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CONTINUOUS DEPENDENCE OF SOLUTIONS FOR ILL-POSED EVOLUTION PROBLEMS

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Abstract. We prove Hölder-continuous dependence results for the difference between certain ill-posed and well-posed evolution problems in a Hilbert space. Specifically, given a positive self-adjoint operator D in a Hilbert space, we consider the ill-posed evolution problem

$$\begin{gathered}\frac{du(t)}{dt} = A(t,D)u(t) \quad 0 \le t < T \\ u(0) = \chi.\end{gathered}$$

We determine functions $f : [0,T] \times [0,\infty) \to \mathbb{R}$ for which solutions of the well-posed problem

$$\begin{gathered}\frac{dv(t)}{dt} = f(t,D)v(t) \quad 0 \le t < T \\ v(0) = \chi\end{gathered}$$

approximate known solutions of the original ill-posed problem, thereby establishing continuous dependence on modelling for the problems under consideration.

1. Introduction

Let D be a positive self-adjoint operator in a Hilbert space H, and consider the evolution problem

$$\begin{gathered}\frac{du(t)}{dt} = A(t,D)u(t) \quad 0 \le t < T \\ u(0) = \chi\end{gathered} \tag{1.1}$$

where χ is an arbitrary element of H and

$$A(t,D) = \sum_{j=1}^{k} a_j(t)D^j,$$

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with $a_j : [0,T] \to \mathbb{R}$ continuous and nonnegative for each $1 \le j \le k$. In general, (1.1) is ill-posed with formal solution given by

$$u(t) = \exp\Big\{ \sum_{j=1}^{k} \Big(\int_0^t a_j(s)ds \Big) D^j \Big\} \chi.$$

Now let $f : [0,T] \times [0,\infty) \to \mathbb{R}$ be a function continuous in t and Borel in λ, and consider the evolution problem

$$\begin{gathered} \frac{dv(t)}{dt} = f(t,D)v(t) \quad 0 \le t < T \\ v(0) = \chi \,, \end{gathered} \tag{1.2}$$

where for each $t \in [0,T]$, $f(t,D)$ is defined by means of the functional calculus for self-adjoint operators in H. In particular, since D is positive, self-adjoint, the spectrum $\sigma(D)$ of D is contained in $[0,\infty)$ and for each $t \in [0,T]$,

$$f(t,D)x = \int_0^\infty f(t,\lambda) dE(\lambda)x,$$

for $x \in \mathrm{Dom}(f(t,D)) = \{x \in H : \int_0^\infty |f(t,\lambda)|^2 d(E(\lambda)x,x) < \infty\}$, where $\{E(\cdot)\}$ denotes the resolution of the identity for the self-adjoint operator D.

We determine conditions on f so that (1.2) is well-posed and such that solutions of (1.2) approximate known solutions of (1.1). In this way, we illustrate how we might stabilize problems against errors that arise when formulating mathematical models such as (1.1) in attempts to describe some physical process. Ames and Hughes [3] established such structural stability results for the problem in the autonomous case, that is when $A(t) = A$ is a positive self-adjoint operator in H, independent of t. This paper generalizes such work and yields a comparable result in the time-dependent case. Namely, if $u(t)$ and $v(t)$ are solutions of (1.1) and (1.2) respectively, we prove the Hölder-continuous approximation

$$\|u(t) - v(t)\| \le C\beta^{1-\frac{t}{T}} M^{t/T},$$

where $0 < \beta < 1$, and C and M are constants independent of β. Our approximation establishes continuous dependence on modelling, meaning "small" changes to our model yield a "small" change in the corresponding solution.

In Section 2, we establish conditions under which (1.2) is well-posed using stable families of generators of semigroups and Kato's stability conditions [8, 11]; our work also utilizes Tanaka's results on evolution problems [14]. In Section 3, we present our approximation theorem which achieves Hölder-continuous dependence on modelling. Finally, Section 4 demonstrates our theorem with examples.

Below, for a closed operator A in a Banach space X, $\rho(A)$ will denote the resolvent set of A, and for $\lambda \in \rho(A)$, $R(\lambda; A)$ will denote the resolvent operator $R(\lambda; A) = (\lambda I - A)^{-1}$ in X.

2. The Well-Posed Evolution Problem

We first clarify what we mean by solutions of evolution problems as well as the notions of ill-posed and well-posed evolution problems.

Definition 2.1 ([11])**.** Let X be a Banach space and for every $t \in [0,T]$, let $A(t)$ be a linear operator in X. The initial value problem

$$\begin{gathered} \frac{du(t)}{dt} = A(t)u(t) \quad 0 \le s \le t < T \\ u(s) = x \end{gathered} \tag{2.1}$$

is called an *evolution problem.* An X-valued function $u : [s,T] \to X$ is a *classical solution* of (2.1) if u is continuous on $[s,T]$, $u(t) \in \mathrm{Dom}(A(t))$ for $s < t < T$, u is continuously differentiable on (s,T), and u satisfies (2.1).

Theorem 2.2 ([11, Theorem 5.1.1])**.** *Let X be a Banach space and for every $t \in [0,T]$, let $A(t)$ be a bounded linear operator on X. If the function $t \mapsto A(t)$ is continuous in the uniform operator topology then for every $x \in X$, the initial value problem* (2.1) *has a unique classical solution u.*

The proof of Theorem 2.2 (cf. [11, Theorem 5.1.1]) shows that the mapping $S : C([s,T] : X) \to C([s,T] : X)$ defined by

$$(Su)(t) = x + \int_s^t A(\tau)u(\tau)d\tau$$

is a well-defined mapping with a unique fixed point u. It is easily shown that u is then a unique classical solution of (2.1). In this case we define the *solution operator* of (2.1) by

$$U(t,s)x = u(t) \quad \text{for } 0 \le s \le t \le T.$$

Theorem 2.3 ([11, Theorem 5.1.2])**.** *Let $U(t,s)$ be the solution operator associated with* (2.1) *where $A(t)$ is a bounded linear operator on X for each $t \in [0,T]$ and $t \mapsto A(t)$ is continuous in the uniform operator topology. Then for every $0 \le s \le t \le T$, $U(t,s)$ is a bounded linear operator such that*

(i) $\|U(t,s)\| \le e^{\int_s^t \|A(\tau)\|d\tau}$.
(ii) $U(t,t) = I$, $U(t,s) = U(t,r)U(r,s)$ *for* $0 \le s \le r \le t \le T$.
(iii) $(t,s) \mapsto U(t,s)$ *is continuous in the uniform operator topology for* $0 \le s \le t \le T$.
(iv) $\frac{\partial U(t,s)}{\partial t} = A(t)U(t,s)$ *for* $0 \le s \le t \le T$.
(v) $\frac{\partial U(t,s)}{\partial s} = -U(t,s)A(s)$ *for* $0 \le s \le t \le T$.

We now turn to the notions of well-posedness and ill-posedness for evolution problems.

Definition 2.4 ([6, Definition 7.1])**.** The evolution problem (2.1) is called *well-posed* in $0 \le t < T$ if the following two assumptions are satisfied:

(i) (Existence of solutions for sufficiently many initial data) There exists a dense subspace Y of X such that for every $s \in [0,T)$ and every $x \in Y$, there exists a classical solution $u(t)$ of (2.1).
(ii) (Continuous dependence of solutions on their initial data) There exists a strongly continuous $B(X)$-valued function $U(t,s)$ defined in $0 \le s \le t \le T$ such that if $u(t)$ is a classical solution of (2.1), then
$$u(t) = U(t,s)x.$$

Equation (2.1) is called *ill-posed* if it is not well-posed.

It is clear that under the hypotheses of Theorem 2.2, that Equation (2.1) is well-posed with unique classical solution given by $u(t) = U(t,s)\chi$ where $U(t,s), 0 \leq s \leq t \leq T$, is the solution operator given by Theorem 2.2. Otherwise, we use the construction of an evolution system and the theory of stable families of operators to obtain well-posedness of (2.1). We explore stability conditions for (2.1) first developed by Kato [8, 11], and later by Tanaka [14].

Definition 2.5 ([11, Definition 5.1.3])**.** A two parameter family of bounded linear operators $U(t,s)$, $0 \leq s \leq t \leq T$, on a Banach space X is called an *evolution system* if the following two conditions are satisfied:

(i) $U(s,s) = I$, $U(t,r)U(r,s) = U(t,s)$ for $0 \leq s \leq r \leq t \leq T$.
(ii) $(t,s) \mapsto U(t,s)$ is strongly continuous for $0 \leq s \leq t \leq T$.

Definition 2.6 ([11, Def. 5.2.1])**.** Let X be a Banach space. A family $\{A(t)\}_{t\in[0,T]}$ of infinitesimal generators of C_0 semigroups on X is called *stable* if there are constants $M \geq 1$ and ω (called the stability constants) such that

$$\rho(A(t)) \supseteq (\omega, \infty) \quad \text{for } t \in [0,T]$$

and

$$\| \prod_{j=1}^{k} R(\lambda; A(t_j))\| \leq M(\lambda - \omega)^{-k} \quad \text{for } \lambda > \omega$$

and every finite sequence $0 \leq t_1 \leq t_2, \dots, t_k \leq T$, $k = 1, 2, \dots$.

Remark. If for $t \in [0,T]$, $A(t)$ is the infinitesimal generator of a C_0 semigroup $\{S_t(s)\}_{s\geq 0}$ satisfying $\|S_t(s)\| \leq e^{\omega s}$, then by the Hille-Yosida theorem (cf. [11]), the family $\{A(t)\}_{t\in[0,T]}$ is stable with constants $M = 1$ and ω.

We now use the theory of stable families of operators to gain well-posedness of (2.1) in the following way. Let X and Y be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|_Y$ respectively. Assume that Y is densely and continuously imbedded in X, that is Y is a dense subspace of X and there is a constant C such that

$$\|y\| \leq C\|y\|_Y \quad \text{for } y \in Y.$$

For each $t \in [0,T]$, let $A(t)$ be the infinitesimal generator of a C_0 semigroup $\{S_t(s)\}_{s\geq 0}$ on X. Assume the following conditions (cf. [8, 11]):

(H1) $\{A(t)\}_{t\in[0,T]}$ is a stable family with stability constants M, ω.
(H2) For each $t \in [0,T]$, Y is an invariant subspace of $S_t(s), s \geq 0$, the restriction $\tilde{S}_t(s)$ of $S_t(s)$ to Y is a C_0 semigroup in Y, and the family $\{\tilde{A}(t)\}_{t\in[0,T]}$ of parts $\tilde{A}(t)$ of $A(t)$ in Y, is a stable family in Y.
(H3) For $t \in [0,T]$, $\mathrm{Dom}(A(t)) \supseteq Y$, $A(t)$ is a bounded operator from Y into X, and $t \mapsto A(t)$ is continuous in the $B(Y,X)$ norm $\|\cdot\|_{Y\to X}$.

Theorem 2.7 ([8, Theorem 4.1], [11, Theorem 5.3.1])**.** *For each $t \in [0,T]$, let $A(t)$ be the infinitesimal generator of a C_0 semigroup $\{S_t(s)\}_{s\geq 0}$ on X. If the family $\{A(t)\}_{t\in[0,T]}$ satisfies conditions* (H1)–(H3)*, then there exists a unique evolution system $U(t,s)$, $0 \leq s \leq t \leq T$, in X satisfying*

(E1) $\|U(t,s)\| \leq Me^{\omega(t-s)}$ *for* $0 \leq s \leq t \leq T$.
(E2) $\frac{\partial^+}{\partial t}U(t,s)y\big|_{t=s} = A(s)y$ *for* $y \in Y$, $0 \leq s \leq T$.
(E3) $\frac{\partial}{\partial s}U(t,s)y = -U(t,s)A(s)y$ *for* $y \in Y$, $0 \leq s \leq t \leq T$,

where the derivative from the right in (E2) and the derivative in (E3) are in the strong sense in X.

This theorem will help in obtaining a certain kind of classical solution of (2.1) in the case where the family $\{A(t)\}_{t\in[0,T]}$ of infinitesimal generators of C_0 semigroups on X satisfies conditions (H1)–(H3).

Definition 2.8 ([11, Definition 5.4.1])**.** Let X and Y be Banach spaces such that Y is densely and continuously imbedded in X and let $\{A(t)\}_{t\in[0,T]}$ be a family of infinitesimal generators of C_0 semigroups on X satisfying the assumptions (H1)–(H3). A function $u \in C([s,T]:Y)$ is a *Y-valued solution* of (2.1) if $u \in C^1((s,T):X)$ and u satisfies (2.1) in X.

Remark. A Y-valued solution u of (2.1) is a classical solution of (2.1) such that $u(t) \in Y \subseteq \mathrm{Dom}(A(t))$ for $t \in [s,T]$ and $u(t)$ is continuous in the stronger Y-norm rather than merely in the X-norm.

Theorem 2.9 ([11, Thm. 5.4.3])**.** *Let $\{A(t)\}_{t\in[0,T]}$ satisfy the conditions of Theorem 2.7 and let $U(t,s)$, $0 \le s \le t \le T$ be the evolution system given in Theorem 2.7. If*

- (E4) *$U(t,s)Y \subseteq Y$ for $0 \le s \le t \le T$ and*
- (E5) *For $x \in Y$, $U(t,s)x$ is continuous in Y for $0 \le s \le t \le T$,*

then for every $x \in Y$, $U(t,s)x$ is the unique Y-valued solution of (2.1).

We now use the above theory of stable families of generators to give criteria for well-posedness of the evolution problem (1.2). Let (2.2) denote the initial value problem (1.2) with 0 replaced by s for $s \in [0,T)$; i.e.,

$$\begin{gathered}\frac{dv(t)}{dt} = f(t,D)v(t) \quad 0 \le s \le t < T \\ v(s) = \chi.\end{gathered} \tag{2.2}$$

We determine conditions on f so that the family of operators $\{f(t,D)\}_{t\in[0,T]}$ is stable and such that (2.2) is well-posed.

Proposition 2.10. *Let $f : [0,T] \times [0,\infty) \to \mathbb{R}$ be continuous in t and Borel in λ. Assume there exist $\omega \in \mathbb{R}$ such that $f(t,\lambda) \le \omega$ for all $(t,\lambda) \in [0,T] \times [0,\infty)$ and a Borel function $r : [0,\infty) \to [0,\infty)$ such that $|f(t,\lambda)| \le r(\lambda)$ and $\mathrm{Dom}(f(t,D)) = \mathrm{Dom}(r(D))$ for all $t \in [0,T]$. Set $Y = \mathrm{Dom}(r(D))$ and let $\|\cdot\|_Y$ denote the graph norm associated with the operator $r(D)$. Further, assume $t \mapsto f(t,D)$ is continuous in the $B(Y,H)$ norm $\|\cdot\|_{Y\to H}$. Then (2.2) is well-posed and for $\chi \in Y$, $V(t,s)\chi = e^{\int_s^t f(\tau,D)d\tau}\chi$ is a unique Y-valued solution of* (2.2).

Proof. By [5, Theorem XII.2.6], $r(D)$ is a closed operator in H with dense domain. Set $Y = \mathrm{Dom}(r(D))$ and endow Y with the graph norm $\|\cdot\|_Y$ given by

$$\|y\|_Y = \|y\| + \|r(D)y\|$$

for all $y \in Y$. Since $r(D)$ is a closed operator, it follows that $(Y, \|\cdot\|_Y)$ is a Banach space. It is also clear that Y is densely and continuously imbedded in H. Since $f(t,\lambda) \le \omega$ for all $(t,\lambda) \in [0,T] \times [0,\infty)$, we have that for each $t \in [0,T]$, $f(t,D)$ is the infinitesimal generator of the C_0 semigroup $\{S_t(s)\}_{s\ge 0}$ on H given by $S_t(s) = e^{sf(t,D)}$. We show that the family $\{f(t,D)\}_{t\in[0,T]}$ satisfies conditions (H1)–(H3).

Let $t \in [0, T]$, $x \in H$. Then

$$\|e^{sf(t,D)}x\|^2 = \int_0^\infty |e^{sf(t,\lambda)}|^2 d(E(\lambda)x, x) \le (e^{s\omega})^2 \int_0^\infty d(E(\lambda)x, x) = (e^{s\omega})^2 \|x\|^2,$$

showing that $\|S_t(s)\| = \|e^{sf(t,D)}\| \le e^{\omega s}$. Thus, $\{f(t, D)\}_{t\in[0,T]}$ is a stable family with stability constants $M = 1$ and ω, and so (H1) is satisfied.

Next, let $t \in [0, T]$, $y \in Y$. For any $s \ge 0$, since $y \in Y = \mathrm{Dom}(r(D))$, we have

$$\int_0^\infty |r(\lambda)e^{sf(t,\lambda)}|^2 d(E(\lambda)y, y) \le (e^{s\omega})^2 \int_0^\infty |r(\lambda)|^2 d(E(\lambda)y, y) < \infty.$$

Thus, $S_t(s)y \in \mathrm{Dom}(r(D))$ and so Y is an invariant subspace of $S_t(s)$. Let $\tilde{S}_t(s)$ be the restriction of $S_t(s)$ to Y. For any positive constant c, for $0 \le s \le c$,

$$|r(\lambda)(e^{sf(t,\lambda)} - 1)|^2 \le |r(\lambda)|^2 (e^{c\omega} + 1)^2 \in L^1(E(\cdot)y, y).$$

Therefore, by Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned}
\lim_{s\to 0^+} \|r(D)(S_t(s) - I)y\|^2 &= \lim_{s\to 0^+} \int_0^\infty |r(\lambda)(e^{sf(t,\lambda)} - 1)|^2 d(E(\lambda)y, y) \\
&= \int_0^\infty \lim_{s\to 0^+} |r(\lambda)(e^{sf(t,\lambda)} - 1)|^2 d(E(\lambda)y, y) = 0,
\end{aligned}$$

and so

$$\begin{aligned}
\|\tilde{S}_t(s)y - y\|_Y &= \|\tilde{S}_t(s)y - y\| + \|r(D)(\tilde{S}_t(s)y - y)\| \\
&= \|S_t(s)y - y\| + \|r(D)(S_t(s) - I)y\| \\
&\to 0 \quad \text{as } s \to 0^+.
\end{aligned}$$

Thus, $\tilde{S}_t(s)$ is a C_0 semigroup on Y.

Next, consider the family $\{\tilde{f}(t, D)\}_{t\in[0,T]}$ of parts $\tilde{f}(t, D)$ of $f(t, D)$ in Y. For each $t \in [0, T]$, $\tilde{f}(t, D)$ is defined by

$$\mathrm{Dom}(\tilde{f}(t, D)) = \{x \in \mathrm{Dom}(f(t, D)) \cap Y : f(t, D)x \in Y\}$$

and

$$\tilde{f}(t, D)x = f(t, D)x \quad \text{for } x \in \mathrm{Dom}(\tilde{f}(t, D)).$$

It is seen [11, Theorem 4.5.5] that $\tilde{f}(t, D)$ is the infinitesimal generator of the C_0 semigroup $\tilde{S}_t(s)$. Moreover, for $y \in Y$,

$$\begin{aligned}
\|\tilde{S}_t(s)y\|_Y &= \|\tilde{S}_t(s)y\| + \|r(D)\tilde{S}_t(s)y\| \\
&= \|S_t(s)y\| + \|r(D)S_t(s)y\| \\
&\le e^{s\omega}\|y\| + e^{s\omega}\|r(D)y\| \\
&= e^{s\omega}\|y\|_Y.
\end{aligned}$$

Thus, $\|\tilde{S}_t(s)\|_Y \le e^{\omega s}$ for all $t \in [0, T]$ and so the family $\{\tilde{f}(t, D)\}_{t\in[0,T]}$ is stable with stability constants $\tilde{M} = 1$ and ω. We have shown that (H2) is satisfied.

Finally, let $t \in [0, T]$. Since $|f(t, \lambda)| \le r(\lambda)$, we have for $y \in Y$,

$$\int_0^\infty |f(t, \lambda)|^2 d(E(\lambda)y, y) \le \int_0^\infty |r(\lambda)|^2 d(E(\lambda)y, y) < \infty.$$

Thus $\mathrm{Dom}(f(t, D)) \supseteq Y$. Also, for $y \in Y$,

$$\|f(t, D)y\| \le \|y\| + \|f(t, D)y\| \le \|y\| + \|r(D)y\| = \|y\|_Y,$$

showing that $f(t,D)$ is a bounded operator from Y into H. By assumption, $t \mapsto f(t,D)$ is continuous in the $B(Y,H)$ norm $\|\cdot\|_{Y\to H}$ and so (H3) is satisfied. By Theorem 2.7, there exists a unique evolution system $V(t,s)$, $0 \le s \le t \le T$, in H satisfying conditions (E1)-(E3) with the operators $f(t,D)$, $t \in [0,T]$, and $M = 1$ in the condition (E1); that is we have

$$\begin{gathered}
\|V(t,s)\| \le e^{\omega(t-s)} \quad \text{for } 0 \le s \le t \le T,\\
\frac{\partial^+}{\partial t} V(t,s)y\big|_{t=s} = f(s,D)y \quad \text{for } y \in Y,\ 0 \le s \le T,\\
\frac{\partial}{\partial s} V(t,s)y = -V(t,s)f(s,D)y \quad \text{for } y \in Y,\ 0 \le s \le t \le T,
\end{gathered}$$

where the derivatives are in the strong sense in H. It can be shown using the Spectral Theorem that $e^{\int_s^t f(\tau,D)d\tau}$ is such an evolution system, and so by uniqueness we must have $V(t,s) = e^{\int_s^t f(\tau,D)d\tau}$. It is also readily seen that $V(t,s) = e^{\int_s^t f(\tau,D)d\tau}$ satisfies (E4) and (E5). Therefore, by Theorem 2.9, for every $\chi \in Y$, $V(t,s)\chi = e^{\int_s^t f(\tau,D)d\tau}\chi$ is the unique Y-valued solution of (2.2).

Finally, suppose v_1 is a classical solution of (2.2). Then $v_1(q) \in \mathrm{Dom}(f(q,D)) = \mathrm{Dom}(r(D))$ for $q \in (s,T)$. As $V(t,s)$, $0 \le s \le t \le T$, satisfies condition (E3) with the operators $f(t,D)$, $t \in [0,T]$, the function $q \mapsto V(t,q)v_1(q)$ is then differentiable and

$$\begin{aligned}
\frac{\partial}{\partial q} V(t,q)v_1(q) &= -V(t,q)f(q,D)v_1(q) + V(t,q)\frac{d}{dq}v_1(q)\\
&= -V(t,q)f(q,D)v_1(q) + V(t,q)f(q,D)v_1(q) = 0.
\end{aligned}$$

Thus $V(t,q)v_1(q)$ is constant for $q \in (s,t)$. Since v_1 is a classical solution, the function $V(t,q)v_1(q)$ is also continuous for $q \in [s,t]$. Thus we have

$$v_1(t) = V(t,t)v_1(t) = V(t,s)v_1(s) = V(t,s)\chi.$$

Thus condition (ii) of Definition 2.4 is satisfied and we see that (2.2) is well-posed with unique classical solution given by $v(t) = V(t,s)\chi$. □

3. The Approximation Theorem

In order that solutions of (1.2) approximate known solutions of (1.1), we will require additional conditions on f. The following definition is inspired by results obtained by Ames and Hughes [3, Definition 1] for continuous dependence on modelling in the autonomous case, that is when $A(t) = A$ is independent of t.

Definition 3.1. Let $f : [0,T] \times [0,\infty) \to \mathbb{R}$ be a function continuous in t and Borel in λ and assume the hypotheses of Proposition 2.10. Then f is said to satisfy the *approximation condition with polynomial* p or simply *Condition* $(\mathcal{A},p)$ if there exist a constant β, with $0 < \beta < 1$, and a nonzero polynomial $p(\lambda)$ independent of β such that for each $t \in [0,T]$, $\mathrm{Dom}(p(D)) \subseteq \mathrm{Dom}(A(t,D)) \cap \mathrm{Dom}(f(t,D))$, and

$$\|(-A(t,D) + f(t,D))\psi\| \le \beta\|p(D)\psi\|,$$

for all $\psi \in \mathrm{Dom}(p(D))$.

Now assume f satisfies Condition $(\mathcal{A},p)$. For each $t \in [0,T]$, set

$$g(t,\lambda) = -A(t,\lambda) + f(t,\lambda),$$

and for each $n \geq |\omega|$, set

$$e_n = \{\lambda \in [0, \infty) : \max_{t \in [0,T]} |g(t, \lambda)| \leq n\}.$$

Then

$$\begin{aligned} \lambda \in e_n &\Rightarrow \max_{t \in [0,T]} |g(t, \lambda)| \leq n \\ &\Rightarrow |g(t, \lambda)| \leq n \quad \forall t \in [0, T] \\ &\Rightarrow A(t, \lambda) \leq n + f(t, \lambda) \quad \forall t \in [0, T]. \end{aligned}$$

Since $A(t, \lambda) \geq 0$ and $f(t, \lambda) \leq \omega$ for all $(t, \lambda) \in [0, T] \times [0, \infty)$, we have that on e_n,

$$\max_{t \in [0,T]} |A(t, \lambda)| \leq n + \omega.$$

Since $f(t, \lambda) = A(t, \lambda) + g(t, \lambda)$, it then follows that on e_n,

$$\max_{t \in [0,T]} |f(t, \lambda)| \leq 2n + \omega.$$

Set $E_n = E(e_n)$ and let $\psi \in H$ be arbitrary. Consider the following three evolution problems:

$$\begin{gathered} \frac{du_n(t)}{dt} = A(t, D)E_n u_n(t) \quad 0 \leq s \leq t < T \\ u_n(s) = \psi, \end{gathered} \tag{3.1}$$

$$\begin{gathered} \frac{dv_n(t)}{dt} = f(t, D)E_n v_n(t) \quad 0 \leq s \leq t < T \\ v_n(s) = \psi, \end{gathered} \tag{3.2}$$

$$\begin{gathered} \frac{dw_n(t)}{dt} = g(t, D)E_n w_n(t) \quad 0 \leq s \leq t < T \\ w_n(s) = \psi. \end{gathered} \tag{3.3}$$

Problems (3.1)–(3.3), as we will see, are well-posed due to the action of E_n and their solutions will aid in approximating known solutions of the ill-posed problem (1.1).

Lemma 3.2. *For each $t \in [0, T]$, $A(t, D)E_n$ is a bounded operator on H such that*

$$\|A(t, D)E_n\| \leq n + \omega,$$

and (3.1) *has a unique classical solution $u_n(t) = U_n(t, s)\psi$. The solution operator $U_n(t, s)$ is a bounded operator on H with*

$$\|U_n(t, s)\| \leq e^{T(n+\omega)}$$

for all s, t such that $0 \leq s \leq t \leq T$. Furthermore, if ψ is replaced by $\psi_n = E_n\psi$ in (3.1)*, then*

$$U_n(t, s)\psi_n = e^{\int_s^t A(\tau, D) d\tau} \psi_n.$$

Proof. Fix $t \in [0, T]$. For all $x \in H$, by [5, Theorem XII.2.6],

$$\begin{aligned} \|A(t, D)E_n x\|^2 &= \int_0^\infty |A(t, \lambda)|^2 d(E(\lambda)E_n x, E_n x) \\ &= \int_{e_n} |A(t, \lambda)|^2 d(E(\lambda)x, x) \\ &\leq (n + \omega)^2 \int_{e_n} d(E(\lambda)x, x) \end{aligned}$$

$$\begin{aligned}
&\le (n+\omega)^2 \int_0^\infty d(E(\lambda)x, x) \\
&= (n+\omega)^2 \|x\|^2,
\end{aligned}$$

showing that $A(t,D)E_n$ is a bounded operator on H with $\|A(t,D)E_n\| \le n+\omega$.

Next, let $t_0 \in [0,T]$. Since e_n is a bounded subset of $[0,\infty)$, we have that $D^jE_n \in B(H)$ for each $1 \le j \le k$. Then by continuity of a_j for each $1 \le j \le k$, we have

$$\begin{aligned}
\|A(t,D)E_n - A(t_0,D)E_n\| &= \|\sum_{j=1}^{k}(a_j(t) - a_j(t_0))D^jE_n\| \\
&\le \sum_{j=1}^{k} |a_j(t) - a_j(t_0)|\, \|D^jE_n\| \to 0 \quad \text{as } t \to t_0,
\end{aligned}$$

showing that $t \mapsto A(t,D)E_n$ is continuous in the uniform operator topology. It follows from Theorem 2.2 that (3.1) has a unique classical solution $u_n(t) = U_n(t,s)\psi$. That

$$\|U_n(t,s)\| \le e^{T(n+\omega)}$$

follows directly from Theorem 2.3 (i) and the fact that $\|A(t,D)E_n\| \le n+\omega$ for all $t \in [0,T]$.

Next, set $\psi_n = E_n\psi$ and let (3.4) denote the evolution problem (3.1) with ψ replaced by ψ_n; i.e.,

$$\begin{gathered}
\frac{du_n(t)}{dt} = A(t,D)E_nu_n(t) \quad 0 \le s \le t < T, \\
u_n(s) = \psi_n.
\end{gathered} \tag{3.4}$$

Using the Spectral Theorem it can be shown that $e^{\int_s^t A(\tau,D)d\tau}\psi_n$ is a classical solution of (3.4). In particular, using properties of the projection operator E_n, we have

$$\begin{aligned}
\frac{d}{dt}e^{\int_s^t A(\tau,D)d\tau}\psi_n &= A(t,D)e^{\int_s^t A(\tau,D)d\tau}\psi_n \\
&= A(t,D)E_ne^{\int_s^t A(\tau,D)d\tau}\psi_n,
\end{aligned}$$

and

$$e^{\int_s^s A(\tau,D)d\tau}\psi_n = \psi_n.$$

Therefore, by uniqueness guaranteed by Theorem 2.2, we have

$$U_n(t,s)\psi_n = e^{\int_s^t A(\tau,D)d\tau}\psi_n.$$

□

Lemma 3.3. *For each $t \in [0,T]$, $f(t,D)E_n$ is a bounded operator on H such that*

$$\|f(t,D)E_n\| \le 2n+\omega,$$

and (3.2) *has a unique classical solution $v_n(t) = V_n(t,s)\psi$. The solution operator $V_n(t,s)$ is a bounded operator on H with*

$$\|V_n(t,s)\| \le e^{T(2n+\omega)}$$

for all s,t such that $0 \le s \le t \le T$. Furthermore, if ψ is replaced by $\psi_n = E_n\psi$ in (3.2)*, then*

$$V_n(t,s)\psi_n = e^{\int_s^t f(\tau,D)d\tau}\psi_n.$$

Proof. Using the fact that on e_n, $\max_{t\in[0,T]} |f(t,\lambda)| \le 2n+\omega$, it is easily shown that for each $t \in [0,T]$, $f(t,D)E_n$ is a bounded operator on H such that $\|f(t,D)E_n\| \le 2n+\omega$. Next, let $t_0 \in [0,T]$. Since $E_nH \subseteq \mathrm{Dom}(f(t,D)) = \mathrm{Dom}(r(D))$ for all $t \in [0,T]$, we have $r(D)E_n \in B(H)$, and so

$$\begin{aligned}
&\|f(t,D)E_n - f(t_0,D)E_n\| \\
&= \sup_{x\in H,\,\|x\|\le 1} \|(f(t,D) - f(t_0,D))E_nx\| \\
&\le \sup_{x\in H,\,\|x\|\le 1} \|f(t,D) - f(t_0,D)\|_{Y\to H}\|E_nx\|_Y \\
&= \sup_{x\in H,\,\|x\|\le 1} \|f(t,D) - f(t_0,D)\|_{Y\to H}(\|E_nx\| + \|r(D)E_nx\|) \\
&\le \|f(t,D) - f(t_0,D)\|_{Y\to H}(\|E_n\| + \|r(D)E_n\|) \to 0 \quad \text{as } t \to t_0
\end{aligned}$$

by the assumption that $t \mapsto f(t,D)$ is continuous in the $B(Y,H)$ norm $\|\cdot\|_{Y\to H}$. Therefore, $t \mapsto f(t,D)E_n$ is continuous in the uniform operator topology. It follows from Theorem 2.2 that (3.2) has a unique classical solution $v_n(t) = V_n(t,s)\psi$. That

$$\|V_n(t,s)\| \le e^{T(2n+\omega)}$$

follows directly from Theorem 2.3 (i) and the fact that $\|f(t,D)E_n\| \le 2n+\omega$ for all $t \in [0,T]$. The rest of the proof is similar to that of Lemma 3.2. ☐

Lemma 3.4. *For each $t \in [0,T]$, $g(t,D)E_n$ is a bounded operator on H such that*

$$\|g(t,D)E_n\| \le n,$$

and (3.3) *has a unique classical solution $w_n(t) = W_n(t,s)\psi$. The solution operator $W_n(t,s)$ is a bounded operator on H with*

$$\|W_n(t,s)\| \le e^{Tn}$$

for all s,t such that $0 \le s \le t \le T$. Furthermore, if ψ is replaced by $\psi_n = E_n\psi$ in (3.3)*, then*

$$W_n(t,s)\psi_n = e^{\int_s^t g(\tau,D)d\tau}\psi_n.$$

Proof. Using the fact that on e_n, $\max_{t\in[0,T]} |g(t,\lambda)| \le n$, it is easily shown that for each $t \in [0,T]$, $g(t,D)E_n$ is a bounded operator on H such that $\|g(t,D)E_n\| \le n$. Also, by the relation $g(t,D)E_n = -A(t,D)E_n + f(t,D)E_n$, it follows that $t \mapsto g(t,D)E_n$ is continuous in the uniform operator topology. Therefore, by Theorem 2.2, (3.3) has a unique classical solution $w_n(t) = W_n(t,s)\psi$. That

$$\|W_n(t,s)\| \le e^{Tn}$$

follows directly from Theorem 2.3 (i) and the fact that $\|g(t,D)E_n\| \le n$ for all $t \in [0,T]$. The rest of the proof is similar to that of Lemma 3.2. ☐

Corollary 3.5. *Let $\psi \in H$ and $\psi_n = E_n\psi$. Then*

$$U_n(t,s)W_n(t,s)\psi_n = V_n(t,s)\psi_n = W_n(t,s)U_n(t,s)\psi_n$$

for all $0 \le s \le t \le T$.

The corollary above follows immediately from Lemmas 3.2, 3.3, and 3.4, and from properties of the functional calculus for unbounded self-adjoint operators [5, Corollary XII.2.7].

We now have all the necessary machinery to prove our approximation theorem. Our strategy will be to extend the solutions $u_n(t)$ of (3.1) with $\psi = \chi_n$, and $v_n(t)$

of (3.2) with $\psi = \chi_n$, into the complex strip $S = \{t + i\eta : t \in [0, T],\ \eta \in \mathbb{R}\}$, and eventually employ Hadamard's Three Lines Theorem (cf. [12]). To make use of such extensions we will need the following results. Our approach is motivated by work of Agmon and Nirenberg [1].

Definition 3.6 ([12, Definition 11.1])**.** Let $\phi(\alpha)$ be a complex function defined in a plane open set Ω. Assume all partial derivatives of ϕ exist and are continuous. Define the *Cauchy-Riemann operator* $\bar{\partial}$ as

$$\bar{\partial} = \frac{1}{2}\Big(\frac{\partial}{\partial t} + i\frac{\partial}{\partial \eta}\Big),$$

where $\alpha = t + i\eta$.

Theorem 3.7 ([12, Theorem 11.2])**.** *Suppose $\phi(\alpha)$ is a complex function in Ω such that all partial derivatives of ϕ exist and are continuous. Then ϕ is analytic in Ω if and only if the Cauchy-Riemann equation*

$$\bar{\partial}\phi(\alpha) = 0$$

holds for every $\alpha \in \Omega$.

Lemma 3.8 ([1])**.** *Let $\phi(z)$ be a complex function with $z = x + iy$. Assume $\phi(z)$ is continuous and bounded on $S = \{z = x + iy : x \in [0, T], y \in \mathbb{R}\}$. For $\alpha = t + i\eta \in S$, define*

$$\Phi(\alpha) = -\frac{1}{\pi}\int\int_S \phi(z)\Big(\frac{1}{z - \alpha} + \frac{1}{\bar{z} + 1 + \alpha}\Big)\,dx\,dy.$$

Then $\Phi(\alpha)$ is absolutely convergent, $\bar{\partial}\Phi(\alpha) = \phi(\alpha)$, and there exists a constant K such that

$$\int_{-\infty}^{\infty} \big|\frac{1}{z - \alpha} + \frac{1}{\bar{z} + 1 + \alpha}\big| dy \le K\Big(1 + \log\frac{1}{|x - t|}\Big)$$

if $x \neq t$.

We now state and prove our approximation theorem.

Theorem 3.9. *Let D be a positive self-adjoint operator acting on H and let $A(t, D)$ be defined as above for all $t \in [0, T]$. Let f satisfy Condition $(\mathcal{A}, p)$, and assume that there exists a constant γ, independent of β, ω, and t such that $g(t, \lambda) \le \gamma$, for all $(t, \lambda) \in [0, T] \times [0, \infty)$. Then if $u(t)$ and $v(t)$ are classical solutions of (1.1) and (1.2) respectively, and if there exist constants $M', M'', M''' \ge 0$ such that $\|u(T)\| \le M'$, $\|p(D)\chi\| \le M''$, and $\|p(D)A(t, D)u(T)\| \le M'''$ for all $t \in [0, T]$, then there exist constants C and M independent of β such that for $0 \le t < T$,*

$$\|u(t) - v(t)\| \le C\beta^{1 - \frac{t}{T}} M^{t/T}.$$

Proof. Let $\chi_n = E_n\chi$ and set $S = \{t + i\eta : t \in [0, T],\ \eta \in \mathbb{R}\}$. Letting $\psi = \chi_n$ and $s = 0$ in (3.1) and (3.2), we extend the solutions $u_n(t)$ and $v_n(t)$ into the complex strip S in the following way. Since $A(0, D)$ and $f(0, D)$ are self-adjoint, $e^{i\eta A(0,D)}$ and $e^{i\eta f(0,D)}$ are bounded operators on H for all $\eta \in \mathbb{R}$, and so we define

$$u_n(\alpha) = e^{i\eta A(0,D)} U_n(t, 0)\chi_n,$$
$$v_n(\alpha) = e^{i\eta f(0,D)} V_n(t, 0)\chi_n,$$

for $\alpha = t + i\eta \in S$. Finally define $\phi_n : S \to H$ by

$$\phi_n(\alpha) = u_n(\alpha) - v_n(\alpha).$$

We first determine $\bar{\partial}\phi_n(\alpha)$. Since $e^{i\eta A(0,D)}$ and $e^{i\eta f(0,D)}$ are bounded operators on H, and $A(t,D)$ and $f(t,D)$ are bounded when acting on E_nH, we have

$$\begin{aligned}\frac{\partial}{\partial t}\phi_n(\alpha) &= \frac{\partial}{\partial t}e^{i\eta A(0,D)}U_n(t,0)\chi_n - \frac{\partial}{\partial t}e^{i\eta f(0,D)}V_n(t,0)\chi_n \\ &= e^{i\eta A(0,D)}\frac{d}{dt}U_n(t,0)\chi_n - e^{i\eta f(0,D)}\frac{d}{dt}V_n(t,0)\chi_n \\ &= e^{i\eta A(0,D)}A(t,D)U_n(t,0)\chi_n - e^{i\eta f(0,D)}f(t,D)V_n(t,0)\chi_n \\ &= A(t,D)u_n(\alpha) - f(t,D)v_n(\alpha).\end{aligned}$$

Next, by standard properties of semigroups of linear operators (cf. [11, Theorem 1.2.4 (c)]), since $U_n(t,0)\chi_n \in \mathrm{Dom}(A(0,D))$, and $V_n(t,0)\chi_n \in \mathrm{Dom}(f(0,D))$, we have

$$\begin{aligned}\frac{\partial}{\partial \eta}\phi_n(\alpha) &= \frac{\partial}{\partial \eta}e^{i\eta A(0,D)}U_n(t,0)\chi_n - \frac{\partial}{\partial \eta}e^{i\eta f(0,D)}V_n(t,0)\chi_n \\ &= iA(0,D)e^{i\eta A(0,D)}U_n(t,0)\chi_n - if(0,D)e^{i\eta f(0,D)}V_n(t,0)\chi_n \\ &= i(A(0,D)u_n(\alpha) - f(0,D)v_n(\alpha)).\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{\partial}\phi_n(\alpha) &= \frac{1}{2}\left(\frac{\partial}{\partial t}\phi_n(\alpha) + i\frac{\partial}{\partial \eta}\phi_n(\alpha)\right) \\ &= \frac{1}{2}[(A(t,D)u_n(\alpha) - f(t,D)v_n(\alpha)) - (A(0,D)u_n(\alpha) - f(0,D)v_n(\alpha))] \\ &= \frac{1}{2}[(A(t,D) - A(0,D))u_n(\alpha) - (f(t,D) - f(0,D))v_n(\alpha)].\end{aligned} \tag{3.5}$$

Since, in general, this quantity is not identically zero, ϕ_n is not analytic and so we cannot apply the Three Lines Theorem to ϕ_n. To amend this, we introduce a related function. Define

$$\Phi_n(\alpha) = -\frac{1}{\pi}\int\int_S e^{z^2}\bar{\partial}\phi_n(z)\Big(\frac{1}{z-\alpha} + \frac{1}{\bar{z}+1+\alpha}\Big)\,dx\,dy,$$

where $z = x+iy$ and $\alpha = t+i\eta$ are in S. To apply Lemma 3.8, we show that $e^{z^2}\bar{\partial}\phi_n(z)$ is bounded and continuous on S. Let $z = x+iy \in S$ be arbitrary. We have from (3.5),

$$\begin{aligned}\|e^{z^2}\bar{\partial}\phi_n(z)\| &= \frac{1}{2}|e^{z^2}|\ \|(A(x,D) - A(0,D))u_n(z) - (f(x,D) - f(0,D))v_n(z)\| \\ &\le \frac{1}{2}e^{T^2}(\|A(x,D)u_n(z)\| + \|A(0,D)u_n(z)\| \\ &\quad + \|f(x,D)v_n(z)\| + \|f(0,D)v_n(z)\|).\end{aligned}$$

Since $\|e^{iyA(0,D)}\| = 1$, we have by Lemma 3.2 and properties of E_n,

$$\begin{aligned}\|A(x,D)u_n(z)\| &= \|A(x,D)e^{iyA(0,D)}U_n(x,0)\chi_n\| \\ &= \|A(x,D)E_ne^{iyA(0,D)}U_n(x,0)\chi_n\| \\ &\le (n+\omega)\|e^{iyA(0,D)}U_n(x,0)\chi_n\| \\ &\le (n+\omega)\|U_n(x,0)\chi_n\| \\ &\le (n+\omega)e^{T(n+\omega)}\|\chi_n\|.\end{aligned}$$

Note that since $x \in [0,T]$ is arbitrary, the same bound holds for $\|A(0,D)u_n(z)\|$. Similarly, $\|e^{iyf(0,D)}\| = 1$, and using Lemma 3.3,

$$
\begin{aligned}
\|f(x,D)v_n(z)\| &= \|f(x,D)e^{iyf(0,D)}V_n(x,0)\chi_n\| \\
&= \|f(x,D)E_n e^{iyf(0,D)}V_n(x,0)\chi_n\| \\
&\le (2n+\omega)\|e^{iyf(0,D)}V_n(x,0)\chi_n\| \\
&\le (2n+\omega)\|V_n(x,0)\chi_n\| \\
&\le (2n+\omega)e^{T(2n+\omega)}\|\chi_n\|.
\end{aligned}
$$

Again, since $x \in [0,T]$ is arbitrary, the same bound holds for $\|f(0,D)v_n(z)\|$. Setting

$$C_n = (n+\omega)e^{T(n+\omega)} + (2n+\omega)e^{T(2n+\omega)},$$

we have

$$\|e^{z^2}\bar{\partial}\phi_n(z)\| \le e^{T^2}C_n\|\chi_n\|, \tag{3.6}$$

showing that $e^{z^2}\bar{\partial}\phi_n(z)$ is indeed bounded on S. It can also easily be shown using continuity of $A(t,D)E_n$ and $f(t,D)E_n$ in the $B(H)$ norm, continuity of $U_n(t,s)$ and $V_n(t,s)$ in the $B(H)$ norm (Theorem 2.3 (iii)), and strong continuity of the groups $\{e^{iyA(0,D)}\}_{y\in\mathbb{R}}$ and $\{e^{iyf(0,D)}\}_{y\in\mathbb{R}}$, that $e^{z^2}\bar{\partial}\phi_n(z)$ is continuous on S. Having satisfied the hypotheses of Lemma 3.8, it follows that

$$\Phi_n(\alpha) = -\frac{1}{\pi}\int\int_S e^{z^2}\bar{\partial}\phi_n(z)\Big(\frac{1}{z-\alpha} + \frac{1}{\bar{z}+1+\alpha}\Big)\,dx\,dy$$

is absolutely convergent,

$$\bar{\partial}\Phi_n(\alpha) = e^{\alpha^2}\bar{\partial}\phi_n(\alpha),$$

and there exists a constant K such that

$$\int_{-\infty}^{\infty}\Big|\frac{1}{z-\alpha} + \frac{1}{\bar{z}+1+\alpha}\Big|dy \le K\Big(1 + \log\frac{1}{|x-t|}\Big)$$

for $x \ne t$.

We now construct a candidate for the Three Lines Theorem. Define $\Psi_n : S \to H$ by

$$\Psi_n(\alpha) = e^{\alpha^2}\phi_n(\alpha) - \Phi_n(\alpha).$$

For α in the interior of S, using the product rule and results from Lemma 3.8,

$$
\begin{aligned}
\bar{\partial}\Psi_n(\alpha) &= \bar{\partial}[e^{\alpha^2}\phi_n(\alpha)] - \bar{\partial}\Phi_n(\alpha) \\
&= [(\bar{\partial}e^{\alpha^2})\phi_n(\alpha) + e^{\alpha^2}\bar{\partial}\phi_n(\alpha)] - \bar{\partial}\Phi_n(\alpha) \\
&= [(0)\phi_n(\alpha) + \bar{\partial}\Phi_n(\alpha)] - \bar{\partial}\Phi_n(\alpha) = 0.
\end{aligned}
$$

Therefore, by Theorem 3.7, Ψ_n is analytic on the interior of S. Next, for $\alpha = t + i\eta \in S$, from (3.6) and the results from Lemma 3.8,

$$
\begin{aligned}
\|\Phi_n(\alpha)\| &= \Big\| -\frac{1}{\pi}\int\int_S e^{z^2}\bar{\partial}\phi_n(z)\Big(\frac{1}{z-\alpha} + \frac{1}{\bar{z}+1+\alpha}\Big)\,dx\,dy\Big\| \\
&\le \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^T e^{T^2}C_n\|\chi_n\|\Big|\frac{1}{z-\alpha} + \frac{1}{\bar{z}+1+\alpha}\Big|\,dx\,dy \\
&= \frac{1}{\pi}e^{T^2}C_n\|\chi_n\|\int_0^T\Big(\int_{-\infty}^{\infty}\Big|\frac{1}{z-\alpha} + \frac{1}{\bar{z}+1+\alpha}\Big|dy\Big)dx
\end{aligned}
$$

$$\le \frac{K}{\pi} e^{T^2} C_n \|\chi_n\| \int_0^T \big(1 + \log \frac{1}{|x-t|}\big) dx.$$

Also, using Lemmas 3.2 and 3.3,

$$\begin{aligned}
\|\phi_n(\alpha)\| &= \|e^{i\eta A(0,D)} U_n(t,0)\chi_n - e^{i\eta f(0,D)} V_n(t,0)\chi_n\| \\
&\le (\|U_n(t,0)\| + \|V_n(t,0)\|)\|\chi_n\| \\
&\le (e^{T(n+\omega)} + e^{T(2n+\omega)})\|\chi_n\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\Psi_n(\alpha)\| &= \|e^{\alpha^2}\phi_n(\alpha) - \Phi_n(\alpha)\| \\
&\le |e^{\alpha^2}|\, \|\phi_n(\alpha)\| + \|\Phi_n(\alpha)\| \\
&\le e^{T^2}(e^{T(n+\omega)} + e^{T(2n+\omega)})\|\chi_n\| \\
&\quad + \frac{K}{\pi} e^{T^2} C_n \|\chi_n\| \big\{ \max_{t\in[0,T]} \int_0^T \big(1 + \log \frac{1}{|x-t|}\big) dx \big\},
\end{aligned}$$

proving that Ψ_n is bounded on S. From (3.6) and the results from Lemma 3.8, it follows via a dominated convergence argument that Φ_n is continuous on S. It is also easily shown that ϕ_n is continuous on S, and therefore Ψ_n is continuous on S.

We have shown that Ψ_n is bounded and continuous on S, and analytic on the interior of S. It follows from the Cauchy-Schwarz Inequality that for arbitrary $h \in H$, the mapping

$$\alpha \mapsto (\Psi_n(\alpha), h)$$

from S into $\mathbb{C}$, where $(\cdot,\cdot)$ denotes the inner product in H, has the same properties. Therefore, by the Three Lines Theorem,

$$|(\Psi_n(\alpha), h)| \le M(0)^{1-\frac{t}{T}} M(T)^{t/T},$$

for $t \in [0,T]$, where $\alpha = t + i\eta$ and

$$M(t) = \max_{\eta\in\mathbb{R}} |(\Psi_n(t+i\eta), h)|.$$

We aim to find bounds on $M(0)$ and $M(T)$. First, for $\eta \in \mathbb{R}$,

$$\begin{aligned}
|(\Psi_n(i\eta), h)| &\le \|\Psi_n(i\eta)\|\|h\| \\
&= \|e^{-\eta^2}\Big(e^{i\eta A(0,D)} U_n(0,0)\chi_n - e^{i\eta f(0,D)} V_n(0,0)\chi_n\Big) - \Phi_n(i\eta)\|\|h\| \\
&= \|e^{-\eta^2}\Big(e^{i\eta A(0,D)}\chi_n - e^{i\eta f(0,D)}\chi_n\Big) - \Phi_n(i\eta)\|\|h\| \\
&\le \Big(e^{-\eta^2}\|e^{i\eta A(0,D)}\chi_n - e^{i\eta f(0,D)}\chi_n\| + \|\Phi_n(i\eta)\|\Big)\|h\|.
\end{aligned}$$

Now, since $\|e^{i\eta A(0,D)}\| = 1$,

$$\begin{aligned}
\|e^{i\eta A(0,D)}\chi_n - e^{i\eta f(0,D)}\chi_n\| &= \|e^{i\eta A(0,D)}\chi_n - e^{i\eta A(0,D)} e^{i\eta g(0,D)}\chi_n\| \\
&= \|e^{i\eta A(0,D)}(I - e^{i\eta g(0,D)})\chi_n\| \\
&\le \|(I - e^{i\eta g(0,D)})\chi_n\|.
\end{aligned}$$

For $\psi \in \mathrm{Dom}(g(0,D))$ and $\eta \in \mathbb{R}$, we have by standard properties of semigroups (cf. [11, Theorem 1.2.4]) that

$$\|(I - e^{i\eta g(0,D)})\psi\| = \| - i\int_0^\eta e^{isg(0,D)} g(0,D)\psi ds\| \le |\eta| \|g(0,D)\psi\|.$$

Note $\chi_n \in \mathrm{Dom}(A(0,D)) \cap \mathrm{Dom}(f(0,D)) \subseteq \mathrm{Dom}(g(0,D))$ and since e_n is a bounded subset of $[0,\infty)$, $\chi_n \in \mathrm{Dom}(p(D))$. Thus we have by Condition $(\mathcal{A}, p)$ and the above inequality that

$$\|e^{i\eta A(0,D)}\chi_n - e^{i\eta f(0,D)}\chi_n\| \le \beta|\eta| \|p(D)\chi_n\|.$$

Next we would like a bound on $\|\Phi_n(i\eta)\|$ in terms of β. Let $z = x + iy \in S$. Then from (3.5),

$$\begin{aligned} 2\|\bar\partial\phi_n(z)\| &= \|(A(x,D) - A(0,D))u_n(z) - (f(x,D) - f(0,D))v_n(z)\| \\ &\le \|A(x,D)u_n(z) - f(x,D)v_n(z)\| + \|A(0,D)u_n(z) - f(0,D)v_n(z)\|. \end{aligned}$$

Now,

$$\begin{aligned} &\|A(x,D)u_n(z) - f(x,D)v_n(z)\| \\ &= \|A(x,D)e^{iyA(0,D)}U_n(x,0)\chi_n - f(x,D)e^{iyf(0,D)}V_n(x,0)\chi_n\| \\ &\le \|A(x,D)e^{iyA(0,D)}U_n(x,0)\chi_n - A(x,D)e^{iyf(0,D)}U_n(x,0)\chi_n\| \qquad (3.7) \\ &\quad + \|A(x,D)e^{iyf(0,D)}U_n(x,0)\chi_n - A(x,D)e^{iyf(0,D)}V_n(x,0)\chi_n\| \qquad (3.8) \\ &\quad + \|A(x,D)e^{iyf(0,D)}V_n(x,0)\chi_n - f(x,D)e^{iyf(0,D)}V_n(x,0)\chi_n\|. \qquad (3.9) \end{aligned}$$

Set $\psi_n = A(x,D)U_n(x,0)\chi_n$. We note that $\psi_n \in \mathrm{Dom}(A(t,D)) \cap \mathrm{Dom}(f(t,D)) \subseteq \mathrm{Dom}(g(t,D))$ for all $t \in [0,T]$, and $\psi_n \in \mathrm{Dom}(p(D))$. Then expression (3.7) is equal to

$$\|e^{iyA(0,D)}\psi_n - e^{iyf(0,D)}\psi_n\| \le \beta|y| \|p(D)\psi_n\|,$$

as above.

Next, using Theorem 2.3, Lemma 3.4, and the assumption that $g(t,\lambda) \le \gamma$ for all $(t,\lambda) \in [0,T] \times [0,\infty)$, we have

$$\begin{aligned} \|(I - W_n(x,0))\psi_n\| &= \|(W_n(x,x) - W_n(x,0))\psi_n\| \\ &= \|\int_0^x \frac{\partial}{\partial s} W_n(x,s)\psi_n ds\| \\ &= \|\int_0^x (-W_n(x,s)g(s,D)E_n)\psi_n ds\| \\ &\le \int_0^x \|W_n(x,s)g(s,D)\psi_n\| ds \\ &\le \int_0^x (1 + e^{\gamma T})\|g(s,D)\psi_n\| ds. \end{aligned}$$

It then follows from Corollary 3.5 and Condition $(\mathcal{A}, p)$ that expression (3.8) is equal to

$$\begin{aligned} &\|e^{iyf(0,D)}(U_n(x,0) - V_n(x,0))A(x,D)\chi_n\| \\ &\le \|(U_n(x,0) - V_n(x,0))A(x,D)\chi_n\| \\ &= \|(U_n(x,0) - W_n(x,0)U_n(x,0))A(x,D)\chi_n\| \\ &= \|(I - W_n(x,0))\psi_n\| \end{aligned}$$

$$\begin{aligned}
&\le \int_0^x (1+e^{\gamma T})\|g(s,D)\psi_n\| ds \\
&\le \int_0^x (1+e^{\gamma T})\beta \|p(D)\psi_n\| ds \\
&\le \beta T(1+e^{\gamma T})\|p(D)\psi_n\|.
\end{aligned}$$

Finally, since $U_n(x,0)\chi_n \in \mathrm{Dom}(p(D))$, Corollary 3.5 and Condition $(\mathcal{A},p)$ imply that expression (3.9) is equal to

$$\begin{aligned}
&\|e^{iyf(0,D)}V_n(x,0)(-A(x,D)+f(x,D))\chi_n\| \\
&\le \|V_n(x,0)(-A(x,D)+f(x,D))\chi_n\| \\
&= \|W_n(x,0)U_n(x,0)(-A(x,D)+f(x,D))\chi_n\| \\
&\le (1+e^{\gamma T})\|(-A(x,D)+f(x,D))U_n(x,0)\chi_n\| \\
&\le \beta(1+e^{\gamma T})\|p(D)U_n(x,0)\chi_n\|.
\end{aligned}$$

Therefore, we have shown

$$\begin{aligned}
&\|A(x,D)u_n(z) - f(x,D)v_n(z)\| \\
&\le \beta(1+e^{\gamma T})\left((|y|+T)\|p(D)A(x,D)U_n(x,0)\chi_n\| + \|p(D)U_n(x,0)\chi_n\|\right).
\end{aligned}$$

Since x is arbitrary, it follows similarly that

$$\begin{aligned}
&\|A(0,D)u_n(z) - f(0,D)v_n(z)\| \\
&\le \beta(1+e^{\gamma T})\left((|y|+T)\|p(D)A(0,D)U_n(x,0)\chi_n\| + \|p(D)U_n(x,0)\chi_n\|\right).
\end{aligned}$$

From the assumptions $\|u(T)\| \le M'$ and $\|p(D)A(t,D)u(T)\| \le M'''$ for all $t \in [0,T]$, it follows that $\|p(D)u(T)\| \le N'$ for some constant $N' \ge 0$. It then follows from these estimates that for $z = x+iy \in S$,

$$\|\bar{\partial}\phi_n(z)\| \le \beta(1+e^{\gamma T})\left((|y|+T)M''' + N'\right),$$

so that by Lemma 3.8,

$$\begin{aligned}
&\|\Phi_n(i\eta)\| \\
&= \| -\frac{1}{\pi}\int\int_S e^{z^2}\bar{\partial}\phi_n(z)\Big(\frac{1}{z-i\eta} + \frac{1}{\bar{z}+1+i\eta}\Big)\,dx\,dy\| \\
&\le \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^T |e^{z^2}|\|\bar{\partial}\phi_n(z)\|\Big|\frac{1}{z-i\eta} + \frac{1}{\bar{z}+1+i\eta}\Big|\,dx\,dy \\
&\le \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^T e^{x^2-y^2}\beta(1+e^{\gamma T})\left((|y|+T)M''' + N'\right)\Big|\frac{1}{z-i\eta} + \frac{1}{\bar{z}+1+i\eta}\Big|\,dx\,dy \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^T e^{x^2}\beta(1+e^{\gamma T})\Big((|y|e^{-y^2} + Te^{-y^2})M''' + e^{-y^2}N'\Big) \\
&\quad \times \Big|\frac{1}{z-i\eta} + \frac{1}{\bar{z}+1+i\eta}\Big|\,dx\,dy \\
&\le \beta\Big[\frac{1}{\pi}e^{T^2}(1+e^{\gamma T})\left((1+T)M''' + N'\right)\int_0^T \Big(\int_{-\infty}^{\infty}\Big|\frac{1}{z-i\eta} + \frac{1}{\bar{z}+1+i\eta}\Big|dy\Big)dx\Big] \\
&\le \beta\Big[\frac{K}{\pi}e^{T^2}(1+e^{\gamma T})\left((1+T)M''' + N'\right)\int_0^T \big(1+\log\frac{1}{x}\big)dx\Big].
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&M(0)\\
&= \max_{\eta\in\mathbb{R}} |(\Psi_n(i\eta),h)|\\
&\le \max_{\eta\in\mathbb{R}} \Big(e^{-\eta^2}\|e^{i\eta A(0,D)}\chi_n - e^{i\eta f(0,D)}\chi_n\| + \|\Phi_n(i\eta)\|\Big)\|h\|\\
&\le \max_{\eta\in\mathbb{R}} \Big(\beta|\eta|e^{-\eta^2}\|p(D)\chi_n\| + \|\Phi_n(i\eta)\|\Big)\|h\|\\
&\le \beta\Big(M'' + \Big[\frac{K}{\pi}e^{T^2}(1+e^{\gamma T})\,((1+T)M''' + N')\int_0^T \big(1+\log\frac{1}{x}\big)dx\Big]\Big)\|h\|.
\end{aligned}
\tag{3.10}
$$

Next, for $\eta \in \mathbb{R}$,

$$
\begin{aligned}
&|(\Psi_n(T+i\eta),h)|\\
&\le \|\Psi_n(T+i\eta)\|\|h\|\\
&= \|e^{(T+i\eta)^2}\Big(e^{i\eta A(0,D)}U_n(T,0)\chi_n - e^{i\eta f(0,D)}V_n(T,0)\chi_n\Big) - \Phi_n(T+i\eta)\|\|h\|\\
&\le \Big(e^{T^2-\eta^2}\|e^{i\eta A(0,D)}U_n(T,0)\chi_n - e^{i\eta f(0,D)}V_n(T,0)\chi_n\| + \|\Phi_n(T+i\eta)\|\Big)\|h\|.
\end{aligned}
$$

Using the assumption that $\|u(T)\| \le M'$,

$$
\begin{aligned}
&\|e^{i\eta A(0,D)}U_n(T,0)\chi_n - e^{i\eta f(0,D)}V_n(T,0)\chi_n\|\\
&\le \|U_n(T,0)\chi_n\| + \|V_n(T,0)\chi_n\|\\
&= \|U_n(T,0)\chi_n\| + \|W_n(T,0)U_n(T,0)\chi_n\|\\
&\le (1+(1+e^{\gamma T}))\|U_n(T,0)\chi_n\|\\
&\le (2+e^{\gamma T})M'.
\end{aligned}
$$

Next, the assumptions $\|u(T)\| \le M'$ and $\|p(D)A(t,D)u(T)\| \le M'''$ for all $t \in [0,T]$ imply that $\|A(t,D)u(T)\| \le N''$ for all $t \in [0,T]$, for some constant $N'' \ge 0$. Thus, for $z = x+iy \in S$, we have

$$
\begin{aligned}
&\|(A(x,D)-A(0,D))u_n(z)\|\\
&\le \|e^{iyA(0,D)}A(x,D)U_n(x,0)\chi_n\| + \|e^{iyA(0,D)}A(0,D)U_n(x,0)\chi_n\|\\
&\le \|A(x,D)U_n(x,0)\chi_n\| + \|A(0,D)U_n(x,0)\chi_n\|\\
&\le 2N''.
\end{aligned}
$$

Meanwhile, using Condition $(\mathcal{A},p)$ and the fact that $0<\beta<1$,

$$
\begin{aligned}
&\|(f(x,D)-f(0,D))v_n(z)\|\\
&\le \|e^{iyf(0,D)}f(x,D)V_n(x,0)\chi_n\| + \|e^{iyf(0,D)}f(0,D)V_n(x,0)\chi_n\|\\
&\le \|f(x,D)V_n(x,0)\chi_n\| + \|f(0,D)V_n(x,0)\chi_n\|\\
&\le \|A(x,D)V_n(x,0)\chi_n\| + \|(-A(x,D)+f(x,D))V_n(x,0)\chi_n\|\\
&\quad + \|A(0,D)V_n(x,0)\chi_n\| + \|(-A(0,D)+f(0,D))V_n(x,0)\chi_n\|\\
&\le \|V_n(x,0)A(x,D)\chi_n\| + \|V_n(x,0)A(0,D)\chi_n\| + 2\beta\|p(D)V_n(x,0)\chi_n\|\\
&\le (1+e^{\gamma T})(\|A(x,D)U_n(x,0)\chi_n\| + \|A(0,D)U_n(x,0)\chi_n\| + 2\|p(D)U_n(x,0)\chi_n\|)\\
&\le 2(1+e^{\gamma T})(N''+N').
\end{aligned}
$$

Therefore, for $z = x + iy \in S$, from (3.5),

$$\begin{aligned}\|\bar\partial\phi_n(z)\| &\le \frac{1}{2}\left(\|(A(x,D)-A(0,D))u_n(z)\| + \|(f(x,D)-f(0,D))v_n(z)\|\right)\\ &\le N'' + (1+e^{\gamma T})(N''+N')\end{aligned}$$

so that by Lemma 3.8,

$$\begin{aligned}&\|\Phi_n(T+i\eta)\|\\ &= \Big\| -\frac{1}{\pi}\int\int_S e^{z^2}\bar\partial\phi_n(z)\left(\frac{1}{z-(T+i\eta)} + \frac{1}{\bar z + 1 + (T+i\eta)}\right)dx\,dy\Big\|\\ &\le \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^T |e^{z^2}|\|\bar\partial\phi_n(z)\|\Big|\frac{1}{z-(T+i\eta)} + \frac{1}{\bar z+1+(T+i\eta)}\Big|\,dx\,dy\\ &\le \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^T e^{T^2}(N''+(1+e^{\gamma T})(N''+N'))\\ &\quad\times\Big|\frac{1}{z-(T+i\eta)} + \frac{1}{\bar z+1+(T+i\eta)}\Big|\,dx\,dy\\ &= \frac{1}{\pi}e^{T^2}(N''+(1+e^{\gamma T})(N''+N'))\\ &\quad\times\int_0^T\Big(\int_{-\infty}^{\infty}\Big|\frac{1}{z-(T+i\eta)} + \frac{1}{\bar z+1+(T+i\eta)}\Big|dy\Big)dx\\ &\le \frac{K}{\pi}e^{T^2}(N''+(1+e^{\gamma T})(N''+N'))\int_0^T\Big(1+\log\frac{1}{|x-T|}\Big)dx.\end{aligned}$$

Thus,

$$\begin{aligned}&M(T)\\ &= \max_{\eta\in\mathbb{R}}|(\Psi_n(T+i\eta),h)|\\ &\le \max_{\eta\in\mathbb{R}}\Big(e^{T^2-\eta^2}(2+e^{\gamma T})M' + \frac{K}{\pi}e^{T^2}(N''+(1+e^{\gamma T})(N''+N'))\\ &\quad\times\int_0^T\Big(1+\log\frac{1}{|x-T|}\Big)dx\Big)\|h\|\\ &\le \Big(e^{T^2}(2+e^{\gamma T})M' + \frac{K}{\pi}e^{T^2}(N''+(1+e^{\gamma T})(N''+N'))\\ &\quad\times\int_0^T\Big(1+\log\frac{1}{|x-T|}\Big)dx\Big)\|h\|.\end{aligned}\tag{3.11}$$

It follows from (3.10) and (3.11) that there exist constants C' and M, independent of β, such that for $0 \le t < T$,

$$|(\Psi_n(t),h)| \le (C'\beta\|h\|)^{1-\frac{t}{T}}(M\|h\|)^{t/T} = (C'\beta)^{1-\frac{t}{T}}M^{t/T}\|h\|.$$

Taking the supremum over all $h \in H$ with $\|h\| \le 1$, we have constants C and M, independent of β, such that for $0 \le t < T$,

$$\|\Psi_n(t)\| \le C\beta^{1-\frac{t}{T}}M^{t/T}.$$

Consequently,

$$\begin{aligned}\|u_n(t)-v_n(t)\| &= \|\phi_n(t)\|\\ &= e^{-t^2}\|\Psi_n(t)+\Phi_n(t)\|\end{aligned}$$

$$\begin{aligned}
&\le \|\Psi_n(t)\| + \|\Phi_n(t)\| \\
&\le C\beta^{1-\frac{t}{T}} M^{t/T} + \|\Phi_n(t)\|.
\end{aligned}$$

It follows from an earlier estimate on $\|\Phi_n(i\eta)\|$, that

$$\|\Phi_n(t)\| \le \beta\Big[\frac{K}{\pi} e^{T^2}(1+e^{\gamma T})\,((1+T)M''' + N')\int_0^T \big(1+\log\frac{1}{|x-t|}\big)dx\Big].$$

Setting

$$K' = \frac{K}{\pi} e^{T^2}(1+e^{\gamma T})\,((1+T)M''' + N')\Big\{\max_{t\in[0,T]}\int_0^T \big(1+\log\frac{1}{|x-t|}\big)dx\Big\},$$

we have

$$\begin{aligned}
\|u_n(t) - v_n(t)\| &\le C\beta^{1-\frac{t}{T}} M^{t/T} + \|\Phi_n(t)\| \\
&\le C\beta^{1-\frac{t}{T}} M^{t/T} + \beta K' \\
&= \Big(C + \beta^{t/T} K' M^{-\frac{t}{T}}\Big)\,\beta^{1-\frac{t}{T}} M^{t/T} \\
&\le C\beta^{1-\frac{t}{T}} M^{t/T},
\end{aligned}$$

for a possibly different constant C. Letting $n \to \infty$, we have found constants C and M, independent of β, such that for $0 \le t < T$,

$$\|u(t) - v(t)\| \le C\beta^{1-\frac{t}{T}} M^{t/T},$$

as desired. □

4. Examples

Below, we give examples illustrating our approximation theorem. Each is a general case of the following universal example. Let $H = L^2(\mathbb{R}^n)$ and $D = -\Delta$ where Δ denotes the Laplacian defined by

$$\Delta h = \sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2}.$$

The operator $-\Delta$ is a positive self-adjoint operator on $L^2(\mathbb{R}^n)$ and so we compare the ill-posed evolution problem

$$\begin{gathered}
\frac{\partial}{\partial t} u(t,x) = A(t,-\Delta)u(t,x), \quad (t,x) \in [0,T) \times \mathbb{R}^n \\
u(0,x) = h(x), \quad x \in \mathbb{R}^n
\end{gathered} \tag{4.1}$$

in $L^2(\mathbb{R}^n)$ to the well-posed approximate problem

$$\begin{gathered}
\frac{\partial}{\partial t} v(t,x) = f(t,-\Delta)v(t,x), \quad (t,x) \in [0,T) \times \mathbb{R}^n \\
v(0,x) = h(x), \quad x \in \mathbb{R}^n.
\end{gathered} \tag{4.2}$$

Example 4.1. As initially defined, let

$$A(t,D) = \sum_{j=1}^{k} a_j(t) D^j.$$

Let $0 < \epsilon < 1$ and set $B_j = \max_{t\in[0,T]} |a_j(t)|$ for each $1 \le j \le k$. Consider the problem

$$\begin{gathered}\frac{\partial}{\partial t} v(t,x) = A(t,-\Delta)v(t,x) - \epsilon(-\Delta)^{k+1}v(t,x), \quad (t,x) \in [0,T) \times \mathbb{R}^n \\ v(0,x) = h(x), \quad x \in \mathbb{R}^n.\end{gathered}$$

Motivated by approximations used by Lattes and Lions, Miller, and Ames and Hughes [2, 3, 9, 10], we define $f : [0,T] \times [0,\infty) \to \mathbb{R}$ by

$$f(t,\lambda) = \sum_{j=1}^{k} a_j(t)\lambda^j - \epsilon\lambda^{k+1}.$$

Then for each $(t,\lambda) \in [0,T] \times [0,\infty)$,

$$f(t,\lambda) \le h(\lambda) := \sum_{j=1}^{k} B_j\lambda^j - \epsilon\lambda^{k+1}.$$

The polynomial $h(\lambda)$ has at most $k+1$ real roots. If $h(\lambda)$ has no real roots on $[0,\infty)$, then $h(\lambda) < 0$ for all $\lambda \ge 0$. Otherwise, let R be the maximum of all such roots of $h(\lambda)$ on $[0,\infty)$. Then $h(\lambda)$ is bounded above on $[0,R]$ and is negative on (R,∞). Therefore, in any case, there exists $\omega \in \mathbb{R}$ such that $h(\lambda) \le \omega$ for all $\lambda \ge 0$. Consequently,

$$f(t,\lambda) \le \omega$$

for all $(t,\lambda) \in [0,T] \times [0,\infty)$. Also

$$|f(t,\lambda)| \le r(\lambda) := \sum_{j=1}^{k} B_j\lambda^j + \epsilon\lambda^{k+1}$$

for all $(t,\lambda) \in [0,T] \times [0,\infty)$. We set $Y = \mathrm{Dom}(r(D))$ and let $\|\cdot\|_Y$ denote the graph norm associated with the operator $r(D)$. We note that $Y = \mathrm{Dom}(r(D)) = \mathrm{Dom}(f(t,D))$ for all $t \in [0,T]$, and that Y is the Sobolev space $W^{2(k+1),2}(\mathbb{R}^n)$, consisting of functions $h \in L^2(\mathbb{R}^n)$ whose derivatives, in the sense of distributions, of order $j \le 2(k+1)$ are in $L^2(\mathbb{R}^n)$ (cf. [11, Chapter 7.1]).

Now, let $t_0 \in [0,T]$. It follows from the definition of $r(D)$ that $D^j \in B(Y,H)$ for each $1 \le j \le k$. Then since a_j is continuous for each $1 \le j \le k$,

$$\begin{aligned}\|f(t,D) - f(t_0,D)\|_{Y\to H} &= \|(A(t,D) - \epsilon D^{k+1}) - (A(t_0,D) - \epsilon D^{k+1})\|_{Y\to H} \\ &= \|A(t,D) - A(t_0,D)\|_{Y\to H} \\ &= \|\sum_{j=1}^{k}(a_j(t) - a_j(t_0))D^j\|_{Y\to H} \\ &\le \sum_{j=1}^{k}|a_j(t) - a_j(t_0)|\, \|D^j\|_{Y\to H} \to 0 \quad \text{as } t \to t_0,\end{aligned}$$

showing that $t \mapsto f(t,D)$ is continuous in the $B(Y,H)$ norm $\|\cdot\|_{Y\to H}$.

Next, set

$$p(\lambda) = \lambda^{k+1}.$$

Then $\mathrm{Dom}(p(D)) \subseteq \mathrm{Dom}(D^j)$ for all $1 \le j \le k+1$ so that

$$\mathrm{Dom}(p(D)) \subseteq \mathrm{Dom}(A(t,D)) \cap \mathrm{Dom}(f(t,D)) = \mathrm{Dom}(f(t,D))$$

for each $t \in [0,T]$. Furthermore, for $\psi \in \mathrm{Dom}(p(D))$ and $t \in [0,T]$,

$$\|(-A(t,D) + f(t,D))\psi\| = \|(-\epsilon D^{k+1})\psi\| = \epsilon\|D^{k+1}\psi\|.$$

Thus f satisfies Condition $(\mathcal{A},p)$ with $r(\lambda) = \sum_{j=1}^{k} B_j\lambda^j + \epsilon\lambda^{k+1}$, $\beta = \epsilon$, and $p(\lambda) = \lambda^{k+1}$. Moreover, $g(t,\lambda) = -\epsilon\lambda^{k+1} \le 0$ for all $(t,\lambda) \in [0,T] \times [0,\infty)$, so we may choose $\gamma = 0$. Theorem 3.9 then yields the result

$$\|u(t) - v(t)\| \le C\beta^{1-\frac{t}{T}} M^{t/T}$$

for $0 \le t < T$, where $u(t)$ and $v(t)$ are solutions of (4.1) and (4.2) respectively.

Example 4.2. As in Example 4.1, let

$$A(t,D) = \sum_{j=1}^{k} a_j(t)D^j.$$

Let $0 < \epsilon < 1$ and set $B_j = \max_{t\in[0,T]} |a_j(t)|$ for each $1 \le j \le k$. Consider the problem

$$\begin{gathered}\frac{\partial}{\partial t}v(t,x) - A(t,-\Delta)v(t,x) + \epsilon(-\Delta)^k\frac{\partial}{\partial t}v(t,x) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^n\\ v(0,x) = h(x), \quad x \in \mathbb{R}^n.\end{gathered}$$

Motivated by work of Showalter [13], we define $f : [0,T] \times [0,\infty) \to \mathbb{R}$ by

$$f(t,\lambda) = \frac{\sum_{j=1}^{k} a_j(t)\lambda^j}{1+\epsilon\lambda^k}.$$

Then for each $(t,\lambda) \in [0,T] \times [0,\infty)$,

$$f(t,\lambda) \le \frac{\sum_{j=1}^{k} B_j\lambda^j}{1+\epsilon\lambda^k}.$$

The rational function $r(\lambda) = \frac{\sum_{j=1}^{k} B_j\lambda^j}{1+\epsilon\lambda^k}$ is continuous on $[0,\infty)$ and tends to $\frac{B_k}{\epsilon}$ as $\lambda \to \infty$. Therefore, there exists $\omega \in \mathbb{R}$ such that $r(\lambda) \le \omega$ for all $\lambda \ge 0$. Consequently,

$$f(t,\lambda) \le \omega$$

for all $(t,\lambda) \in [0,T] \times [0,\infty)$. As $r(\lambda)$ is a bounded Borel function on $[0,\infty)$, the Spectral Theorem yields that $r(D)$ is a bounded everywhere-defined operator on H. Thus, we may choose $Y = \mathrm{Dom}(r(D)) = H$.

Now, let $t_0 \in [0,T]$. It follows from the definition of $r(D)$ that $D^j(I+\epsilon D^k)^{-1} \in B(H)$ for each $1 \le j \le k$. Then since a_j is continuous for each $1 \le j \le k$,

$$\begin{aligned}\|f(t,D) - f(t_0,D)\| &= \|A(t,D)(I+\epsilon D^k)^{-1} - A(t_0,D)(I+\epsilon D^k)^{-1}\|\\ &= \|\sum_{j=1}^{k}(a_j(t) - a_j(t_0))D^j(I+\epsilon D^k)^{-1}\|\\ &\le \sum_{j=1}^{k} |a_j(t) - a_j(t_0)|\ \|D^j(I+\epsilon D^k)^{-1}\| \to 0 \quad \text{as } t \to t_0,\end{aligned}$$

showing that $t \mapsto f(t,D)$ is continuous in the $B(H)$ norm.

Next, set

$$p(\lambda)=\sum_{j=1}^{k} B_j\lambda^{k+j}.$$

Then for each $t\in[0,T]$, $\mathrm{Dom}(p(D))\subseteq \mathrm{Dom}(A(t,D))$. Next, note D^k is positive since D is, and so $\frac{1}{\epsilon}\in\rho(-D^k)$. Therefore, in view of the fact that

$$(I+\epsilon D^k)^{-1}=\frac{1}{\epsilon}R\big(\frac{1}{\epsilon};-D^k\big),$$

we see that for each $t\in[0,T]$,

$$\mathrm{Dom}(f(t,D))=\mathrm{Dom}(A(t,D)(I+\epsilon D^k)^{-1})=H\supseteq \mathrm{Dom}(A(t,D)).$$

Therefore,

$$\mathrm{Dom}(p(D))\subseteq \mathrm{Dom}(A(t,D))\cap \mathrm{Dom}(f(t,D))$$

for all $t\in[0,T]$. Next, fix $t\in[0,T]$ and assume $\psi\in\mathrm{Dom}(p(D))$. Set

$$y=A(t,D)(I+\epsilon D^k)^{-1}\psi.$$

Since $-D^k$ generates a C_0 semigroup of contractions, we have by the Hille-Yosida Theorem [11, Theorem 1.3.1] that

$$\|(I+\epsilon D^k)^{-1}\|=\|\frac{1}{\epsilon}R\Big(\frac{1}{\epsilon};-D^k\Big)\|\leq \frac{1}{\epsilon}\Big(\frac{1}{1/\epsilon}\Big)=1.$$

Thus,

$$\begin{aligned}
\|(-A(t,D)+f(t,D))\psi\| &= \|(-A(t,D)+A(t,D)(I+\epsilon D^k)^{-1})\psi\|\\
&= \|-\epsilon D^k y\|\\
&= \epsilon\|\Big(\sum_{j=1}^{k} a_j(t)D^{k+j}\Big)(I+\epsilon D^k)^{-1}\psi\|\\
&\leq \epsilon\|\Big(\sum_{j=1}^{k} a_j(t)D^{k+j}\Big)\psi\|\\
&\leq \epsilon\|\Big(\sum_{j=1}^{k} B_jD^{k+j}\Big)\psi\|.
\end{aligned}$$

Then f satisfies Condition $(\mathcal{A},p)$ with $r(\lambda)=\frac{\sum_{j=1}^{k}B_j\lambda^j}{1+\epsilon\lambda^k}$, $\beta=\epsilon$, and $p(\lambda)=\sum_{j=1}^{k}B_j\lambda^{k+j}$. Moreover, $g(t,\lambda)=-A(t,\lambda)+A(t,\lambda)(1+\epsilon\lambda^k)^{-1}\leq 0$ for all $(t,\lambda)\in[0,T]\times[0,\infty)$, so we may choose $\gamma=0$. Again, Theorem 3.9 yields the result

$$\|u(t)-v(t)\|\leq C\beta^{1-\frac{t}{T}}M^{t/T}$$

for $0\leq t<T$, where $u(t)$ and $v(t)$ are solutions of (4.1) and (4.2) respectively.

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