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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR THE NONCOERCIVE NEUMANN P-LAPLACIAN

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Abstract. We consider a nonlinear Neumann problem driven by the p-Laplacian differential operator with a nonsmooth potential (hemivariational inequality). Using variational techniques based on the smooth critical point theory and the second deformation theorem, we prove an existence theorem and a multiplicity theorem, under hypothesis that in general do not imply the coercivity of the Euler functional.

1. Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with C^2 boundary, ∂Z. This article concerns the existence and multiplicity of nontrivial solutions for the following nonlinear Neumann problem with a nonsmooth potential (hemivariational inequality):

$$\begin{gathered} -\Delta_p x(z) \in \partial j(z, x(z)) \quad \text{a.e. in } Z, \\ \frac{\partial x}{\partial n} = 0 \quad \text{on } \partial Z. \end{gathered} \tag{1.1}$$

Here $\Delta_p x = \operatorname{div}(\|Dx\|_{\mathbb{R}^N}^{p-2} Dx)$ $(1 < p < \infty)$ is the p-Laplacian differential operator, j is a measurable potential function, and for almost all $z \in Z$ the function $x \mapsto j(z, x)$ is locally Lipschitz and in general nonsmooth. By $\partial j(z, x)$ we denote the generalized (Clarke) subdifferential of the locally Lipschitz function $x \mapsto j(z, x)$. Our aim, in this work, is to prove existence and multiplicity results for (1.1), under hypotheses which do not guarantee the coercivity of the Euler functional.

The question of existence of multiple nontrivial solutions has been studied extensively in the context of Dirichlet problems driven by the p-Laplacian and there are several such papers in the literature, using a variety of hypotheses and techniques. In contrast, the Neumann p-Laplacian case, in some sense, is lagging behind. Recently there have been some multiplicity results within the Neumann setting. We mention the works [1, 4, 13, 17]. In these works, the authors establish the existence

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of infinitely many solutions for certain nonlinear elliptic Neumann problems, by imposing a kind of oscillatory behavior on the nonlinear term. In all these works, it is crucial the assumption that $p > N$ (low dimensional problems). It is well-known that this dimensionality condition implies that the Sobolev space $W^{1,p}(Z)$ is embedded compactly in $C(\bar{Z})$ and this fact is used extensively by the authors, in the aforementioned works. The works [3] and [9], consider nonlinear Neumann eigenvalue problems and prove a "three solutions theorem", using an abstract multiplicity result of Ricceri [16]. Again the condition $p > N$ is present in these works, as it is in the recent work of Wu-Tan [18], but their approach is based on minimax techniques from critical point theory. In all the aforementioned works, the potential function is smooth; i.e., $j(z, \cdot) \in C^1(\mathbb{R})$. Neumann problems involving the p-Laplacian and a nonsmooth potential were investigated in [2, 10, 14]. In these works, the assumptions on the potential j imply that the Euler functional, or a suitable truncation of it, is coercive. In [2], it is assumed that $p \geq 2$ and the approach is degree theoretic. In [10] and [14], the approach is variational based on the nonsmooth critical point theorem (e.g., see [5, 11, 15]). Here, our hypotheses on the nonsmooth potential j do not necessarily imply the coercivity of the Euler functional and our method of proof is based on the nonsmooth second deformation theorem, due to Corvellec [8].

This paper is organized as follows. In Section 2, we recall various notions and results which will be used later. In Section 3, we prove an existence theorem for a generalized version of (1.1). Finally, in Section 4, by strengthening our hypotheses on j, we establish a multiplicity result for (1.1).

2. Mathematical Background

The nonsmooth critical point theory, which we will use in the variational arguments of this paper, is based mainly on the subdifferential theory for locally Lipschitz functions. So, we start by recalling some basic notions from this theory. Details can be found in [7].

Let X be a Banach space. By X^* we denote the topological dual of X and by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). If $\varphi : X \to \mathbb{R}$ is a locally Lipschitz function, then the *generalized directional derivative* $\varphi^0(x; h)$ of φ at $x \in X$, in the direction of $h \in X$, is defined by

$$\varphi^0(x; h) = \limsup_{x' \to x,\, \lambda \downarrow 0} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to see that $h \mapsto \varphi^0(x; h)$ is sublinear continuous and so, it is the support function of a nonempty, convex and w^*-compact set $\partial\varphi(x) \subset X^*$ defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq \varphi^0(x; h) \text{ for all } h \in X\}.$$

If $\varphi \in C^1(X)$, then φ is locally Lipschitz and $\partial\varphi(x) = \{\varphi'(x)\}$. Similarly, if $\varphi : X \to \mathbb{R}$ is continuous convex, then φ is locally Lipschitz and the generalized subdifferential of φ, coincides with the subdifferential in the sense of convex analysis, given by

$$\partial_c\varphi(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq \varphi(y) - \varphi(x) \text{ for all } y \in X\}.$$

We say that $x \in X$ is a *critical point* of the locally Lipschitz function $\varphi : X \to \mathbb{R}$, if $0 \in \partial\varphi(x)$. In this case, $c = \varphi(x)$ is a *critical value* of φ. It is easy to check that, if $x \in X$ is a local extremum of φ (i.e., $x \in X$ is a local minimum or a local maximum), then $x \in X$ is a critical point of φ.

Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, we say that φ satisfies the *Palais-Smale condition* at the level $c \in \mathbb{R}$ (the "PS_c-condition" for short), if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\varphi(x_n) \to c$ and $m(x_n) \to 0$ as $n \to +\infty$, with $m(x_n) = \inf\{\|x^*\| : x^* \in \partial\varphi(x_n)\}$, has a strongly convergent subsequence. We say that φ satisfies the "PS-condition", if it satisfies the PS_c-condition at every level $c \in \mathbb{R}$.

For a locally Lipschitz function $\varphi : X \to \mathbb{R}$ and $c \in \mathbb{R}$, we define the sets:

$$\dot{\varphi}^c = \{x \in X : \varphi(x) < c\}, \quad \varphi^c = \{x \in X : \varphi(x) \leq c\},$$
$$K_c = \{x \in X : 0 \in \partial\varphi(x),\, \varphi(x) = c\}.$$

The next theorem, due to Corvellec [8], is a partial extension to a nonsmooth setting of the so-called "second deformation theorem" (see [6, p.23] and [12, p.628]) In fact, the result of Corvellec is formulated in a more general framework, namely, for continuous (or even lower semicontinuous) functions on a metric space, using the notion of weak slope (see [8], [12, Section 1.3.5], and [15, Section 2.3]). For our purposes, the following particular version of the result suffices.

Theorem 2.1. *If X is a Banach space, $\varphi : X \to \mathbb{R}$ is locally Lipschitz, $-\infty < a < b < +\infty$, φ satisfies the PS_c-condition for every $c \in [a, b)$, φ has no critical values in $[a, b)$, and K_a is a finite set consisting of local minima, then there exists a continuous deformation $h : [0, 1] \times \dot{\varphi}^b \to \dot{\varphi}^b$ such that:*

(a) $h(t, \cdot)\big|_{K_a} = id\big|_{K_a}$ *for all* $t \in [0, 1]$;
(b) $h(1, \dot{\varphi}^b) \subseteq \dot{\varphi}^a \cup K_a$;
(c) $\varphi(h(t, x)) \leq \varphi(x)$ *for all* $t \in [0, 1]$ *and all* $x \in \dot{\varphi}^b$.

In particular, this theorem implies that the set $\dot{\varphi}^a \cup K_a$ is a weak deformation retract of $\dot{\varphi}^b$. In the smooth version of the second deformation theorem, the conclusion is that φ^a is a strong deformation retract of $\varphi^b \backslash K_b$.

3. Existence Theorem

We shall prove an existence theorem, for the following more general version of problem (1.1),

$$\begin{gathered} -\Delta_p\, x(z) \in \partial j(z, x(z)) + h(z) \quad \text{a.e. in } Z, \\ \frac{\partial x}{\partial n} = 0 \quad \text{on } \partial Z, \end{gathered} \tag{3.1}$$

where $h \in L^\infty(Z)$ satisfies

$$\int_Z h(z)\, dz = 0. \tag{3.2}$$

To prove an existence theorem for (3.1), we will need the following hypotheses on the nonsmooth potential j:

(H1) $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z, 0) = 0$ a.e. on Z and
  (i) for all $x \in \mathbb{R}$, $z \mapsto j(z, x)$ is measurable;
  (ii) for almost all $z \in Z$, $x \mapsto j(z, x)$ is locally Lipschitz;
  (iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z, x)$, we have
$$|u| \leq a(z) + c|x|^{r-1}$$
  with $a \in L^\infty(Z)_+$, $c > 0$ and $1 \leq r < p^*$;

(iv) there exists $\xi \in L^1(Z)_+$ such that $j(z,x) \le \xi(z)$ for a.a. $z \in Z$ and all $x \in \mathbb{R}$;

(v) there exists $c_0 \in \mathbb{R}\backslash\{0\}$ such that $\int_Z j(z,c_0)\,dz > 0$.

Here, p^* is the usual Sobolev critical exponent

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } N > p, \\ +\infty & \text{if } N \le p. \end{cases}$$

Example 3.1. The following potential function j satisfies hypotheses (H1), where for the sake of simplicity we drop the z-dependence,

$$j(x) = \begin{cases} s(x)c\left(|x|^r - |x|^q\right) & \text{if } |x| \le 1, \\ s(x)\left(\frac{1}{x^2} - \ln|x| - 1\right) & \text{if } |x| > 1, \end{cases}$$

where $s(x) \equiv 1$ or $s(x) = sign(x) + 2$, $c > 0$ and $1 \le r \le p < q$. In the latter case, where s is nonconstant, j has no symmetry properties. Moreover, if $1 < r$ and $c = \frac{3}{q-r} > 0$, then $j \in C^1(\mathbb{R})$.

Example 3.2. The following potential function j satisfies hypotheses (H1), where again for the sake of simplicity we drop the z-dependence,

$$j(x) = \begin{cases} |x|^r & \text{if } |x| \le 1, \\ \frac{1}{x^2}\ln(|x|) + 1 & \text{if } |x| > 1, \end{cases}$$

where $1 \le r \le p$. Note that the corresponding Euler functional is noncoercive.

In what follows, we set

$$\beta = \int_Z \limsup_{|x|\to\infty} j(z,x)\,dz.$$

By hypothesis (H1)(iv), we have $\beta \in \mathbb{R} \cup \{-\infty\}$.

It is worth pointing out, that hypotheses (H1) incorporate, in our framework of analysis, problems which are strongly resonant with respect to the principal eigenvalue $\lambda_0 = 0$ of the Neumann p-Laplacian. For this reason, we do not expect the PS-condition to be satisfied globally (i.e., at all levels). This will be confirmed in the sequel (see Proposition 3.5). But to be able to reach that result, we shall need some preparation.

So, we consider the following auxiliary Neumann problem:

$$\begin{gathered} -\Delta_p x(z) = h(z) \quad \text{a.e. in } Z, \\ \frac{\partial x}{\partial n} = 0 \quad \text{on } \partial Z, \end{gathered} \tag{3.3}$$

with a $h \in L^\infty(Z)$ that satisfies (3.2). We consider the direct sum decomposition

$$W^{1,p}(Z) = \mathbb{R} \oplus V \quad \text{with } V = \big\{\hat{x} \in W^{1,p}(Z) : \int_Z \hat{x}(z)\,dz = 0\big\}. \tag{3.4}$$

Then, we have the following simple result.

Proposition 3.3. *Problem* (3.3) *has a unique solution* $\hat{x}_0 \in C^1(\bar{Z}) \cap V$.

Proof. Let $n : W^{1,p}(Z) \to \mathbb{R}$ be the C^1-functional defined by

$$\eta(z) = \frac{1}{p}\|Dx\|_p^p - \int_Z hx\,dz$$

for all $x \in W^{1,p}(Z)$. Every $x \in W^{1,p}(Z)$ can be written in a unique way as

$$x = \bar{x} + \hat{x}$$

with $\bar{x} \in \mathbb{R}$ and $\hat{x} \in V$ (see (3.4)).

Because of (3.2), we see that $\eta|_{\mathbb{R}} = 0$. Let $\hat{\eta} = \eta|_V$ (i.e., $\hat{\eta}$ is the restriction of η on V). By virtue of the Poincaré-Wirtinger inequality, we see that $\hat{\eta}$ is coercive on V. Moreover, it is clear that $\hat{\eta}$ is sequentially weakly lower semicontinuous on V. So, by the Weierstrass theorem, we can find $\hat{x}_0 \in V$ such that $-\infty < \hat{m}_0 = \hat{\eta}(\hat{x}_0) = \inf_V \hat{\eta}$, which implies

$$\hat{\eta}'(\hat{x}_0) = 0 \quad \text{in } V^*. \tag{3.5}$$

Let $p_V : W^{1,p}(Z) \to V$ be the projection operator onto V. It exists since V is finite codimensional. Using the chain rule, we have

$$\eta'(x) = p_V^* \hat{\eta}'(p_V(x)) = p_V^* \hat{\eta}'(\hat{x}) \quad \text{for all } x \in W^{1,p}(Z). \tag{3.6}$$

In what follows, by $\langle \cdot, \cdot \rangle_V$ we denote the duality brackets for the pair (V^*, V). Then for any $x, y \in W^{1,p}(Z)$, we have

$$\begin{aligned} \langle \eta'(x), y \rangle &= \langle p_V^* \hat{\eta}'(p_V(x)), y \rangle \quad \text{(see (3.6))} \\ &= \langle \hat{\eta}'(p_V(x)), p_V(y) \rangle_V \end{aligned}$$

which implies

$$\langle \eta'(\hat{x}_0), y \rangle = \langle \eta'(\hat{x}_0), p_V(y) \rangle_V = 0. \tag{3.7}$$

Because $y \in W^{1,p}(Z)$ was arbitrary, from (3.7) it follows that $\eta'(\hat{x}_0) = 0$ in $W^{1,p}(Z)$, so

$$A(\hat{x}_0) = h, \tag{3.8}$$

where $A : W^{1,p}(Z) \to W^{1,p}(Z)^*$ is the nonlinear operator defined by

$$\langle A(x), y \rangle = \int_Z \|Dx\|_{\mathbb{R}^N}^{p-2} (Dx, Dy)_{\mathbb{R}^N} \, dz \tag{3.9}$$

for all $x, y \in W^{1,p}(Z)$. Evidently, A is strictly monotone (strongly monotone, if $p \geq 2$) and continuous. From (3.8), using the nonlinear Green's identity and the nonlinear regularity theory (e.g., see [12]), we infer that $\hat{x}_0 \in C^1(\bar{Z})$ and it solves (3.3). Moreover, the strict monotonicity of $A|_V$ implies that $\hat{x}_0 \in V$ is unique in V. □

From [14, Proposition 12], we have the following useful fact about the nonlinear map $A : W^{1,p}(Z) \to W^{1,p}(Z)^*$ defined by (3.9).

Proposition 3.4. *If $A : W^{1,p}(Z) \to W^{1,p}(Z)^*$ is defined by (3.9), then A is maximal monotone and of type $(S)_+$; i.e., if $x_n \xrightarrow{w} x$ in $W^{1,p}(Z)$ and*

$$\limsup_{n \to +\infty} \langle A(x_n), x_n - x \rangle \leq 0,$$

then $x_n \to x$ in $W^{1,p}(Z)$.

The next proposition illustrates the failure of the global PS-condition already mentioned earlier. So, let $\varphi_1 : W^{1,p}(Z) \to \mathbb{R}$ be the Euler functional for (3.1), defined by

$$\varphi_1(x) = \frac{1}{p} \|Dx\|_p^p - \int_Z j(z, x(z)) \, dz - \int_Z h(z) x(z) \, dz$$

for all $x \in W^{1,p}(Z)$. From [7, p.83], we know that φ_1 is Lipschitz continuous on bounded sets, hence it is locally Lipschitz.

Proposition 3.5. *If hypotheses* (H1) *hold and* $c < \eta(\hat{x}_0) - \beta$, *then* φ_1 *satisfies the* PS_c*-condition.*

Proof. Consider a sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ such that

$$\varphi_1(x_n) \to c \text{ as } n \to +\infty \quad \text{with } c < \eta(\hat{x}_0) - \beta, \tag{3.10}$$

$$m_1(x_n) = \inf\{\|x^*\| : x^* \in \partial\varphi_1(x_n)\} \to 0 \quad \text{as } n \to \infty. \tag{3.11}$$

Because $\partial\varphi_1(x_n) \subseteq W^{1,p}(Z)^*$ is w-compact and the norm functional in a Banach space is weakly lower semicontinuous, we can find $x_n^* \in \partial\varphi_1(x_n)$ such that $m_1(x_n) = \|x_n^*\|$. We know that

$$x_n^* = A(x_n) - u_n - h, \tag{3.12}$$

with $u_n \in N(x_n) = \{u \in L^{r'}(Z) : u(z) \in \partial j(z, x_n(z)) \text{ a.e. on } Z\}$ and $\frac{1}{r} + \frac{1}{r'} = 1$ (see [7, p. 83]). Also, we have $x_n = \bar{x}_n + \hat{x}_n$ with $\bar{x}_n \in \mathbb{R}$ and $\hat{x}_n \in V$. From (3.10) and (3.2), we can find $M_1 > 0$ such that

$$\begin{aligned} M_1 \geq \varphi_1(x_n) &= \frac{1}{p}\|D\hat{x}_n\|_p^p - \int_Z j(z, x(z))\,dz - \int_Z h(z)\hat{x}_n(z)\,dz \\ &\geq \frac{1}{p}\|D\hat{x}_n\|_p^p - \|\xi\|_1 - c_1\|D\hat{x}_n\|_p \end{aligned} \tag{3.13}$$

for some $c_1 > 0$ and all $n \geq 1$. Here, we have used the Poincaré-Wirtinger inequality and hypothesis (H1)(iv). From (3.13) and the Poincaré-Wirtinger inequality again, we infer that

$$\{\hat{x}_n\}_{n\geq 1} \subseteq W^{1,p}(Z) \text{ is bounded.} \tag{3.14}$$

Because of (3.14) and by passing to a suitable subsequence if necessary, we may assume that

$$|\hat{x}_n(z)| \leq k(z) \tag{3.15}$$

for a.a. $z \in Z$, all $n \geq 1$, with $k \in L^r(Z)_+$.

Suppose that $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ is not bounded. We may assume that $\|x_n\| \to \infty$ and so, because of (3.14), we must have $|\bar{x}_n| \to \infty$. Then

$$|x_n(z)| \geq |\bar{x}_n| - |\hat{x}(z)| \geq |\bar{x}_n| - k(z)$$

for a.a. $z \in Z$ (see (3.15)), hence

$$|x_n(z)| \to +\infty \quad \text{as} \quad n \to \infty$$

for a.a. $z \in Z$. From (3.13), we see that

$$M_1 \geq \varphi_1(x_n) = \eta(\hat{x}_n) - \int_Z j(z, x_n(z))\,dz \geq \eta(\hat{x}_0) - \int_Z j(z, x_n(z))\,dz,$$

(see the proof of Proposition 3.3). Passing to the limit as $n \to +\infty$ and using (3.10), we obtain

$$\begin{aligned} M_1 \geq c &\geq \eta(\hat{x}_0) - \limsup_{n\to\infty} \int_Z j(z, x_n(z))\,dz \\ &\geq \eta(\hat{x}_0) - \int_Z \limsup_{n\to\infty} j(z, x_n(z))\,dz \quad \text{(by Fatou's lemma, see (H1)(iv))} \\ &= \eta(\hat{x}_0) - \beta, \end{aligned}$$

which contradicts the choice of c (see (3.10)). This proves that $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ is bounded. Hence, we may assume that

$$x_n \xrightarrow{w} w \text{ in } W^{1,p}(Z) \quad \text{and} \quad x_n \to x \text{ in } L^r(Z).$$

From (3.11) and (3.12), we have, with $\epsilon_n \downarrow 0$,

$$\big|\langle A(x_n), x_n - x\rangle - \int_Z u_n(x_n - x)\,dz - \int_Z h(x_n - x)\,dz\big| \le \epsilon_n \|x_n - x\|. \tag{3.16}$$

Clearly

$$\int_Z u_n(x_n - x)\,dz \to 0, \quad \int_Z h(x_n - x)\,dz \to 0.$$

So, if in (3.16) we pass to the limit as $n \to \infty$, then we obtain

$$\lim_{n\to\infty} \langle A(x_n), x_n - x\rangle = 0.$$

thus by virtue of Proposition 3.4, we have that $x_n \to x$ in $W^{1,p}(Z)$. This proves that φ satisfies the PS_c-condition for all $c < \eta(\hat{x}_0) - \beta$. □

Now we are ready for the existence result concerning Problem (3.1).

Theorem 3.6. *If hypotheses* (H1) *hold and* $\beta < \int_Z j(z, \hat{x}_0(z))\,dz$, *then* (3.1) *admits a nontrivial solution* $y_0 \in C^1(\bar{Z})$.

Proof. Recall that

$$\varphi_1(x) = \eta(\hat{x}) - \int_Z j(z, x(z))\,dz$$

for all $x \in W^{1,p}(Z)$ ($\hat{x} = p_V(x)$). From the proof of Proposition 3.3, we know that $\hat{x}_0 \in V$ is a minimizer of the functional η. Therefore,

$$\varphi_1(x) \ge \eta(\hat{x}_0) - \int_Z j(z, x(z))\,dz \ge \eta(\hat{x}_0) - \|\xi\|_1$$

for all $x \in W^{1,p}(Z)$ (see hypothesis (H1)(iv)). Hence φ_1 is bounded below and so $-\infty < \hat{m}_1 = \inf\big\{\varphi_1 \in W^{1,p}(Z)\big\}$. Also

$$-\infty < \hat{m}_1 \le \varphi_1(\hat{x}_0) = \eta(\hat{x}_0) - \int_Z j(z, x_0(z))\,dz, < \eta(x_0) - \beta$$

by hypothesis. Then, by virtue of Proposition 3.5, φ_1 satisfies the $PS_{\hat{m}_1}$-condition. So, from [11, p.144], we infer that there exists $y_0 \in W^{1,p}(Z)$ such that

$$\varphi_1(y_0) = \hat{m}_1 = \inf\{\varphi_1(x) : x \in W^{1,p}(Z)\} \le \varphi_1(c_0) = -\int_Z j(z, c_0)\,dz < 0 = \varphi_1(0)$$

(see hypothesis (H1)(v)). It follows that $y_0 \ne 0$. Also $\varphi'(y_0) = 0$, which implies

$$A(y_0) = u_0 + h \quad \text{with } u_0 \in N(y_0). \tag{3.17}$$

From (3.17) as before, using the nonlinear Green's identity and nonlinear regularity theory, we infer that $y_0 \in C^1(\bar{Z})$ and solves (3.1). □

We remark that in Example 3.2, we have $j(x) > 0$ for $x \ne 0$, so the hypotheses of Theorem 3.6 are satisfied for $h \equiv 0$.

4. Multiplicity Theorem

In this section, we return to (1.1), where $h \equiv 0$, hence $\hat{x}_0 = 0$ and $\eta(x_0) = 0$. To prove a multiplicity theorem for (1.1), we need to strengthen the hypotheses on the nonsmooth potential j as follows:

(H2) $j : Z \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $j(z, 0) \to 0$ a.e. on Z, and satisfies hypotheses (H1)(i)–(v) and

(vi) $\beta = \int_Z \limsup_{|x|\to\infty} j(z, x)\, dz < 0$ and there exists $\eta \in L^\infty(Z)_+$, $\eta \neq 0$ such that
$$\eta(z) \le \liminf_{x\to 0} \frac{j(z, x)}{|x|^p}$$
uniformly for a.a. $z \in Z$;

(vii) $j(z, x) \le \frac{\lambda_1}{p}|x|^p$ for a.a. $z \in Z$ and all $x \in \mathbb{R}$ and with $\lambda_1 > 0$ being the first nonzero eigenvalue of $(-\Delta_p\, , W^{1,p}(Z))$ (i.e., the second eigenvalue).

The Euler functional $\varphi : W^{1,p}(Z) \to \mathbb{R}$ for (1.1) is defined by
$$\varphi(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z j(z, x(z))\, dz$$
for all $x \in W^{1,p}(Z)$. We know that φ is Lipschitz continuous on bounded sets, hence it is locally Lipschitz (see [7], p.83).

Theorem 4.1. *If hypotheses* (H2) *hold, then* (1.1) *has at least two nontrivial solutions $y_0, v_0 \in C^1(\bar{Z})$.*

Proof. As we already mentioned, since $h \equiv 0$, we have $\hat{x}_0 = 0$ and so $j(z, \hat{x}_0(z)) = 0$ a.e. on Z. Then hypothesis (H2) permits the use of Theorem 3.6, which gives a nontrivial solution $y_0 \in C^1(\bar{Z})$ for (1.1).

Hypothesis (H2)(vi) implies that, for $\epsilon > 0$, we can find $\delta = \delta(\epsilon) > 0$ such that
$$j(z, x) \ge (\eta(z) - \epsilon)\, |x|^p \tag{4.1}$$
for a.a. Z and all $|x| \le \delta$. If $\xi \in \mathbb{R}$ with $0 < |\xi| \le \delta$, then
$$\varphi(\xi) = -\int_Z j(z, \xi)\, dz \le \int_Z (\epsilon - \eta(z)) dz\, |\xi|^p, \tag{4.2}$$
(see (4.1)). If we choose $0 < \epsilon < \frac{1}{|Z|_N} \int_Z \eta(z)\, dz$ (by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$), then from (4.2), we infer that $\varphi(\xi) < 0$, so
$$\mu_r = \max_{\partial B_r \cap \mathbb{R}} \varphi < 0 \tag{4.3}$$
for all $0 < r \le \delta$. Let
$$C(p) = \{x \in W^{1,p}(Z) : \int_Z |x(z)|^{p-2} x(z)\, dz = 0\}.$$
Then, for every $x \in C(p)$, we have
$$\begin{aligned} \varphi(x) &= \frac{1}{p}\|Dx\|_p^p - \int_Z j(z, x(z))\, dz \\ &\ge \frac{1}{p}\|Dx\|_p^p - \frac{\lambda_1}{p}\|Dx\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p \ge 0, \end{aligned}$$

since $x \in C(p)$ (see [14]) and by using (H2)(vii). Hence,

$$\inf_{C(p)} \varphi = 0. \tag{4.4}$$

We introduce the set

$$\Gamma_r = \{\gamma \in C(\bar{B}_r \cap \mathbb{R}, W^{1,p}(Z)) : \gamma|_{\partial B_r \cap \mathbb{R}} = id|_{\partial B_r \cap \mathbb{R}}\}, \quad 0 < r \leq \delta,$$

and define the minimax quantity

$$\hat{c}_r = \inf_{\gamma \in \Gamma_r} \max_{x \in \bar{B}_r \cap \mathbb{R}} \varphi(\gamma(x)). \tag{4.5}$$

Consider the map $\sigma : W^{1,p}(Z) \to \mathbb{R}$ defined by

$$\sigma(u) = \int_Z |u|^{p-2} u \, dz.$$

Evidently, σ is continuous and for $r, -r \in \gamma(\bar{B}_r \cap \mathbb{R})$, $\gamma \in \Gamma_r$, we have

$$\sigma(-r) < 0 < \sigma(r).$$

So, by Bolzano's theorem, we can find $u \in \gamma(\bar{B}_r \cap \mathbb{R})$ (recall that the set is connected), such that

$$\sigma(u) = \int_Z |u|^{p-2} u \, dz = 0,$$

hence $u \in C(p)$. Therefore, we have $u \in \gamma(\bar{B}_r \cap \mathbb{R}) \cap C(p)$, which implies

$$\hat{c}_r \geq 0 \tag{4.6}$$

(see (4.4) and (4.5)). Suppose that $\{0, y_0\}$ are the only critical points of φ. Set

$$b = 0 = \varphi(0) \text{ and } a = \hat{m} = \inf \varphi = \varphi(y_0)$$

(see the proof of Theorem 3.6). We know that $a < b$ (also from Theorem 3.6). Moreover, by virtue of Proposition 3.5, φ satisfies the PS_c-condition for every $c \in (a, b)$ (recall that $\eta(\hat{x}_0) = 0$ and by hypothesis (H2)(vi), $\beta < 0$). Also $K_a = \{y_0\}$ and y_0 is a minimizer of φ. Finally, by hypothesis, φ has no critical values in (a, b). Therefore, we can apply Theorem 2.1 and obtain a continuous deformation $h : 0, 1 \times \dot{\varphi}^b \to \dot{\varphi}^b$ such that $h(t, \cdot)|_{K_a} = id|_{K_a}$ for all $t \in [0, 1]$ and

$$h(1, \dot{\varphi}^b) \subseteq \dot{\varphi}^a \cup K_a = \{y_0\} \tag{4.7}$$

(since $\dot{\varphi}^a = \emptyset$) and

$$\varphi(h(t, x)) \leq \varphi(x) \quad \text{for all } t \in [0, 1] \text{ and all } x \in \dot{\varphi}^b. \tag{4.8}$$

We consider the map $\gamma_0 : \bar{B}_r \cap \mathbb{R} \to W^{1,p}(Z)$ defined by

$$\gamma_0(x) = \begin{cases} y_0 & \text{if } \|x\| \leq \frac{r}{2}, \\ h\big(\frac{2(r - \|x\|)}{r}, \frac{rx}{\|x\|}\big) & \text{if } \|x\| > \frac{r}{2}, \end{cases} \tag{4.9}$$

for all $x \in \bar{B}_r \cap \mathbb{R}$. If $\|x\| = \frac{r}{2}$, then $h\big(\frac{2(r-\|x\|)}{r}, \frac{rx}{\|x\|}\big) = h(1, 2x) = y_0$ (see (4.7) and (4.3)). Hence, it follows that γ_0 is continuous.

If $\|x\| = r$, then $\gamma_0(x) = h(0, x) = x$ (since h is a deformation). Therefore, $\gamma \in \Gamma$. Moreover, from (4.9) and (4.8) and since $\varphi(y_0) = a \leq \mu_r < 0$ (see (4.3)), we have

$$\varphi(\gamma_0(x)) \leq \mu_r < 0 \quad \text{for all } x \in \bar{B}_r \cap \mathbb{R},$$

which implies

$$\hat{c}_r < 0. \tag{4.10}$$

(see (4.5) and recall $\gamma_0 \in \Gamma$). Comparing (4.3) and (4.10), we reach a contradiction. This means that there is one more critical point $v_0 \notin \{0, y_0\}$ of φ. Then, as before, we check that $v_0 \in W^{1,p}(Z)$ is a solution for (1.1) and nonlinear regularity theory implies that $v_0 \in C^1(\bar{Z})$. □

References

[1] G. Anello; *Existence of infinitely many weak solutions for a Neumann problem*, Nonlin. Anal. **57**(2004), 199–209.

[2] G. Barletta and N. S. Papageorgiou; *A multiplicity theorem for the Neumann p-Laplacian with an asymmetric nonsmooth potential*, J. Global Optim. **39**(2007), 365–392.

[3] G. Bonanno and P. Candito; *Three solutions to a Neumann problem for the elliptic equations involving the p-Laplacian*, Archiv der Math **80**(2003), 424–429.

[4] F. Cammaroto, A. Chinni, and B. DiBella; *Some multiplicity results for quasilinear Neumann problems*, Archiv der Math **86**(2006), 154–162.

[5] S. Carl, V. K. Le, and D. Motreanu; *Nonsmooth Variational Problems and their Inequalities: Comparison Principles and Applications*, Springer, New York (2007).

[6] K.-C. Chang; *Infinite Dimensional Morse Theory and Multiple Solution Problems*, Birkhäuser, Boston (1993).

[7] F. H. Clarke; *Optimization and Nonsmooth Analysis*, Wiley Interscience, New York (1983).

[8] J.-N. Corvellec; *On the second deformation lemma*, Topol. Meth. Nonlin. Anal. **17**(2001), 55–66.

[9] F. Faraci; *Multiplicity results for a Neumann problem involving the p-Laplacian*, J. Math. Anal. Appl. **277**(2003), 180–188.

[10] M. Filippakis, L. Gasinski, and N. S. Papageorgiou; *Multiplicity results for nonlinear Neumann problems*, Canad. J. Math. **58**(2006), 64–92.

[11] L. Gasinski and N. S. Papageorgiou; *Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems*, Chapman & Hall / CRC, Boca Raton (2005).

[12] L. Gasinski and N.S. Papageorgiou; *Nonlinear Analysis*, Chapman & Hall / CRC, Boca Raton (2006).

[13] S. Marano and D. Motreanu; *Infinitely many critical points of nondifferentiable functions and applications to a Neumann-type problem involving the p-Laplacian*, J. Diff. Eqns **182**(2002), 108–120.

[14] D. Motreanu and N. S. Papageorgiou; *Existence and multiplicity for Neumann problems*, J. Diff. Eqns **232**(2007), 1–35.

[15] D. Motreanu and V. Radulescu; *Variational and Nonvariational Methods in Nonlinear Analysis and Boundary Value Problems*, Kluwer, Dordrecht (2003).

[16] B. Ricceri; *On a three critical points theorem*, Archiv der Math **75**(2000), 220–226.

[17] B. Ricceri; *Infinitely many solutions of the Neumann problem for elliptic equations involving the p-Laplacian*, Bull London Math Soc **33**(2001), 331–340.

[18] X. Wu and K. K. Tan; *On the existence and multiplicty of solutions of Neumann boundary problems for quasilinear elliptic equations*, Nonlin. Anal. **65**(2006), 1334–1347.

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