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PRÜFER TRANSFORMATION FOR THE p-LAPLACIAN

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ABSTRACT. Prüfer transformation is a useful tool for study of second-order ordinary differential equations. There are many possible extensions of the original Prüfer transformation. We focus on a transformation suitable for study of boundary value problems for the p-Laplacian in the resonant case. The purpose of this paper is to establish its basic properties in deep detail.

1. Introduction

In most of the literature, the Prüfer transformation is viewed as a technique introducing the polar coordinates (or their modifications) in the phase plane. The original Prüfer's paper [9] dealt with the Sturm-Liouville theory for the second-order linear equation

$$(k(t)u')' + (l(t) + \lambda r(t))u = 0. \tag{1.1}$$

Prüfer studied nodal properties of the corresponding eigenfunctions via oscillation theory. His famous transformation originated in the proof of his "Oszillationstheorem". At the beginning of the proof, he wrote:

> For a fixed value of λ, let v and u be solutions of the system, equivalent to (1.1),
>
> $$\begin{aligned} v' &= -(l(t) + \lambda r(t))u, \\ u' &= \frac{1}{k(t)}v. \end{aligned} \tag{1.2}$$
>
> If one puts u, v into coordinates in a phase plane, the solution $u(t)$, $v(t)$ appears as a curve, the coordinates of which are continuous and differentiable functions of t. Similarly, if one introduces polar coordinates by
>
> $$v = \varrho \cos\varphi, \quad u = \varrho \sin\varphi, \tag{1.3}$$
>
> the polar coordinates of the curve are again continuous and differentiable functions of t, as long as $\varrho \neq 0$, and if unnecessary π-jumps

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in φ are omitted. These functions satisfy the equations

$$\begin{aligned} \varrho' &= \Big(\frac{1}{k(t)} - l(t) - \lambda r(t)\Big) \varrho \sin\varphi \cos\varphi, \\ \varphi' &= \frac{1}{k(t)} \cos^2\varphi + (l(t) + \lambda r(t)) \sin^2\varphi. \end{aligned} \tag{1.4}$$

From the system (1.4), one can easily deduce that increasing of $l(t) + \lambda r(t)$ moves the zeros of the solution u of a Cauchy problem for (1.1) towards the initial point. Indeed, let us focus on the second equation in (1.4). The points t where $\varphi(t) = n\pi$, $n \in \mathbb{Z}$, are zeros of u. Take a solution φ of the corresponding Cauchy problem. Obviously, if we increase the expression $l(t) + \lambda r(t)$, then φ increases right to the initial point and decreases left to it. This moves the zeros of u towards the initial point. See [9] for details.

Elbert [6] was interested in Sturm's comparison theory for the second-order **quasilinear** equation

$$-(\Phi(u'))' - q(t)\Phi(u) = 0 \tag{1.5}$$

where $\Phi(s) = |s|^{p-2}s$, $s > 0$, $\Phi(0) = 0$ and $1 < p < \infty$ is a constant. Choosing $p = 2$, (1.5) reduces to the linear equation (1.1). The equation (1.5) is equivalent to the system

$$\begin{aligned} v' &= -q(t)\Phi(u), \\ u' &= \Phi^{-1}(v). \end{aligned} \tag{1.6}$$

To this end, Elbert modified the Prüfer transformation to

$$\begin{aligned} v &= \Phi(\varrho \cos_p \varphi), \\ u &= \varrho \sin_p \varphi \end{aligned} \tag{1.7}$$

where $\sin_p$ is a solution of (1.5) with $q \equiv p - 1$, $\sin_p(0) = 0$ and $\sin_p'(0) = 1$, and $\cos_p = \sin_p'$. Similarly as above, $\varrho > 0$ is determined uniquely and φ uniquely up to a multiple of $2\pi_p$ where

$$\pi_p = \frac{2\pi}{p \sin\frac{\pi}{p}}$$

is the first positive zero of $\sin_p$. In this case, the pair ϱ, φ is a solution of the system

$$\begin{aligned} \varrho' &= \Big(1 - \frac{q(t)}{p-1}\Big) \varrho \Phi(\sin_p\varphi) \cos_p\varphi, \\ \varphi' &= |\cos_p\varphi|^p + \frac{q(t)}{p-1} |\sin_p\varphi|^p. \end{aligned} \tag{1.8}$$

Let us illustrate the advantages of the generalized Prüfer transformation (1.7) on the question of unique solvability of the Cauchy problem for (1.6). Obviously, the right-hand side of (1.6) is not Lipschitz continuous when $p \neq 2$ since $\Phi'(0) = +\infty$ for $1 < p < 2$ and $(\Phi^{-1})'(0) = +\infty$ for $p > 2$.

The one-to-one correspondence between the solution v, u of (1.6) and the solution ϱ, φ of (1.8) (up to a multiple of $2\pi_p$ in the case of φ) makes the unique solvability of the corresponding Cauchy problems for (1.6) and (1.8) equivalent. The right-hand side of (1.8) is not Lipschitz continuous either (the argument in [6] is incorrect). Indeed, if $1 < p < 2$, then $\Phi(\sin_p\varphi)$ has an infinite derivative at $\varphi = n\pi_p$, $n \in \mathbb{Z}$. If $p > 2$, then $\cos_p\varphi$ has an infinite derivative at $\varphi = (n + 1/2)\pi_p$, $n \in \mathbb{Z}$.

However, existence of a unique solution of the Cauchy problem for (1.8) can be easily proved since the Lipschitz continuity fails only in the first equation and, moreover, ϱ does not appear in the second equation. This allows us to solve the equations separately. Indeed, the second equation has a unique solution φ (satisfying an initial condition). Substituting this concrete function for φ in the first equation, we get a linear first-order equation for ϱ that (together with an initial condition) has a unique solution, too.

The transformation (1.7) is also useful for the study of oscillatory properties of solutions of second-order quasilinear equation — see [5] and the references therein.

We see that the one-to-one correspondence between the solutions of (1.6) and (1.8) is important. Nevertheless, it is used in [6] with no proof. Several other authors used the Prüfer's transformation to study boundary value problems for the p-Laplacian — see Bennewitz [2] and Yang [11] and [12], and also for the radially symmetric p-Laplacian in $\mathbb{R}^n$ — see Reichel and Walter [10] and Brown and Reichel [3] and [4]. To our knowledge, the only authors who prove the one-to-one correspondence are Reichel and Walter in [10]. Precisely said, they prove only the "nontrivial" part, i.e., given a pair u, v, there exists a unique ϱ and a unique φ up to a multiple of $2\pi_p$ satisfying a relation similar to (1.7). However, their proof contains several minor incorrectnesses. For example, they claim that [10, Equation (9)] which is similar to the first equation in (1.7) defines φ up to a multiple of $2\pi_p$. But $\cos_p$ is an even function, and so the equation defines φ also up to the sign. It turns out that if we want to determine φ up to a multiple of $2\pi_p$, we have to combine both equations in (1.7). Moreover, Reichel and Walter use $\sin_p''$ in their computations (e.g., [10, first equation on page 55]) that does not exist everywhere when $p > 2$. Hence they actually prove that ϱ and φ satisfy a transformed system almost everywhere only, not proving that ϱ and φ are absolutely continuous. Our aim is to provide a thorough correct proof of the one-to-one correspondence in this paper.

The function $\sin_p$ that, together with its derivative $\cos_p$, appears in the transformation (1.7) is the principal eigenfunction of the eigenvalue problem

$$\begin{gathered} -(\Phi(u'))' - (p-1)\lambda\Phi(u) = 0 \quad \text{in } (0,\pi_p), \\ u(0) = u(\pi_p) = 0, \end{gathered} \tag{1.9}$$

corresponding to the principal eigenvalue $\lambda_1 = 1$. Manásevich and Takáč [7] studied solvability of a resonant nonhomogeneous problem (1.9), i.e., with a given function at the right-hand side of the equation, and with λ equal to the k-th eigenvalue of (1.9) $\lambda_k = k^p$, $k \in \mathbb{N}$ (nonlinear Fredholm alternative). For this purpose, it is more useful to substitute the corresponding k-th eigenfunction $t \mapsto \frac{1}{k}\sin_p(kt)$ and its derivative $t \mapsto \cos_p(kt)$ for $\sin_p$ and $\cos_p$ in (1.7) to get the transformation

$$\begin{aligned} v &= \Phi(\varrho\cos_p(k\varphi)), \\ u &= \varrho\frac{1}{k}\sin_p(k\varphi). \end{aligned} \tag{1.10}$$

Notice that it is not just to replace φ by $k\varphi$ in (1.7) since $t \mapsto \cos_p(kt)$ is not a derivative of $t \mapsto \sin_p(kt)$! In fact, using (1.10), (1.6) would be equivalent to a system essentially different from (1.8).

In this paper, we further generalize (1.10) to a transformation suitable for study of resonant problems with jumping nonlinearity. We write the transformation in the form

$$\begin{aligned} v &= \Phi(\varrho C(\varphi)), \\ u &= \varrho S(\varphi) \end{aligned} \tag{1.11}$$

where $C = S'$ and S is the unique solution (see [1, Theorem 2 and Corollary 4]) of

$$-(\Phi(u'))' - (p-1)\left(\mu\Phi(u^+) - \nu\Phi(u^-)\right) = 0 \tag{1.12}$$

where $\mu, \nu > 0$, $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$, satisfying $S(0) = 0$ and $S'(0) = 1$. Notice that if $\mu = \nu = \lambda$ in (1.12), then it reduces to the equation in (1.9). It is easily seen that S is a $(\mu^{-1/p} + \nu^{-1/p})\pi_p$-periodic function and

$$S(t) = \begin{cases} \mu^{-1/p}\sin_p(\mu^{1/p}t) & \text{for } t \in [0, \mu^{-1/p}\pi_p], \\ \nu^{-1/p}\sin_p(\nu^{1/p}t) & \text{for } t \in (-\nu^{-1/p}\pi_p, 0). \end{cases}$$

If we consider a constant $\varrho > 0$, then the planar curve $\varphi \mapsto (v, u)$ given by (1.11), $\varphi \in (-\nu^{-1/p}\pi_p, \mu^{-1/p}\pi_p]$, is sketched in Figure 1.

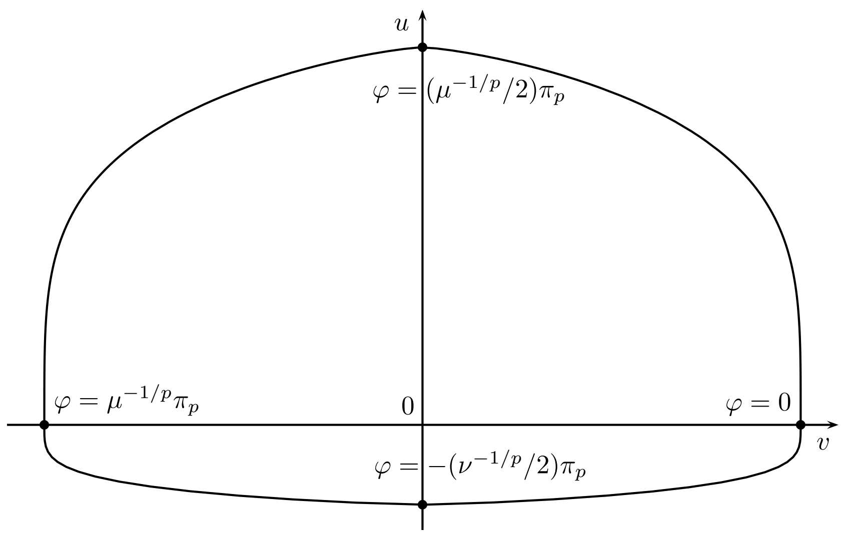

Figure 1. Generalized polar coordinates given by (1.11) for $p = 4$, $\mu = 1$, $\nu = 500$ and $\varrho = 1$.

2. Main Results

Given an interval $\mathcal{I} \subset \mathbb{R}$, let X denote the vector space of all real continuous functions on $\mathcal{I}$ and X^+ its subset of positive continuous functions. By $X/(a\mathbb{Z})$ we denote the quotient space of classes of continuous functions on $\mathcal{I}$ which differ by a multiple of $a > 0$. If $\mathcal{I}$ is compact, then X equipped with the sup-norm is the Banach space $C(\mathcal{I})$.

Theorem 2.1. *Let $p > 1$ and $\mathcal{I} \subset \mathbb{R}$ be an interval. There exists a bijection*

$$\Pi = (\Pi_1, \Pi_2)\colon \{(v, u) \in X^2 : |v| + |u| > 0 \text{ on } \mathcal{I}\} \to X^+ \times X/(2\pi_p\mathbb{Z})$$

such that for any $v, u, \varphi \in X$ and $\varrho \in X^+$, (1.7) holds on $\mathcal{I}$ if and only if $\varrho = \Pi_1(v, u)$ and $\varphi \in \Pi_2(v, u)$.

If $v, u, \varphi \in X$ and $\varrho \in X^+$ are such that (1.7) holds on $\mathcal{I}$, then

- *if v' and u' exist at a point $t \in \mathcal{I}$, then ϱ' and φ' exist at t, too, and*

$$\begin{aligned} \varrho' &= \frac{\varrho^{2-p}}{p-1} \cos_p \varphi v' + \Phi(\sin_p \varphi) u', \\ \varphi' &= -\frac{\varrho^{1-p}}{p-1} \sin_p \varphi v' + \frac{1}{\varrho} \Phi(\cos_p \varphi) u' \end{aligned} \tag{2.1}$$

at t (the derivatives are one-sided when $t \in \partial\mathcal{I}$),

- *both v and u are continuously differentiable on $\mathcal{I}$ if and only if both ϱ and φ are continuously differentiable on $\mathcal{I}$,*
- *if, moreover, $\mathcal{I}$ is compact, then $v, u \in AC(\mathcal{I})$ if and only if $\varrho, \varphi \in AC(\mathcal{I})$ (Here and in the sequel, $AC(\mathcal{I})$ stands for the space of absolutely continuous functions on $\mathcal{I}$).*

If $\mathcal{I}$ is compact, then the mappings

$$\{(\varrho, \varphi) \in (C(\mathcal{I}))^2 : \varrho > 0 \text{ on } \mathcal{I}\} \to \{(v, u) \in (C(\mathcal{I}))^2 : |v| + |u| > 0 \text{ on } \mathcal{I}\}$$

and

$$\{(\varrho, \varphi) \in (C^1(\mathcal{I}))^2 : \varrho > 0 \text{ on } \mathcal{I}\} \to \{(v, u) \in (C^1(\mathcal{I}))^2 : |v| + |u| > 0 \text{ on } \mathcal{I}\}$$

which map (ϱ, φ) on (v, u) if and only if (1.7) holds, are a local C^1-diffeomorphism and a local homeomorphism, respectively, at each $(\varrho, \varphi) \in (C(\mathcal{I}))^2$ and $(\varrho, \varphi) \in (C^1(\mathcal{I}))^2$, respectively, $\varrho > 0$.

Example 2.2. Let $p > 1$ and $\mathcal{I} \subset \mathbb{R}$ be an interval. Let v and u be continuous and $|v| + |u| > 0$ on $\mathcal{I}$. Then Theorem 2.1 yields that there exists a unique continuous $\varrho > 0$ a unique (up to a multiple of $2\pi_p$) continuous φ such that (1.7) holds. Moreover, v, u is a classical solution of (1.6) on $\mathcal{I}$ if and only if ϱ, φ is a classical solution of (1.8) on $\mathcal{I}$ (we combine (1.6) and (1.7) with (2.1)). If $\mathcal{I}$ is compact, then the same holds for the Carathéodory solution instead of the classical one.

Theorem 2.3. *Let $p > 1$, $\mu, \nu > 0$ and $\mathcal{I} \subset \mathbb{R}$ be an interval. There exists a bijection*

$$\Pi = (\Pi_1, \Pi_2) \colon \{(v, u) \in X^2 : |v| + |u| > 0 \text{ on } \mathcal{I}\} \to X^+ \times X/((\mu^{-1/p} + \nu^{-1/p})\pi_p \mathbb{Z})$$

such that for any $v, u, \varphi \in X$ and $\varrho \in X^+$, (1.11) holds on $\mathcal{I}$ if and only if $\varrho = \Pi_1(v, u)$ and $\varphi \in \Pi_2(v, u)$.

If $v, u, \varphi \in X$ and $\varrho \in X^+$ are such that (1.11) holds on $\mathcal{I}$, then

- *if v' and u' exist at a point $t \in \mathcal{I}$, then ϱ' and φ' exist at t, too, and*

$$\begin{aligned} \varrho' &= \frac{\varrho^{2-p}}{p-1} C(\varphi) v' + \big(\mu \Phi(S^+(\varphi)) - \nu \Phi(S^-(\varphi))\big) u', \\ \varphi' &= -\frac{\varrho^{1-p}}{p-1} S(\varphi) v' + \frac{1}{\varrho} \Phi(C(\varphi)) u' \end{aligned} \tag{2.2}$$

at t (the derivatives are one-sided when $t \in \partial\mathcal{I}$),

- *both v and u are continuously differentiable on $\mathcal{I}$ if and only if both ϱ and φ are continuously differentiable on $\mathcal{I}$,*
- *if, moreover, $\mathcal{I}$ is compact, then $v, u \in AC(\mathcal{I})$ if and only if $\varrho, \varphi \in AC(\mathcal{I})$.*

If $\mathcal{I}$ is compact, then the mappings

$$\{(\varrho, \varphi) \in (C(\mathcal{I}))^2 : \varrho > 0 \text{ on } \mathcal{I}\} \to \{(v, u) \in (C(\mathcal{I}))^2 : |v| + |u| > 0 \text{ on } \mathcal{I}\}$$

and

$$\{(\varrho,\varphi)\in (C^1(\mathcal{I}))^2 : \varrho>0 \text{ on } \mathcal{I}\} \to \{(v,u)\in (C^1(\mathcal{I}))^2 : |v|+|u|>0 \text{ on } \mathcal{I}\}$$

which map (ϱ,φ) on (v,u) if and only if (1.11) *holds, are a local C^1-diffeomorphism and a local homeomorphism, respectively, at each $(\varrho,\varphi)\in (C(\mathcal{I}))^2$ and $(\varrho,\varphi)\in (C^1(\mathcal{I}))^2$, respectively, $\varrho>0$.*

Remark. Theorem 2.1 is a special case of Theorem 2.3 for $\mu=\nu=1$.

Example 2.4. Let $p>1$, $\mu,\nu>0$ and $\mathcal{I}\subset\mathbb{R}$ be an interval. Let us consider the equation

$$-(\Phi(u'))' - (p-1)\big(\mu\Phi(u^+) - \nu\Phi(u^-)\big) = f \tag{2.3}$$

which is equivalent to

$$\begin{gathered} v' = -(p-1)\big(\mu\Phi(u^+) - \nu\Phi(u^-)\big) - f,\\ u' = \Phi^{-1}(v). \end{gathered} \tag{2.4}$$

Let v and u be continuous and $|v|+|u|>0$ on $\mathcal{I}$. Then Theorem 2.3 yields that there exists a unique continuous $\varrho>0$ and a unique (up to a multiple of $(\mu^{-1/p}+\nu^{-1/p})\pi_p$) continuous φ such that (1.11) holds. Moreover, v,u is a classical solution of (2.4) on $\mathcal{I}$ if and only if ϱ,φ is a classical solution of

$$\begin{gathered} \varrho' = -\frac{\varrho^{2-p}}{p-1}C(\varphi)f,\\ \varphi' = 1 + \frac{\varrho^{1-p}}{p-1}S(\varphi)f \end{gathered} \tag{2.5}$$

on $\mathcal{I}$ (we combine (2.4) and (1.11) with (2.2)). Again, if $\mathcal{I}$ is compact, then the same holds for the Carathéodory solution instead of the classical one.

If we choose $\mu=\nu=k^p$ in (2.3), we obtain the equation studied by Manásevich and Takáč in [7]. They used a slightly different transformation, but their coordinates r,Θ (see [7, eqs. (23), (24)]) can be expressed in terms of our ϱ,φ as $r=\varrho^{p-1}$ and $\Theta=(\varphi-t)\varrho^{p-1}$. Differentiating these formulas and using (2.5) we easily get formulas [7, eqs. (27), (28)]:

$$\frac{\mathrm{d}\Theta}{\mathrm{d}x} = f(x)\left[\frac{1}{k(p-1)}\sin_p k\left(x+\frac{\Theta}{r}\right) - \frac{\Theta}{r}\cos_p k\left(x+\frac{\Theta}{r}\right)\right]$$

and

$$\frac{\mathrm{d}r}{\mathrm{d}x} = -f(x)\cos_p k\left(x+\frac{\Theta}{r}\right).$$

3. Proof of Theorem 2.1

The reason why we prove Theorem 2.1 in spite of the fact that it is a special case of Theorem 2.3 is that we want to give a very clear and thorough proof in the simpler case avoiding unnecessary technical difficulties. In the next section we prove Theorem 2.3, focusing mainly on the differences between the proofs.

First we show that Π^{-1} is well defined by (1.7). Let $\varrho\in X^+$ and $\tilde\varphi\in X/(2\pi_p\mathbb{Z})$. Choose an arbitrary $\varphi\in\tilde\varphi$. Since both $\sin_p$ and $\cos_p$ are $2\pi_p$-periodic functions, v and u defined by (1.7) are independent of the choice of φ. Let us view (1.7) as a transformation in $\mathbb{R}^2$, i.e., we define a mapping

$$F\colon (0,\infty)\times\mathbb{R}\to\mathbb{R}^2\setminus\{(0,0)\} : F(\varrho,\varphi)=(v,u), \text{ such that (1.7) holds.}$$

Continuity of F implies $v, u \in X$. Since $\varrho > 0$ and $\sin_p$ and $\cos_p$ have no common zeros, we have also $|v(t)| + |u(t)| > 0$ for all $t \in \mathcal{I}$.

To show that Π is well defined, too, we start by inverting F. Notice that F is not injective (it is $2\pi_p$-periodic in the second variable), and so F^{-1} is understood to be a multi-valued function. Let $(v, u) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Using the well-known identity

$$|\cos_p x|^p + |\sin_p x|^p = 1 \quad \forall x \in \mathbb{R},$$

we infer from (1.7)

$$|\Phi^{-1}(v)|^p + |u|^p = |\varrho|^p(|\cos_p \varphi|^p + |\sin_p \varphi|^p) = |\varrho|^p.$$

At this point, we could choose the sign of ϱ (as it was admitted in the Prüfer's paper [9]). But we define F only for $\varrho > 0$, and so if $(\varrho, \varphi) = F^{-1}(v, u)$, then

$$\varrho = \left(|v|^{p/(p-1)} + |u|^p\right)^{1/p} > 0. \tag{3.1}$$

To obtain φ, we deduce from (1.7) that if $v \neq 0$, then

$$\tan_p \varphi = \frac{u}{\Phi^{-1}(v)} \quad \text{where} \quad \tan_p x \overset{\text{def}}{=} \frac{\sin_p x}{\cos_p x},\ x \neq (n + 1/2)\pi_p,\ n \in \mathbb{Z},$$

and if $u \neq 0$, then

$$\operatorname{cotan}_p \varphi = \frac{\Phi^{-1}(v)}{u} \quad \text{where} \quad \operatorname{cotan}_p x \overset{\text{def}}{=} \frac{\cos_p x}{\sin_p x},\ x \neq n\pi_p,\ n \in \mathbb{Z}.$$

Consequently,

$$\begin{gathered}
v > 0 \quad \Longrightarrow \quad \varphi = \arctan_p \frac{u}{\Phi^{-1}(v)} + 2n\pi_p, \quad n \in \mathbb{Z}, \\
v < 0 \quad \Longrightarrow \quad \varphi = \arctan_p \frac{u}{\Phi^{-1}(v)} + (2n+1)\pi_p, \quad n \in \mathbb{Z}, \\
u > 0 \quad \Longrightarrow \quad \varphi = \operatorname{arccotan}_p \frac{\Phi^{-1}(v)}{u} + 2n\pi_p, \quad n \in \mathbb{Z}, \\
u < 0 \quad \Longrightarrow \quad \varphi = \operatorname{arccotan}_p \frac{\Phi^{-1}(v)}{u} + (2n+1)\pi_p, \quad n \in \mathbb{Z},
\end{gathered} \tag{3.2}$$

where $\arctan_p$ is the inverse function to $\tan_p|_{(-\pi_p/2, \pi_p/2)}$ and $\operatorname{arccotan}_p$ is the inverse function to $\operatorname{cotan}_p|_{(0,\pi_p)}$. Obviously, if $uv \neq 0$, then we are free to choose between two formulas, one using $\arctan_p$ and one using $\operatorname{arccotan}_p$. Otherwise, only one of the above four formulas is applicable (we remind that we assume $(v, u) \neq (0, 0)$). Geometrical interpretation of φ in the v, u-plane is found in [6, Figure 2, page 159].

Now that we have formulas (3.1) and (3.2) defining F^{-1}, let v and u be continuous on $\mathcal{I}$, $|v(t)| + |u(t)| > 0$ for all $t \in \mathcal{I}$. The function ϱ is given by (3.1). Clearly, ϱ is continuous and positive on $\mathcal{I}$.

Although φ is given by (3.2), n and the choice of the appropriate formula depend on $t \in \mathcal{I}$. Let us choose a $t_0 \in \mathcal{I}$. Assume $v(t_0) \neq 0$ (for $u(t_0) \neq 0$ we proceed similarly). We determine $\varphi(t_0)$ from the first (if $v(t_0) > 0$) or the second (if $v(t_0) < 0$) formula in (3.2), choosing an arbitrary $n \in \mathbb{Z}$. Now we extend φ to a continuous function on $\mathcal{I}_+ \overset{\text{def}}{=} \mathcal{I} \cap [t_0, \infty)$ in the following way. If $v \neq 0$ on $\mathcal{I}_+$, we use the same formula in (3.2) as in t_0, and also the same n (otherwise φ would not be continuous). Otherwise, let t_1 be the first point in $\mathcal{I}_+$ where $v(t_1) = 0$. We determine $\varphi(t_1)$ from the third or the fourth formula in (3.2), depending on the sign of $u(t_1)$. It is easy to check that there is a unique $n \in \mathbb{Z}$ that we have to use in the respective formula to guarantee left-continuity of φ in t_1. We proceed further in a similar fashion. We extend φ using the same formula and the same n either to the rest of $\mathcal{I}_+$, or up to the first t_2 where $u(t_2)=0$, and so on.

We have to prove that this procedure covers the whole $\mathcal{I}_+$. Assume the contrary, i.e., that $t_i \to T < \sup\mathcal{I}$ as $i\to\infty$. But $v(t_{2i-1})=0$ and $u(t_{2i})=0$, $i\in\mathbb{N}$, and so the continuity of v and u would imply $v(T)=u(T)=0$, a contradiction. Extension of φ to $\mathcal{I}\cap(-\infty,t_0)$ is done analogously. Since the choice of n at t_0 determines a unique continuous φ, we obtain a unique class from $X/(2\pi_p\mathbb{Z})$ and the proof that Π is a well-defined mapping is complete.

To prove (2.1) we use the chain rule, so we need to differentiate both (3.1) and (3.2) with respect to both v and u. From (3.1) we infer

$$\frac{\partial\varrho}{\partial v}=\frac{1}{p}\big(|v|^{p/(p-1)}+|u|^p\big)^{(1-p)/p}\frac{p}{p-1}\Phi^{-1}(v)=\frac{1}{p-1}\varrho^{1-p}\varrho\cos_p\varphi=\frac{\varrho^{2-p}}{p-1}\cos_p\varphi \tag{3.3}$$

and, similarly,

$$\frac{\partial\varrho}{\partial u}=\frac{1}{p}\big(|v|^{p/(p-1)}+|u|^p\big)^{(1-p)/p}p\Phi(u)=\varrho^{1-p}\Phi(\varrho\sin_p\varphi)=\Phi(\sin_p\varphi). \tag{3.4}$$

This proves the first equality in (2.1). To differentiate φ defined by (3.2), we first notice that each of the four formulas is valid on an open set, so if one of them holds at a $t\in\mathcal{I}$, then it holds in a neighborhood of t. Hence it suffices to differentiate all the four formulas separately. The reader is invited to verify

$$\arctan_p' x=\frac{1}{1+|x|^p},\quad x\in\mathbb{R}.$$

Hence the first two formulas in (3.2) yield that if $v\neq 0$, then

$$\begin{aligned}\frac{\partial\varphi}{\partial v}&=\frac{1}{1+\frac{|u|^p}{|v|^{p/(p-1)}}}u\frac{-1}{p-1}|v|^{-1/(p-1)-1}=-\frac{1}{p-1}\frac{u}{|v|^{p/(p-1)}+|u|^p}\\&=-\frac{1}{p-1}\frac{\varrho\sin_p\varphi}{\varrho^p}=-\frac{\varrho^{1-p}}{p-1}\sin_p\varphi\end{aligned}$$

and

$$\frac{\partial\varphi}{\partial u}=\frac{1}{1+\frac{|u|^p}{|v|^{p/(p-1)}}}\frac{1}{\Phi^{-1}(v)}=\frac{v}{|v|^{p/(p-1)}+|u|^p}=\frac{\Phi(\varrho\cos_p\varphi)}{\varrho^p}=\frac{1}{\varrho}\Phi(\cos_p\varphi).$$

If $v=0$, then $u\neq 0$ and we differentiate the last two formulas in (3.2). But we cannot use the chain rule directly unless $p=2$ since

$$\operatorname{arccotan}_p' x=-\frac{|x|^{p-2}}{1+|x|^p},\quad x\neq 0,\quad \operatorname{arccotan}_p' 0=\begin{cases}-\infty & \text{for } 1<p<2,\\ -1 & \text{for } p=2,\\ 0 & \text{for } p>2\end{cases}$$

and $(\Phi^{-1})'(0)=\infty$ for $p>2$. We rewrite the last two formulas in (3.2) in the form

$$\varphi=\operatorname{arccotan}_p\Phi^{-1}\Big(\frac{v}{\Phi(u)}\Big)+m\pi_p,\quad m\in\mathbb{Z}, \tag{3.5}$$

and we derive the derivative of the composite function $x\mapsto\operatorname{arccotan}_p\Phi^{-1}(x)$, $x\in\mathbb{R}$, directly from the derivative of its inverse. The reader is invited to check that

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arccotan}_p\Phi^{-1}(x)=-\frac{1}{(p-1)(1+|x|^{p/(p-1)})},\quad x\in\mathbb{R}. \tag{3.6}$$

Combining (3.5) and (3.6) we get that if $u \neq 0$, then

$$\frac{\partial\varphi}{\partial v} = -\frac{1}{(p-1)\left(1+\frac{|v|^{p/(p-1)}}{|u|^p}\right)}\frac{1}{\Phi(u)} = -\frac{u}{(p-1)(|u|^p+|v|^{p/(p-1)})} = -\frac{\varrho^{1-p}}{p-1}\sin_p\varphi$$

and

$$\frac{\partial\varphi}{\partial u} = -\frac{1}{(p-1)\left(1+\frac{|v|^{p/(p-1)}}{|u|^p}\right)}v(1-p)|u|^{-p} = \frac{v}{|u|^p+|v|^{p/(p-1)}} = \frac{1}{\varrho}\Phi(\cos_p\varphi).$$

This completes the proof of (2.1).

From (2.1) we easily infer that if v' and u' are continuous on $\mathcal{I}$, then ϱ' and φ' are continuous there, too. Indeed, all the derivatives $\frac{\partial\varrho}{\partial v}$, $\frac{\partial\varrho}{\partial u}$, $\frac{\partial\varphi}{\partial v}$ and $\frac{\partial\varphi}{\partial u}$ are continuous functions of t on $\mathcal{I}$. Conversely, if ϱ' and φ' are continuous on $\mathcal{I}$, then the continuity of v' and u' follows from (1.7), precisely said, from the continuity of

$$\begin{gathered}\frac{\partial v}{\partial\varrho} = (p-1)\varrho^{p-2}\Phi(\cos_p\varphi), \quad \frac{\partial v}{\partial\varphi} = -\varrho^{p-1}(p-1)\Phi(\sin_p\varphi), \\ \frac{\partial u}{\partial\varrho} = \sin_p\varphi, \quad \frac{\partial u}{\partial\varphi} = \varrho\cos_p\varphi\end{gathered} \tag{3.7}$$

on $\mathcal{I}$. Notice that we used the identity

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi(\cos_p x) = -(p-1)\Phi(\sin_p x) \quad \forall x\in\mathbb{R} \tag{3.8}$$

which follows from the fact that $\sin_p$ is defined as a solution of (1.5) with $q \equiv p-1$.

Further, we prove that $v, u \in AC(\mathcal{I})$ if and only if $\varrho, \varphi \in AC(\mathcal{I})$ provided $\mathcal{I}$ is compact. Compactness of $\mathcal{I}$ guarantees that ϱ attains a positive minimum there. Hence all the derivatives $\frac{\partial\varrho}{\partial v}$, $\frac{\partial\varrho}{\partial u}$, $\frac{\partial\varphi}{\partial v}$, $\frac{\partial\varphi}{\partial u}$, $\frac{\partial v}{\partial\varrho}$, $\frac{\partial v}{\partial\varphi}$, $\frac{\partial u}{\partial\varrho}$ and $\frac{\partial u}{\partial\varphi}$ are bounded on $\mathcal{I}$. Consequently, ϱ and φ are composite functions of v and u and a Lipschitz continuous function, and vice versa. Since the composition of an absolutely continuous function and a Lipschitz continuous function is absolutely continuous (see [8]), the assertion follows.

The reader is invited to check by definition that the last assertion of Theorem 2.1 follows from the fact that for a compact $\mathcal{I}$, all four derivatives (3.7) are uniformly continuous on

$$\Big\{(\varrho,\varphi)\in\mathbb{R}^2 : \varrho \in \Big(\frac{1}{2}\inf_{t\in\mathcal{I}}\varrho(t), \sup_{t\in\mathcal{I}}\varrho(t) + \frac{1}{2}\inf_{t\in\mathcal{I}}\varrho(t)\Big), \varphi\in\mathbb{R}\Big\}.$$

This completes the proof of Theorem 2.1.

4. Proof of Theorem 2.3

Theorem 2.3 is a generalization of Theorem 2.1. It can be proved using the same ideas, but with additional technical complications. We give just an outline of the main differences so the interested reader can follow the proof of Theorem 2.1.

First of all, we use S and C instead of $\sin_p$ and $\cos_p$. Hence (1.7) becomes (1.11). The dependence of ϱ on v and u takes the form

$$\varrho = \big(|v|^{p/(p-1)} + \mu(u^+)^p + \nu(u^-)^p\big)^{1/p}$$

instead of (3.1) by virtue of the identity

$$|C(x)|^p + \mu(S^+(x))^p + \nu(S^-(x))^p = 1 \quad \forall x\in\mathbb{R}.$$

Finally, (3.2) is replaced by

$$
\begin{aligned}
v > 0 \quad &\Longrightarrow \quad \varphi = \mu^{-1/p} \operatorname{arctan}_p \Big(\mu^{1/p} \frac{u^+}{\Phi^{-1}(v)}\Big) - \nu^{-1/p} \operatorname{arctan}_p \Big(\nu^{1/p} \frac{u^-}{\Phi^{-1}(v)}\Big) \\
&\qquad + (\mu^{-1/p} + \nu^{-1/p}) n\pi_p, \quad n \in \mathbb{Z}, \\
v > 0 \quad &\Longrightarrow \quad \varphi = \mu^{-1/p} \operatorname{arctan}_p \Big(\mu^{1/p} \frac{u^+}{\Phi^{-1}(v)}\Big) - \nu^{-1/p} \operatorname{arctan}_p \Big(\nu^{1/p} \frac{u^-}{\Phi^{-1}(v)}\Big) \\
&\qquad + \big((\mu^{-1/p} + \nu^{-1/p}) n + \mu^{-1/p}\big)\pi_p, \quad n \in \mathbb{Z}, \\
u > 0 \quad &\Longrightarrow \quad \varphi = \mu^{-1/p} \operatorname{arccotan}_p \Big(\mu^{-1/p} \frac{\Phi^{-1}(v)}{u}\Big) \\
&\qquad + (\mu^{-1/p} + \nu^{-1/p}) n\pi_p, \quad n \in \mathbb{Z}, \\
u < 0 \quad &\Longrightarrow \quad \varphi = \nu^{-1/p} \operatorname{arccotan}_p \Big(\nu^{-1/p} \frac{\Phi^{-1}(v)}{u}\Big) \\
&\qquad + \big((\mu^{-1/p} + \nu^{-1/p}) n + \mu^{-1/p}\big)\pi_p, \quad n \in \mathbb{Z}
\end{aligned}
$$

(cf. Figure 1). The reader is invited to differentiate ϱ and φ to prove (2.2). Since it leads to technically complicated calculations, we present an alternative approach, which is less transparent, but more suitable for this case. The function F that maps (ϱ, φ) to (v, u) such that (1.11) holds, is a local diffeomorphism at each point of $(0, \infty) \times \mathbb{R}$. Indeed, its Jacobi matrix is

$$
\begin{aligned}
J_F &= \begin{pmatrix} \partial v/\partial\varrho & \partial v/\partial\varphi \\ \partial u/\partial\varrho & \partial u/\partial\varphi \end{pmatrix} \\
&= \begin{pmatrix} (p-1)\varrho^{p-2}\Phi(C(\varphi)) & -\varrho^{p-1}(p-1)\big(\mu\Phi(S^+(\varphi)) - \nu\Phi(S^-(\varphi))\big) \\ S(\varphi) & \varrho C(\varphi) \end{pmatrix}
\end{aligned}
$$

and $\det J_F = (p-1)\varrho^{p-1} > 0$. Similarly as in the proof of Theorem 2.1, we used the identity

$$
\frac{\mathrm{d}}{\mathrm{d}x}\Phi(C(x)) = -(p-1)\big(\mu\Phi(S^+(x)) - \nu\Phi(S^-(x))\big) \quad \forall x \in \mathbb{R}
$$

which follows directly from the definition of S as a solution of (1.12). Consequently, the Jacobi matrix $J_{F^{-1}}$ of the locally inverse function is

$$
\begin{pmatrix} \partial\varrho/\partial v & \partial\varrho/\partial u \\ \partial\varphi/\partial v & \partial\varphi/\partial u \end{pmatrix} = (J_F)^{-1} = \begin{pmatrix} \dfrac{\varrho^{2-p}}{p-1}C(\varphi) & \mu\Phi(S^+(\varphi)) - \nu\Phi(S^-(\varphi)) \\ -\dfrac{\varrho^{1-p}}{p-1}S(\varphi) & \dfrac{1}{\varrho}\Phi(C(\varphi)) \end{pmatrix}.
$$

This proves (2.2). The rest of the proof of Theorem 2.3 is very similar to the proof of Theorem 2.1, and so we omit it.

5. Counterexamples for noncompact interval

Theorem 2.1 states that if $v, u, \varphi \in X$ and $\varrho \in X^+$ satisfy (1.7) and $\mathcal{I}$ is a compact interval, then

$$
v, u \in AC(\mathcal{I}) \quad \Longleftrightarrow \quad \varrho, \varphi \in AC(\mathcal{I}). \tag{5.1}
$$

The aim of this section is to show on several counterexamples that the equivalence (5.1) is not true unless $\mathcal{I}$ is compact. We will not discuss unbounded $\mathcal{I}$ since it is not clear how to define absolute continuity on an unbounded interval. For example, the standard ε-δ definition does not guarantee Lebesgue integrability of the derivative of the function as it is true on a bounded interval (a simple example of such a function is the identity function $t \mapsto t$, $t \in \mathbb{R}$). Absolute continuity is defined variously in the literature, depending on the concrete purpose.

On the other hand, there can be no confusion with definition of absolute continuity on a bounded open interval since if a function satisfies the standard ε-δ definition on an interval (a, b), then it can be easily extended to an absolutely continuous function on $[a, b]$ defining its value at the end-points by the one-sided limits. However, the assumption $\varrho \in X^+$ guarantees positivity of ϱ (and $|v| + |u|$) at the interior points only. If the limits of ϱ at the end-points are positive, too, then (5.1) still holds. If at least one of the limits is zero, then (5.1) can fail, as we show on the following three counterexamples.

Example 5.1. Let $p = 3/2$, $\mathcal{I} = (0, 1)$,

$$\varrho(t) = t^2\Big(1 + \sin^2 \frac{1}{t}\Big) > 0, \quad \varphi(t) = 0, \quad t \in (0, 1).$$

Figure 2 shows a part of the graph of ϱ. We have $\varrho, \varphi \in AC(\mathcal{I})$. Indeed,

$$\varrho'(t) = 2t\Big(1 + \sin^2 \frac{1}{t}\Big) - \sin \frac{2}{t}, \quad t \in (0, 1),$$

is bounded on $\mathcal{I}$. Hence ϱ is Lipschitz continuous and, consequently, absolutely continuous on $\mathcal{I}$. But (1.7) yields

$$v(t) = t\sqrt{1 + \sin^2 \frac{1}{t}} \quad \Longrightarrow \quad v'(t) = \sqrt{1 + \sin^2 \frac{1}{t}} - \frac{\frac{1}{t}\sin\frac{2}{t}}{2\sqrt{1 + \sin^2 \frac{1}{t}}}, \quad t \in (0, 1).$$

Since $1 \leq \sqrt{1 + \sin^2 \frac{1}{t}} \leq \sqrt{2}$ on $\mathcal{I}$, Lebesgue integrability of v' on $\mathcal{I}$ is equivalent to that of $\frac{1}{t}\sin\frac{2}{t}$. It is readily seen that

$$\int_0^1 \big(\frac{1}{t}\sin\frac{2}{t}\big)^+ \mathrm{d}t = \infty \quad \text{and} \quad \int_0^1 \big(\frac{1}{t}\sin\frac{2}{t}\big)^- \mathrm{d}t = -\infty.$$

Consequently, $v' \notin L^1(0, 1)$ and v cannot be absolutely continuous on $(0, 1)$.

Example 5.2. Let $p = 3$, $\mathcal{I} = (0, 1)$,

$$v(t) = t^2\big(1 + \sin^2 \frac{1}{t}\big) > 0, \quad u(t) = 0, \quad t \in (0, 1).$$

From (3.1) we deduce

$$\varrho(t) = t\sqrt{1 + \sin^2 \frac{1}{t}}, \quad t \in (0, 1).$$

Hence, similarly as in the previous example, $v, u \in AC(\mathcal{I})$, but $\varrho \notin AC(\mathcal{I})$.

Example 5.3. Let $p > 1$, $\mathcal{I} = (0, 1)$,

$$v(t) = t^{2(p-1)}, \quad u(t) = t^2 \sin \frac{1}{t}, \quad t \in (0, 1).$$

Clearly, $v, u \in AC(\mathcal{I})$. Since $v > 0$ on $\mathcal{I}$, we can determine φ from the first formula in (3.2), where we choose $n = 0$. Then

$$\varphi = \arctan_p \sin \frac{1}{t}, \quad t \in (0, 1).$$

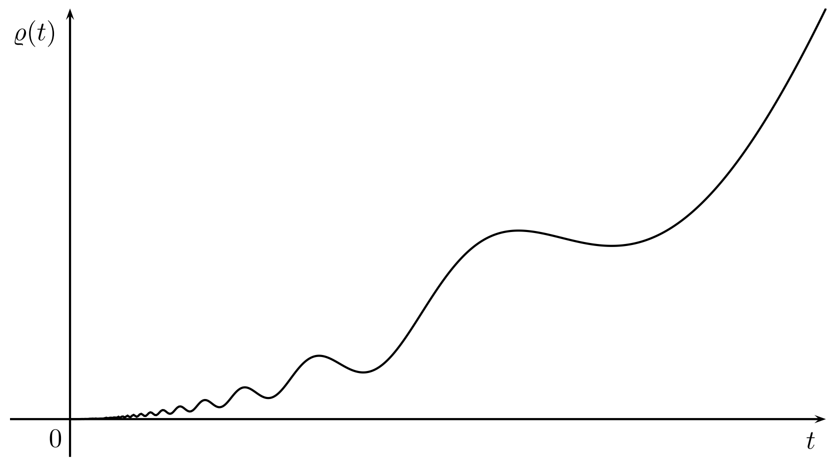

FIGURE 2. Graph of ϱ from Example 5.1 on $(0, 2/5)$.

Obviously,

$$\varphi\Big(\frac{1}{m\pi}\Big) = 0 \quad \text{and} \quad \varphi\Big(\frac{1}{(2m+1/2)\pi}\Big) = \operatorname{arctan}_p 1 > 0, \quad m \in \mathbb{N}.$$

Consequently, $\varphi \notin AC(\mathcal{I})$ since it is not even uniformly continuous there.

We summarize the validity of (5.1) (precisely said, all the four implications $\varrho, \varphi \in AC(\mathcal{I}) \Rightarrow v \in AC(\mathcal{I})$, $\varrho, \varphi \in AC(\mathcal{I}) \Rightarrow u \in AC(\mathcal{I})$, $v, u \in AC(\mathcal{I}) \Rightarrow \varrho \in AC(\mathcal{I})$ and $v, u \in AC(\mathcal{I}) \Rightarrow \varphi \in AC(\mathcal{I})$ separately) for a bounded $\mathcal{I}$ in the below table, distinguishing among $1 < p < 2$, $p = 2$, and $p > 2$.

<table>
<tr><td rowspan="2"></td><td colspan="2">$\varrho, \varphi \in AC(\mathcal{I}) \Rightarrow$</td><td colspan="2">$v, u \in AC(\mathcal{I}) \Rightarrow$</td></tr>
<tr><td>$v \in AC(\mathcal{I})$</td><td>$u \in AC(\mathcal{I})$</td><td>$\varrho \in AC(\mathcal{I})$</td><td>$\varphi \in AC(\mathcal{I})$</td></tr>
<tr><td>$1 < p < 2$</td><td>NO
(Example 5.1)</td><td>YES</td><td>YES</td><td rowspan="3">NO
(Example 5.3)</td></tr>
<tr><td>$p = 2$</td><td>YES</td><td>YES</td><td>YES</td></tr>
<tr><td>$p > 2$</td><td>YES</td><td>YES</td><td>NO
(Example 5.2)</td></tr>
</table>

It is easy to justify all the fields with "YES". First we prove $\varrho, \varphi \in AC(\mathcal{I}) \Rightarrow v \in AC(\mathcal{I})$ for $p \geq 2$. So let us assume $\varrho, \varphi \in AC(\mathcal{I})$ and $p \geq 2$. Since an absolutely continuous function ϱ on a bounded interval is bounded and Φ is Lipschitz continuous on any bounded interval for $p \geq 2$, the function $\varrho \mapsto \Phi(\varrho)$ is bounded and Lipschitz continuous on $[\inf_{t\in\mathcal{I}} \varrho(t), \sup_{t\in\mathcal{I}} \varrho(t)]$. Moreover, $\varphi \mapsto \Phi(\cos_p \varphi)$ is a periodic C^1-function on $\mathbb{R}$ — see (3.8). Consequently, $(\varrho, \varphi) \mapsto \Phi(\varrho \cos_p \varphi)$ is a Lipschitz continuous function on $[\inf_{t\in\mathcal{I}} \varrho(t), \sup_{t\in\mathcal{I}} \varrho(t)] \times \mathbb{R}$ and we deduce from (1.7) that v is a composition of absolute continuous ϱ and φ and a Lipschitz continuous function. So $v \in AC(\mathcal{I})$ by [8].

Second, $\varrho, \varphi \in AC(\mathcal{I}) \Rightarrow u \in AC(\mathcal{I})$ is proved even more easily since $(\varrho, \varphi) \mapsto \varrho \sin_p \varphi$ is Lipschitz continuous on $\mathbb{R}^2$ for any $p > 1$.

Finally, assume $v, u \in AC(\mathcal{I})$ and $1 < p \leq 2$. Hence both v and u are bounded and, by (3.1), ϱ is bounded on $\mathcal{I}$, too. To prove $\varrho \in AC(\mathcal{I})$, notice that (3.1),

[8] and $\varrho \in X^+$ imply that it suffices to prove Lipschitz continuity of $(v,u) \mapsto (|v|^{p/(p-1)} + |u|^p)^{1/p}$ on the bounded set

$$\big\{(v,u) \in \mathbb{R}^2 : 0 < (|v|^{p/(p-1)} + |u|^p)^{1/p} \le \sup_{t\in\mathcal{I}} \varrho(t)\big\}$$

that does not contain the origin. This follows from the fact that, due to $2-p \ge 0$, both its partial derivatives (3.3) and (3.4) are bounded on this set. The proof is complete.

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