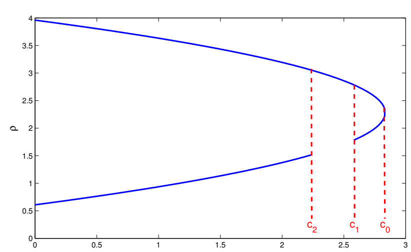

FIGURE 14. ρ vs c for $a = 4.4$, $b = 1$

Theorem 3.9. *For $b = 1$, if $a \in [a_3, a_4)$ (for $a_4 \approx 5.0407$) then there exist $C_i > 0$, $i = 0, 1$, such that if*

1. *$0 \le c \le C_1$ then* (1.5)–(1.7) *has exactly 2 positive solutions.*
2. *$C_1 < c \le C_0$ then* (1.5)–(1.7) *has a unique positive solution.*
3. *$c > C_0$ then* (1.5)–(1.7) *has no positive solution.*

Theorem 3.9 is illustrated in Figure 15.

Theorem 3.10. *For $b = 1$, if $a \in [a_4, a_5)$ (for $a_5 = \pi^2$) then there exist $C_i > 0$, $i = 0, 1, 2$, such that if*

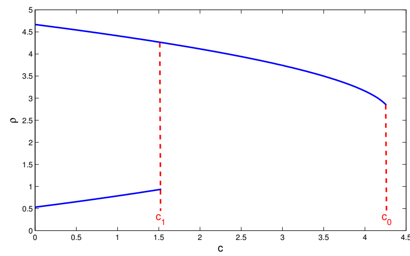

FIGURE 15. ρ vs c for $a = 5.03$, $b = 1$

(1) $0 \le c \le C_2$ *then* (1.5)–(1.7) *has exactly 6 positive solutions.*
(2) $C_2 < c < C_1$ *then* (1.5)–(1.7) *has exactly 5 positive solutions.*
(3) $c = C_1$ *then* (1.5)–(1.7) *has exactly 3 positive solutions.*
(4) $C_1 < c \le C_0$ *then* (1.5)–(1.7) *has a unique positive solution.*
(5) $c > C_0$ *then* (1.5)–(1.7) *has no positive solution.*

Theorem 3.10 is depicted in Figure 16.

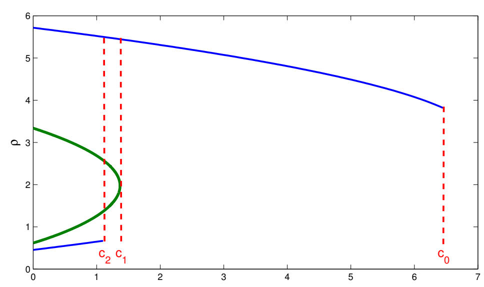

FIGURE 16. ρ vs c for $a = 6$, $b = 1$

Theorem 3.11. *For* $b = 1$*, if* $a \in [a_5, a_6)$ *(some* $a_6 \in (10, 10.1388)$*) then there exist* $C_i > 0$*,* $i = 0, 1, 2, 3$*, such that if*

(1) $0 \le c < C_3$ *then* (1.5)–(1.7) *has exactly 8 positive solutions.*
(2) $c = C_3$ *then* (1.5)–(1.7) *has exactly 7 positive solutions.*
(3) $C_3 < c \le C_2$ *then* (1.5)–(1.7) *has exactly 6 positive solutions.*
(4) $C_2 < c < C_1$ *then* (1.5)–(1.7) *has exactly 5 positive solutions.*
(5) $c = C_1$ *then* (1.5)–(1.7) *has exactly 3 positive solutions.*
(6) $C_1 < c \le C_0$ *then* (1.5)–(1.7) *has a unique positive solution.*
(7) $c > C_0$ *then* (1.5)–(1.7) *has no positive solution.*

Figure 17 shows the bifurcation diagram for $a = 10$, $b = 1$ along with Figure 18, which gives two small cross sections of the diagram.

Theorem 3.12. *For* $b = 1$*, if* $a \in [a_6, a_7)$ *(for* $a_7 \approx 10.1388$*) then there exist* $C_i > 0$*,* $i = 0, 1, 2, 3$*, such that if*

(1) $0 \le c \le C_3$ *then* (1.5)–(1.7) *has exactly 8 positive solutions.*

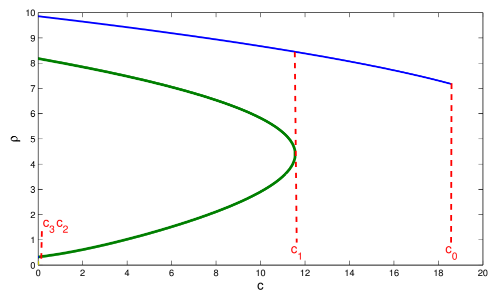

FIGURE 17. ρ vs c for $a = 10$, $b = 1$

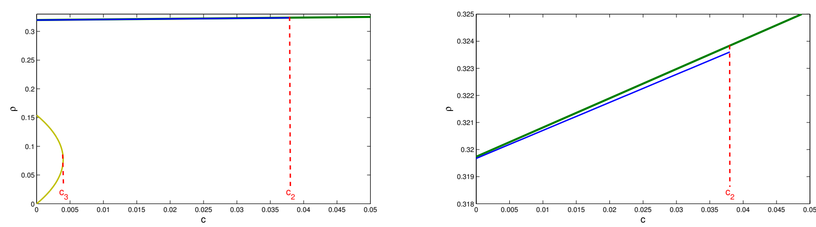

FIGURE 18. ρ vs c cross-sections for $a = 10$, $b = 1$

(2) $C_3 < c < C_2$ *then* (1.5)–(1.7) *has exactly 7 positive solutions.*
(3) $c = C_2$ *then* (1.5)–(1.7) *has exactly 6 positive solutions.*
(4) $C_2 < c < C_1$ *then* (1.5)–(1.7) *has exactly 5 positive solutions.*
(5) $c = C_1$ *then* (1.5)–(1.7) *has exactly 3 positive solutions.*
(6) $C_1 < c \le C_0$ *then* (1.5)–(1.7) *has a unique positive solution.*
(7) $c > C_0$ *then* (1.5)–(1.7) *has no positive solution.*

The bifurcation diagram for $a = 10.1, b = 1$ is depicted in Figures 19 and 20.

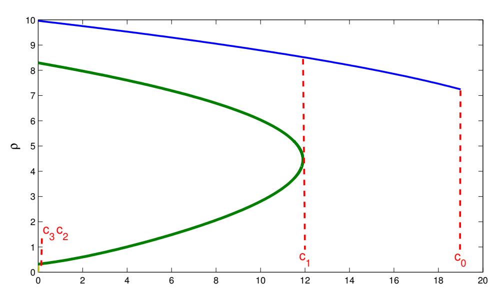

FIGURE 19. ρ vs c for $a = 10.1$, $b = 1$

Theorem 3.13. *For* $b = 1$, *if* $a \in [a_7, a_8]$ *(for* $a_8 = 4\pi^2$*) then there exist* $C_i > 0$, $i = 0, 1, 2, 3$, *such that if*

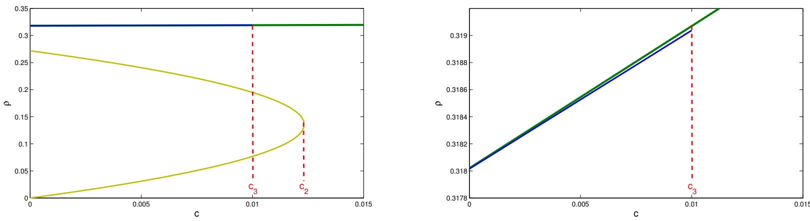

FIGURE 20. ρ vs c cross-sections for $a = 10.1$, $b = 1$

(1) $0 \le c < C_3$ *or* $C_2 \le c < C_1$ *then* (1.5)–(1.7) *has exactly 5 positive solutions.*
(2) $c = C_3$ *then* (1.5)–(1.7) *has exactly 4 positive solutions.*
(3) $C_3 < c < C_2$ *or* $c = C_1$ *then* (1.5)–(1.7) *has exactly 3 positive solutions.*
(4) $C_1 < c \le C_0$ *then* (1.5)–(1.7) *has a unique positive solution.*
(5) $c > C_0$ *then* (1.5)–(1.7) *has no positive solution.*

Figure 21 shows the bifurcation diagram for $a = 11$, $b = 1$.

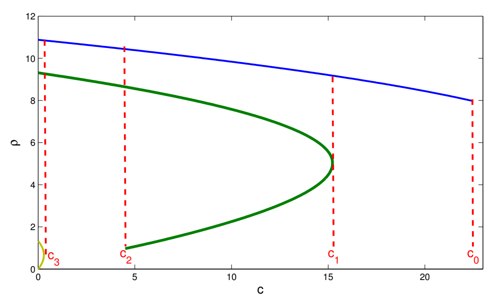

FIGURE 21. ρ vs c for $a = 11$, $b = 1$

Theorem 3.14. *For $b = 1$, if $a \in (a_8, \infty)$ then there exist $C_i > 0$, $i = 0, 1, 2, 3$, such that if*

(1) $C_3 \le c < C_2$ *then* (1.5)–(1.7) *has exactly 5 positive solutions.*
(2) $0 \le c < C_3$ *or* $c = C_2$ *then* (1.5)–(1.7) *has exactly 4 positive solutions.*
(3) $C_2 < c \le C_1$ *then* (1.5)–(1.7) *has exactly 3 positive solutions.*
(4) $C_1 < c \le C_0$ *then* (1.5)–(1.7) *has a unique positive solution.*
(5) $c > C_0$ *then* (1.5)–(1.7) *has no positive solution.*

The bifurcation diagram for $a = 40, b = 1$ is shown in Figure 22.

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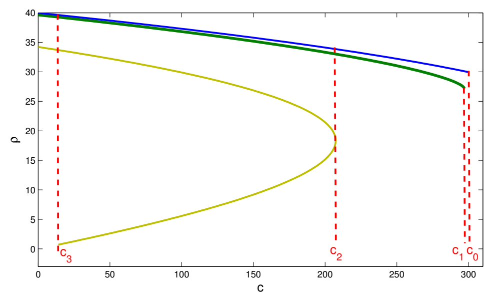

FIGURE 22. ρ vs c for $a = 40$, $b = 1$

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