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REGULARITY OF SOLUTIONS TO DOUBLY NONLINEAR DIFFUSION EQUATIONS

JOCHEN MERKER

Abstract. We prove under weak assumptions that solutions u of doubly nonlinear reaction-diffusion equations

$$\dot{u} = \Delta_p u^{m-1} + f(u)$$

to initial values $u(0) \in L^a$ are instantly regularized to functions $u(t) \in L^\infty$ (ultracontractivity). Our proof is based on a priori estimates of $\|u(t)\|_{r(t)}$ for a time-dependent exponent $r(t)$. These a priori estimates can be obtained in an elementary way from logarithmic Gagliardo-Nirenberg inequalities by an optimal choice of $r(t)$, and they do not only imply ultracontractivity, but provide further information about the long-time behaviour.

1. Introduction

Let us consider the quasilinear parabolic equation

$$\dot{u} = \Delta_p u^{m-1} + f(u)$$

on $(0,\infty) \times \mathbb{R}^n$. Here $u^{m-1} := |u|^{m-2}u$ denotes signed power,

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

is the p-Laplacian and f is a nonlinearity depending on u.

In the semilinear case $p = 2$, $m = 2$, it is well-known that initial values $u(0) \in L^a$ are instantly regularized to $u(t) \in L^\infty$ $(t > 0)$. This property called ultracontractivity is important, because once $u(t) \in L^\infty$ has been established, in a second step often Hölder continuity and differentiability of $u(t)$ can be shown.

Surprisingly, also in the quasilinear case $p \neq 2$, $m \neq 2$, the generated semigroup is ultracontractive. This property can be proved by Moser iteration (see [4, 10]); i.e., step-by-step $\|u(t)\|_{r_i} \le C_i$ is shown for some increasing sequence of indices r_i, while the constants C_i are controlled so that $r_i \to \infty$ and $u(t) \in L^\infty$ can be guaranteed.

However, it is much more favourable to prove ultracontractivity by a priori estimates of the time-dependent Lebesgue norm $\|u(t)\|_{r(t)}$ obtained from logarithmic Gagliardo-Nirenberg inequalities. A discussion of such logarithmic inequalities of

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Sobolev type can be found in [2, 11], and in [5, 3] ultracontractivity was proved in the purely diffusive case $f = 0$ by this method.

In [9] a rather elementary exposition of this method was given. There the method was applied to doubly nonlinear diffusion equations without a nonlinearity, i.e. to the case $f = 0$, and optimal results were obtained by choosing $r(t)$ in an optimal way. Here it is shown how nonlinearities f can be incorporated, and again – compared to Moser iteration – the method of time-dependent exponents shows advantages: It is independent of the domain and boundary conditions, it allows to handle different types of nonlinearities in a flexible and unified way, and the estimates of the time-dependent norm $\|u(t)\|_{r(t)}$ are as optimal as the pregiven estimates of the nonlinearity.

In [7] resp. [12] similar results are obtained in special cases, namely the 1-homogeneous case $m = p'$ without nonlinearity resp. the p-diffusion case $m = 1$ with nonlinearity. However, their a priori estimates are based on logarithmic Sobolev inequalities, while for the general case of doubly nonlinear reaction-diffusion equations you need logarithmic Gagliardo-Nirenberg inequalities.

1.1. **Outline.** Let us summarize this paper: In the second section we describe how the method of time-dependent exponents can be applied to doubly nonlinear reaction-diffusion equations. We formulate logarithmic Gagliardo-Nirenberg inequalities and prove an ordinary differential inequality – depending on the estimates of the nonlinearity – for the time-dependent Lebesgue norm $\|u(t)\|_{r(t)}$ of a solution $u(t)$, where the variable exponent $r(t)$ is arbitrary.

Due to the validity of the differential inequality, $\|u(t)\|_{r(t)}$ is bounded by the solution $h(t)$ of the associated differential equation to the initial value $\|u(0)\|_a$. This solution $h(t)$ depends on $r(t)$, and to obtain optimal bounds of $\|u(t)\|_{r(t)}$, we minimize $h(t)$ w.r.t. the time-dependent exponent $r(t)$. Now, if there are minimizers $r(t)$ which blow up in arbitrarily short time while $h(t)$ stays bounded, then an optimal ultracontractivity estimate of $\|u(t)\|_\infty$ can be proved.

Because this procedure is rather general, in the third section we discuss the case of sublinear nonlinearities in detail. Especially, this example shows that a recalculation of the minimizer r gives better a priori estimates than using the optimal time-dependent exponent obtained in [9] for doubly nonlinear diffusion equations without nonlinearity. The main result about doubly nonlinear diffusions with sublinear nonlinearity is

Theorem 1.1. *Let $n \geq 2$, $p \geq 1$, $m > 1$, $a > 0$ and assume the validity of $a > \max\left(1, \frac{n}{p}\right)(1 - (m-1)(p-1))$. Let f be sublinear; i.e., there is a constant $H > 0$ such that $f(x,u)u \leq H|u|^2$ for all x and u, then every strong solution $u(t) \in L^a(\mathbb{R}^n)$ of $\dot{u} = \Delta_p u^{m-1} + f(u)$ to an initial value $u(0) \in L^a(\mathbb{R}^n)$ is instantely regularized to a function $u(t) \in L^\infty(\mathbb{R}^n)$, $t > 0$.*

More precisely, there is a constant $C_{n,p,m,a}$ such that

$$
\begin{aligned}
\|u(T)\|_\infty &\leq C_{n,p,m,a} \|u(0)\|_a^{\frac{ap}{ap+n((m-1)(p-1)-1)}} \\
&\quad \times (\exp(H((m-1)(p-1)-1)T) - 1)^{-\frac{n^2}{p(ap+n((m-1)(p-1)-1))}} \\
&\quad \times \exp\left(\frac{n^2\,(\exp(H((m-1)(p-1)-1)T)(H((m-1)(p-1)-1)T-1)+1)}{p(ap+n((m-1)(p-1)-1))\,(\exp(H((m-1)(p-1)-1)T)-1)}\right),
\end{aligned}
$$

and consequently the global attractor is contained in a bounded set in L^∞.

1.2. **Remarks.** During the text for the reader's convenience $u(t) \in L^a$ is assumed to be a strong solution, but the method also works for weak solutions, as the arguments in the proof of [12, Lemma 4.2] show. Regarding the existence of weak solutions, let us point out that at least in the doubly degenerated case $p > 2$, $m > 2$, weak solutions with $u(t) \in L^{m'}$ exist. For bounded domains this is proved e.g. in [1, 4, 10], a proof by Faedo-Galerkin method for general domains will be presented in a forthcoming paper.

Further, it is not essential that the equation is considered on $\mathbb{R}^n$. In fact, the independence on the domain is a special feature of the method of time-dependent exponents, so that analogous results are valid for bounded domains $\Omega \subset \mathbb{R}^n$ with Dirichlet or Neumann boundary, and Riemannian manifolds with an Euclidean type Sobolev inequality. However, pay attention to the fact that often the estimates of the nonlinearity f depend on the domain.

2. A priori estimates

Our proof of ultracontractivity of doubly nonlinear diffusion equations relies on logarithmic Gagliardo-Nirenberg inequalities and the pregiven estimates of the nonlinearity. Therefore, let us formulate logarithmic Gagliardo-Nirenberg inequalities, and let us combine these inequalities with the estimates of the nonlinearity to obtain a differential inequality for the time-dependent norm $\|u(t)\|_{r(t)}$ of a solution $u(t)$.

2.1. **Logarithmic Gagliardo-Nirenberg inequalities.** Logarithmic Gagliardo-Nirenberg inequalities can be used to estimate the diffusive part $\Delta_p u^{m-1}$ of doubly nonlinear diffusion equations.

Lemma 2.1 (Logarithmic Gagliardo-Nirenberg inequalities)**.** *The inequalities*

$$\int \frac{|u|^q}{\|u\|_q^q} \log\Big(\frac{|u|^q}{\|u\|_q^q}\Big) dx \le \frac{1}{1-q/p^*} \log\Big(C_{n,p,q}^q \frac{\|\nabla u\|_p^q}{\|u\|_q^q}\Big)$$

are valid for parameters $1 \le p < \infty$, $0 < q < \infty$ with $q/p^* < 1$ and functions $u \in L^q$ on $\mathbb{R}^n$ with $\nabla u \in L^p$. Hereby the constant C depends on n and p only in the case $p < n$, and on n, p and a finite upper bound of q in the case $p \ge n$.

An elementary proof of these inequalities is given in [9]. Now choose p^2/q instead of q and substitute u with $u^{q/p}$ to obtain the reformulation

$$\int \frac{|u|^p}{\|u\|_p^p} \log\Big(\frac{|u|^p}{\|u\|_p^p}\Big) dx \le \frac{p}{q-p^2/p^*} \log\Big(C_{n,p,q}^p \frac{\|\nabla u^{q/p}\|_p^p}{\|u\|_p^q}\Big)$$

valid for parameters $1 \le p < \infty$, $0 < q < \infty$ with $q > p^2/p^*$ and functions $u \in L^p$ with $\nabla u^{q/p} \in L^p$. Equivalently, these inequalities can be formulated in the parametric form

$$\begin{aligned} &\int \frac{|u|^p}{\|u\|_p^p} \log\Big(\frac{|u|^p}{\|u\|_p^p}\Big) dx - \mu \frac{\|\nabla u^{q/p}\|_p^p}{\|u\|_p^p} \\ &\le \frac{p}{q-p^2/p^*}\Big(\log\Big(\frac{pC_{n,p,q}^p}{e(q-p^2/p^*)\mu}\Big) + \log\big((\|u\|_p^{p-q})\big)\Big) \end{aligned} \tag{2.1}$$

valid for all $\mu > 0$ under the same conditions as before.

2.2. **A differential inequality for the time-dependent norm.** The basic differential inequality for the time-dependent Lebesgue norm $\|u(t)\|_{r(t)}$ is obtained from

$$\begin{aligned}\frac{d}{dt}\|u\|_r &= \|u\|_r\Big(-\frac{\dot r}{r^2}\log\Big(\int|u|^r\Big)+\frac{1}{r\int|u|^r}\int|u|^r\big(\dot r\log(|u|)+r\frac{u\dot u}{|u|^2}\big)\Big)\\ &= \|u\|_r\frac{\dot r}{r^2}\Big(\int\frac{|u|^r}{\|u\|_r^r}\log\Big(\frac{|u|^r}{\|u\|_r^r}\Big)+\frac{r^2}{\dot r\|u\|_r^r}\int\dot u u^{r-1}\Big)\end{aligned} \tag{2.2}$$

and $\dot u=\Delta_p u^{m-1}+f(u)$ by applying logarithmic Gagliardo-Nirenberg inequalities to estimate the diffusive part $\Delta_p u^{m-1}$, and the pregiven estimates of the nonlinearity f to estimate the reactive part.

To estimate the diffusive part, note that the p-Laplacian satisfies

$$\begin{aligned}\int(\Delta_p u^{m-1}))u^{r-1} &= -\int\big(|\nabla u^{m-1}|^{p-2}\nabla u^{m-1}\big)\cdot\big(\nabla u^{r-1}\big)\\ &= -(m-1)^{p-1}(r-1)\int|u|^{r+(m-2)(p-1)-2}|\nabla u|^p\\ &= -\frac{p^p(m-1)^{p-1}(r-1)}{|r+(m-1)(p-1)-1|^p}\int|\nabla u^{(r+(m-1)(p-1)-1)/p}|^p\,.\end{aligned}$$

Thus by substituting $w:=u^{r/p}$ and with the abbreviation $q:=p(r+(m-1)(p-1)-1)/r$ the bracket in equation (2.2) is up to the reactive part exactly the left hand side

$$\int\frac{|w|^p}{\|w\|_p^p}\log\Big(\frac{|w|^p}{\|w\|_p^p}\Big)-\frac{p^p(m-1)^{p-1}r^2(r-1)}{\dot r|r+(m-1)(p-1)-1|^p}\frac{\|\nabla w^{q/p}\|_p^p}{\|w\|_p^p}$$

of the parametric form of logarithmic Gagliardo-Nirenberg inequalities (2.1) with parameter μ given by $\mu=\dfrac{p^p(m-1)^{p-1}r^2(r-1)}{\dot r|r+(m-1)(p-1)-1|^p}$.

Hence we impose the restriction $q>p^2/p^*$ and apply the logarithmic Gagliardo-Nirenberg inequalities in their parametric form along with $\|w\|_p^{p-q}=\|u\|_r^{r(p-q)/p}$ to conclude

$$\begin{aligned}\frac{d}{dt}\|u\|_r \le \|u\|_r\frac{\dot r}{r^2}\frac{p}{q-p^2/p^*}\Big(\log\Big(\frac{pC^p}{e(q-p^2/p^*)\mu}\Big)+\frac{r(p-q)}{p}\log(\|u\|_r)\Big)\\ +\frac{\int f(u)u^{r-1}}{\|u\|_r^{r-1}}\,.\end{aligned}$$

This differential inequality depends strongly on the estimate of the nonlinearity f: If the nonlinearity satisfies an estimate of the form $\int f(u)u^{r-1}\le H(r,\|u\|_r)\|u\|_r^r$ with a function H, then the differential equation

$$\dot h=F(t)h\log(h)+G(t)h+H(r,h)h \tag{2.3}$$

corresponds to the differential inequality, and its solution $h(t)$ to the initial value $\|u(0)\|_a$ is a bound for $\|u(t)\|_{r(t)}$. Hereby the functions

$$F(t):=\frac{\dot r}{r}\frac{p-q}{q-p^2/p^*}=\frac{n(1-(m-1)(p-1))\dot r}{r(rp+n((m-1)(p-1)-1))}$$

and

$$G(t):=\frac{\dot r}{r^2}\frac{p}{q-p^2/p^*}\log\Big(\frac{pC^p}{e(q-p^2/p^*)\mu}\Big)$$

$$= \frac{n\dot{r}}{r(rp + n((m-1)(p-1)-1))} \times \log\Big(\frac{C^p\dot{r}(r+(m-1)(p-1)-1)^p}{ep^p(m-1)^{p-1}r(r-1)(r(1-p/p^*)+(m-1)(p-1)-1)}\Big)$$

are the same as in [9].

However, even if $\int f(u)u^{r-1}$ can not be estimated by $\|u\|_r$ only, but in terms of $\|u\|_r$ and $\|\nabla u^{rq/p^2}\|_p^p$ via an inequality of the form

$$\int f(u)u^{r-1} \le \epsilon\|\nabla u^{rq/p^2}\|_p^p + g(r,\epsilon,\|u\|_r)\|u\|_r^r\,,$$

then as long as ϵ is so small that $\mu - \frac{r^2}{\dot{r}}\epsilon$ is positive, the diffusive part compensates the reactive part estimated by $\epsilon\|\nabla u^{rq/p^2}\|_p^p$. Therefore, it is possible to derive a (more complicated) differential inequality for $\|u(t)\|_{r(t)}$ involving not only $r(t)$ but also $\epsilon(t)$ as time-dependent parameter (see [12] for an example where this situation is handeled successfully). For simplicity, in the following let us mainly discuss the case where the differential equation corresponding to the inequality has the form (2.3), although the general procedure does not depend strongly on the form of this ODE.

As $\|u(t)\|_{r(t)}$ is bounded by the solution $h(t)$ to the initial value $\|u(0)\|_a$, let us minimize h w.r.t. r (and possibly ϵ) to obtain an optimal bound of $\|u\|_r$. It depends on the complexity of the ODE for h how this is done in detail: If the ODE for h can be solved explicitly, then $h(T)$ depends on r and $\dot{r}$ via an integral $\int_0^T L(r,\dot{r})\,dt$. Thus minimizing $h(T)$ is equivalent to solving the Euler-Lagrange equations associated to this integral.

Unfortunately, the associated differential equation (2.3) often can not be solved analytically for general nonlinearities f, so there is no formula expressing $h(T)$ in terms of $h(0)$, r and $\dot{r}$. But still it is possible to minimize h on the fly by an optimal choice of r. In fact, denote the differential equation (2.3) by $\dot{h} = F(r(t),\dot{r}(t),h(t))$, then $h(T) = h(0) + \int_0^T F(r(t),\dot{r}(t),h(t))\,dt$ and minimizing $h(T)$ is equivalent to the minimization of $\int_0^T F(r(t),\dot{r}(t),h(t))\,dt$ w.r.t. r under the constraint that h solves the differential equation $\dot{h} = F(r,\dot{r},h)$. Because of $dh = F(r(t),\dot{r}(t),h(t))\,dt$, the infinitesimal dependence of h on r is given by $\frac{dh}{dr} = \frac{F(r,\dot{r},h)}{\dot{r}}$. Thus the corresponding Euler-Lagrange equations for r are $\frac{d}{dt}F_{\dot{r}} = F_r + F_h \cdot F/\dot{r}$. Hence the complete system of ODEs for the variable h and the control parameter r is

$$\begin{gathered}\dot{h} = F(r,\dot{r},h)\\ \frac{d}{dt}F_{\dot{r}} = F_r + \frac{F_h\cdot F}{\dot{r}}\,,\end{gathered}$$

and you are interested in solutions $h(t), r(t)$ to the initial value $h(0) = \|u(0)\|_a$ and the boundary values $r(0) = a$ and $r(T) = b$. In fact, then $\|u(T)\|_b \le h(T)$, where the right hand side depends on $\|u(0)\|_a$, a, b and T only. This is the a priori estimate we searched for, because if the right hand side stays bounded as $b \to \infty$, an inequality of the form $\|u(T)\|_\infty \le C(\|u(0)\|_a, a, T)$ has been proved. This inequality implies that in the (arbitrary small) time T the function $u(0) \in L^a$ is regularized to a function $u(T) \in L^\infty$, i.e. ultracontractivity has been shown. Moreover, the dependence of the right hand side on T gives you information about the long-time behaviour. For example, if the right hand side converges to 0 for

$T \to \infty$, every solutions u approaches 0, and if the right hand side is bounded in T, you have a global attractor in L^∞.

Although it seems unlikely to find an explicit solution h and minimizer r for general estimates of nonlinearities, in [9] an explicit solution is calculated for the purely diffusive case, and in the next section we succeed to calculate an explicit solution for sublinear nonlinearities. But even if you do not find an explicit solution, then still you can try to estimate the right hand side of (2.3) by a right hand side for which you can solve the ODE, and although the obtained estimate for u is worse, it may be enough to conclude ultracontractivity. A further alternative is to study the ODE numerically.

Let us end here our general description of the method of time-dependent exponents applied to doubly nonlinear diffusions. As a worthwhile example in the next section the particular case of a sublinear nonlinearity is discussed in detail.

3. Sublinear Nonlinearities

Sublinear nonlinearities provide an example for the fact that recalculating the minimizer r gives better estimates than simply using the minimizer $r(t) = \frac{1}{At+B} + \frac{n(1-(m-1)(p-1))}{p}$ obtained in [9] for doubly nonlinear diffusion equations without a nonlinearity.

A nonlinearity f is called sublinear, if there is a constant $H > 0$ such that

$$\int f(x,u(x))u(x)^{r-1}\,dx \le H\|u\|_r^r$$

holds for all $r > 0$ and $u \in L^r$. Particularly, a Caratheodory function with $f(x,u)u \le H|u|^2$ for all x and u is sublinear.

In this case $H(r,h) \equiv H$ is constant, and equation (2.3) is equivalent to

$$\dot{\log(h)} = F(t)\log(h) + G(t) + H\,.$$

Under the boundary conditions $r(0) = a$, $r(T) = b$, this equation has the explicit solution

$$\begin{aligned} h(T) = h(0)^{\frac{a(bp+n((m-1)(p-1)-1))}{b(ap+n((m-1)(p-1)-1))}} \quad &\exp\Bigg(\frac{bp+n((m-1)(p-1)-1)}{bn} \\ &\int_0^T \frac{n^2\dot r}{(pr+n((m-1)(p-1)-1))^2} \\ &\times \log\Big(\frac{nC^p(r+(m-1)(p-1)-1)^p\dot r}{ep^p(m-1)^{p-1}r(r-1)(pr+n((m-1)(p-1)-1))}\Big) \\ &+ H\frac{nr}{pr+n((m-1)(p-1)-1)}\,dt\Bigg) \end{aligned} \tag{3.1}$$

Now for the optimal time-dependent exponent $r(t) = \frac{1}{At+B} + \frac{n(1-(m-1)(p-1))}{p}$ of the doubly nonlinear diffusion equation without a nonlinearity the last term in the integral is

$$H\frac{nr}{pr+n((m-1)(p-1)-1)} = H\frac{n}{p} - H\frac{n^2((m-1)(p-1)-1)(At+B)}{p^2}\,,$$

and thus an additional factor $\exp\Big(H\frac{n}{p}T - H\frac{n^2((m-1)(p-1)-1)(AT^2/2+BT)}{p^2}\Big)$ arises in the calculation of $h(T)$ in [9]. With

$$B = \frac{p}{ap+n((m-1)(p-1)-1)}$$
$$AT = \frac{p^2(a-b)}{(ap+n((m-1)(p-1)-1))(bp+n((m-1)(p-1)-1))}\,.$$

this factor becomes

$$\exp\Big(\frac{n}{p}\Big(1-\frac{n((m-1)(p-1)-1)(p(a+b)+2n((m-1)(p-1)-1))}{2p(ap+n((m-1)(p-1)-1))(bp+n((m-1)(p-1)-1))}\Big)HT\Big)$$

and converges as $b\to\infty$ to

$$\exp\Big(\frac{n}{p}\Big(1-\frac{n((m-1)(p-1)-1)}{2p(ap+n((m-1)(p-1)-1))}\Big)HT\Big)\,.$$

Thus with the optimal time-dependent exponent for pure diffusions hypercontractivity and ultracontractivity of solutions can be proved also in the presence of sublinear reaction terms, but merely by an exponential estimate in time

$$\begin{aligned}\|u(T)\|_\infty &\le C_{n,p,m,a}\|u(0)\|_a^{\frac{ap}{ap+n((m-1)(p-1)-1)}}\,T^{-\frac{n}{ap+n((m-1)(p-1)-1)}}\\ &\quad\times\exp\Big(\frac{n}{p}\Big(1-\frac{n((m-1)(p-1)-1)}{2p(ap+n((m-1)(p-1)-1))}\Big)HT\Big)\,.\end{aligned}$$

Especially, this estimate does not provide any useful information about the long-time behaviour of the reaction-diffusion equation.

If instead, we recalculate the optimal time-dependent exponent, the Euler-Lagrange equation corresponding to the minimization of the integral in the explicit solution $h(T)$ is very similar to the equation obtained in [9]. In fact, by similar simplifications as there the Euler-Lagrange equations are equivalent to

$$\frac{\ddot r}{\dot r} = 2\frac{p\dot r}{pr+n((m-1)(p-1)-1)} + H((m-1)(p-1)-1)$$

and thus to

$$\frac{d}{dt}\left(\log(\dot r)\right) = 2\frac{d}{dt}\left(\log(pr+n((m-1)(p-1)-1))\right) + H((m-1)(p-1)-1)\,.$$

Hence we obtain

$$\dot r = -A\exp(H((m-1)(p-1)-1)t)(r+\frac{n((m-1)(p-1)-1)}{p})^2$$

with a constant A. Integrating this equation gives

$$\begin{aligned}&-\frac{1}{r+\frac{n((m-1)(p-1)-1)}{p}}\\ &= -\frac{A}{H((m-1)(p-1)-1)}\exp(H((m-1)(p-1)-1)t) - \frac{B}{H((m-1)(p-1)-1)}\end{aligned}$$

with a constant B, so that finally

$$r(t) = \frac{H((m-1)(p-1)-1)}{A\exp(H((m-1)(p-1)-1)t)+B} - \frac{n((m-1)(p-1)-1)}{p}$$

is obtained. The boundary conditions $r(0)=a$ and $r(T)=b$ imply

$$A\left(\exp(H((m-1)(p-1)-1)T)-1\right)$$

$$= \frac{Hp^2((m-1)(p-1)-1)(a-b)}{(ap+n((m-1)(p-1)-1))\,(bp+n((m-1)(p-1)-1))}$$

and

$$\begin{aligned} B &= Hp((m-1)(p-1)-1) \\ &\quad\times \frac{(bp+n((m-1)(p-1)-1)) + \frac{(b-a)p}{\exp(H((m-1)(p-1)-1)T)-1}}{(ap+n((m-1)(p-1)-1))\,(bp+n((m-1)(p-1)-1))} \end{aligned}$$

Thus minimizers blow up e.g. if $(m-1)(p-1)>1$, as then $A<0$, $B>0$.

Now let us calculate $h(T)$ for the recalculated minimizers r. The integral in formula (3.1) contains as first term

$$\frac{n^2\dot r}{(pr+n((m-1)(p-1)-1))^2} = -\frac{n^2}{p^2}A\exp(H((m-1)(p-1)-1)t)\,,$$

the second term is

$$\begin{aligned} &\log\Big(\tfrac{nC^p(r+(m-1)(p-1)-1)^p\dot r}{ep^p(m-1)^{p-1}r(r-1)(pr+n((m-1)(p-1)-1))}\Big) \\ &= \log\Big(-A\tfrac{nC^pHp((m-1)(p-1)-1)}{ep^{2p}(m-1)^{p-1}(A\exp(H((m-1)(p-1)-1)t)+B)^{p-1}} \\ &\quad\times\tfrac{\exp(H((m-1)(p-1)-1)t)}{(Hp((m-1)(p-1)-1)-n((m-1)(p-1)-1)(A\exp(H((m-1)(p-1)-1)t)+B))} \\ &\quad\times\tfrac{(Hp((m-1)(p-1)-1)+(p-n)((m-1)(p-1)-1)(A\exp(H((m-1)(p-1)-1)t)+B))^p}{(Hp((m-1)(p-1)-1)-(n((m-1)(p-1)-1)+p)(A\exp(H((m-1)(p-1)-1)t)+B))}\Big) \end{aligned}$$

and the third term is

$$H\frac{nr}{pr+n((m-1)(p-1)-1)} = \frac{n}{p}\left(Hp-n(A\exp(H((m-1)(p-1)-1)t)+B)\right).$$

Thus the integral in (3.1) is

$$\begin{aligned} &-\tfrac{n^2}{p^2}A\int_0^T \exp(H((m-1)(p-1)-1)t) \\ &\times\log\Big(-A\tfrac{nC^pHp((m-1)(p-1)-1)}{ep^{2p}(m-1)^{p-1}(A\exp(H((m-1)(p-1)-1)t)+B)^{p-1}} \\ &\times\tfrac{\exp(H((m-1)(p-1)-1)t)}{(Hp((m-1)(p-1)-1)-n((m-1)(p-1)-1)(A\exp(H((m-1)(p-1)-1)t)+B))} \\ &\times\tfrac{(Hp((m-1)(p-1)-1)+(p-n)((m-1)(p-1)-1)(A\exp(H((m-1)(p-1)-1)t)+B))^p}{(Hp((m-1)(p-1)-1)-(n((m-1)(p-1)-1)+p)(A\exp(H((m-1)(p-1)-1)t)+B))}\Big) \\ &+\tfrac{n}{p}\left(Hp-n(A\exp(H((m-1)(p-1)-1)t)+B)\right)\,dt \end{aligned}$$

Substitute $s=\exp(H((m-1)(p-1)-1)t)$, then the differential element is given by $ds = H((m-1)(p-1)-1)\exp(H((m-1)(p-1)-1)t)\,dt$, and we obtain

$$\begin{aligned} &-\tfrac{n^2}{p^2}\tfrac{A}{H((m-1)(p-1)-1)}\int_1^{\exp(H((m-1)(p-1)-1)T)} \\ &\times\log\Big(-A\tfrac{nC^pHp((m-1)(p-1)-1)s}{ep^{2p}(m-1)^{p-1}(As+B)^{p-1}(Hp((m-1)(p-1)-1)-n((m-1)(p-1)-1)(As+B))} \\ &\times\tfrac{(Hp((m-1)(p-1)-1)+(p-n)((m-1)(p-1)-1)(As+B))^p}{(Hp((m-1)(p-1)-1)-(n((m-1)(p-1)-1)+p)(As+B))}\Big) + \tfrac{n}{p}\left(Hp-n(As+B)\right)\,ds \\ &= -\tfrac{n^2}{p^2}\tfrac{A}{H((m-1)(p-1)-1)}\int_1^{\exp(H((m-1)(p-1)-1)T)}\log\Big(-A\tfrac{nC^pHp((m-1)(p-1)-1)}{ep^{2p}(m-1)^{p-1}}\Big) \\ &\quad+\log(s)+p\log\big(Hp((m-1)(p-1)-1) \\ &\quad+(p-n)((m-1)(p-1)-1)(As+B)\big) \\ &\quad-(p-1)\log(As+B)-\log\big(Hp((m-1)(p-1)-1) \end{aligned}$$

$$- n((m-1)(p-1)-1)(As+B)\big) - \log\big(Hp((m-1)(p-1)-1)$$
$$- (n((m-1)(p-1)-1)+p)(As+B)\big) + \tfrac{n}{p}\left(Hp - n(As+B)\right) ds$$

Now compute the integrals of each term and use

$$A + B = \frac{Hp((m-1)(p-1)-1)}{ap + n((m-1)(p-1)-1)}$$
$$A\exp(H((m-1)(p-1)-1)T) + B = \frac{Hp((m-1)(p-1)-1)}{bp + n((m-1)(p-1)-1)}$$

to obtain for the integral in (3.1) the expression

$$\begin{aligned}
& n^2 \frac{b-a}{(ap+n((m-1)(p-1)-1))\,(bp+n((m-1)(p-1)-1))} \\
& \times \Big(\log\Big(\tfrac{H^2 nC^p((m-1)(p-1)-1)^2(b-a)}{ep^{2p-3}(m-1)^{p-1}(ap+n((m-1)(p-1)-1))(bp+n((m-1)(p-1)-1))} \Big) \\
& \quad - \log\big(e^{H((m-1)(p-1)-1)T} - 1\big) \Big) \\
& + n^2 \tfrac{b-a}{(ap+n((m-1)(p-1)-1))(bp+n((m-1)(p-1)-1))} \\
& \times \tfrac{\exp(H((m-1)(p-1)-1)T)(H((m-1)(p-1)-1)T-1)+1}{\exp(H((m-1)(p-1)-1)T)-1} \\
& - \tfrac{n^2}{(p-n)((m-1)(p-1)-1)} \Big(\Big(1 + \tfrac{(p-n)((m-1)(p-1)-1)}{bp+n((m-1)(p-1)-1)}\Big) \\
& \times \Big(\log\Big(Hp((m-1)(p-1)-1) \Big(1 + \tfrac{(p-n)((m-1)(p-1)-1)}{bp+n((m-1)(p-1)-1)}\Big) \Big) - 1\Big) \\
& - \Big(1 + \tfrac{(p-n)((m-1)(p-1)-1)}{ap+n((m-1)(p-1)-1)}\Big) \Big(\log\Big(Hp((m-1)(p-1)-1) \\
& \times \Big(1 + \tfrac{(p-n)((m-1)(p-1)-1)}{ap+n((m-1)(p-1)-1)}\Big) \Big) - 1\Big)\Big) \\
& + \tfrac{n^2}{p^2} \tfrac{p-1}{H^2((m-1)(p-1)-1)^2} \tfrac{b-a}{(ap+n((m-1)(p-1)-1))(bp+n((m-1)(p-1)-1))} \\
& - \tfrac{n}{p} \tfrac{1}{((m-1)(p-1)-1)} \Big(\Big(1 - \tfrac{n((m-1)(p-1)-1)}{bp+n((m-1)(p-1)-1)}\Big) \\
& \times \Big(\log\Big(Hp((m-1)(p-1)-1) \Big(1 - \tfrac{n((m-1)(p-1)-1)}{bp+n((m-1)(p-1)-1)}\Big) \Big) - 1\Big) \\
& - \Big(1 - \tfrac{n((m-1)(p-1)-1)}{ap+n((m-1)(p-1)-1)}\Big) \Big(\log\Big(Hp((m-1)(p-1)-1) \\
& \times \Big(1 - \tfrac{n((m-1)(p-1)-1)}{ap+n((m-1)(p-1)-1)}\Big) \Big) - 1\Big)\Big) \\
& - \tfrac{n}{p} \tfrac{1}{((m-1)(p-1)-1)+p} \Big(\Big(1 - \tfrac{(n((m-1)(p-1)-1)+p)}{bp+n((m-1)(p-1)-1)}\Big) \\
& \times \Big(\log\Big(Hp((m-1)(p-1)-1) \Big(1 - \tfrac{(n((m-1)(p-1)-1)+p)}{bp+n((m-1)(p-1)-1)}\Big) \Big) - 1\Big) \\
& - \Big(1 - \tfrac{(n((m-1)(p-1)-1)+p)}{ap+n((m-1)(p-1)-1)}\Big) \Big(\log\Big(Hp((m-1)(p-1)-1) \\
& \times \Big(1 - \tfrac{(n((m-1)(p-1)-1)+p)}{ap+n((m-1)(p-1)-1)}\Big) \Big) - 1\Big)\Big) \\
& + n^3 \tfrac{1}{((m-1)(p-1)-1)} \tfrac{H((m-1)(p-1)-1)(b-a)}{(ap+n((m-1)(p-1)-1))(bp+n((m-1)(p-1)-1))} \\
& - n^4 H((m-1)(p-1)-1) \tfrac{b-a}{(ap+n((m-1)(p-1)-1))^2(bp+n((m-1)(p-1)-1))^2} \\
& \times \tfrac{p(b+a)+n((m-1)(p-1)-1)}{2}\,.
\end{aligned}$$

Only the first two terms depend on T, and because all the other terms converge for $b \to \infty$ to a constant depending on n, p, m and a, we obtain the estimate

$$\begin{aligned}
\|u(T)\|_\infty &\le C_{n,p,m,a} \|u(0)\|_a^{\frac{ap}{ap+n((m-1)(p-1)-1)}} \\
&\quad \times (\exp(H((m-1)(p-1)-1)T)-1)^{-\frac{n^2}{p(ap+n((m-1)(p-1)-1))}} \\
&\quad \times \exp\Big(\frac{n^2\,(\exp(H((m-1)(p-1)-1)T)(H((m-1)(p-1)-1)T-1)+1)}{p(ap+n((m-1)(p-1)-1))\,(\exp(H((m-1)(p-1)-1)T)-1)}\Big).
\end{aligned}$$

This estimate proves our main theorem.

Note that for large T the last factor increases like

$$\exp\Big(H\frac{n^2((m-1)(p-1)-1)}{p(ap+n((m-1)(p-1)-1))}T\Big),$$

while the first factor decreases like

$$\exp\Big(-H\frac{n^2((m-1)(p-1)-1)}{p(ap+n((m-1)(p-1)-1))}T\Big)$$

in T. Specially, for every $\epsilon > 0$ there is a constant D not depending on time T (but on the indices n, p, m and a and the norm of the initial value $\|u(0)\|_a$) such that $\|u(T)\|_\infty \le D$ for all $T \ge \epsilon$. Therefore, after an arbitrarily short time the function $u(t)$ is contained in a bounded set in L^∞. This estimate is much better than the one obtained before by using the optimal time-dependent exponent of the purely diffusive case. In particular, it shows that the global attractor of doubly nonlinear diffusions with sublinear reaction terms lies in L^∞.

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Jochen Merker
University of Rostock, D-18051 Rostock, Germany
E-mail address: jochen.merker@uni-rostock.de