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ON POSITIVE SOLUTIONS FOR A CLASS OF STRONGLY COUPLED P-LAPLACIAN SYSTEMS

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Dedicated to Jacqueline Fleckinger on the occasion of an international conference in her honor

Abstract. Consider the system

$$\begin{gathered} -\Delta_p u = \lambda f(u,v) \quad \text{in } \Omega \\ -\Delta_q v = \lambda g(u,v) \quad \text{in } \Omega \\ u = 0 = v \quad \text{on } \partial\Omega \end{gathered}$$

where $\Delta_s z = \operatorname{div}(|\nabla z|^{s-2}\nabla z)$, $s > 1$, λ is a non-negative parameter, and Ω is a bounded domain in $\mathbb{R}$ with smooth boundary $\partial\Omega$. We discuss the existence of a large positive solution for λ large when

$$\lim_{x\to\infty} \frac{f(x, M[g(x,x)]^{1/q-1})}{x^{p-1}} = 0$$

for every $M > 0$, and $\lim_{x\to\infty} g(x,x)/x^{q-1} = 0$. In particular, we do not assume any sign conditions on $f(0,0)$ or $g(0,0)$. We also discuss a multiplicity results when $f(0,0) = 0 = g(0,0)$.

1. Introduction

Consider the boundary-value problem

$$\begin{gathered} -\Delta_p u = \lambda f(u,v) \quad \text{in } \Omega \\ -\Delta_q v = \lambda g(u,v) \quad \text{in } \Omega \\ u = 0 = v \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where $\Delta_s z = \operatorname{div}(|\nabla z|^{s-2}\nabla z)$, $s > 1, \lambda$ is a non-negative parameter, and Ω is a bounded domain in $\mathbb{R}$ with smooth boundary $\partial\Omega$.

We are interested in the study of positive solutions to (1.1) when no conditions on $f(0,0), g(0,0)$ are assumed, in particular, they could be negative (semipositone systems). Semipositive problems are mathematically challenging area in the study of positive solutions (see [2] and [5]). For a review on semipositone problems, see [3]. In this paper we make the following assumptions:

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(H1) $f, g \in C^1((0,\infty)\times(0,\infty)) \cap C([0,\infty)\times[0,\infty))$ be monotone functions such that $f_u, f_v, g_u, g_v \geq 0$ and $\lim_{u,v\to\infty} f(u,v) = \lim_{u,v\to\infty} g(u,v) = \infty$.

(H2) $\lim_{x\to\infty} \dfrac{f(x, M[g(x,x)]^{1/q-1})}{x^{p-1}} = 0$ for every $M > 0$.

(H3) $\lim_{x\to\infty} \dfrac{g(x,x)}{x^{q-1}} = 0$.

We establish the following existence and multiplicity results:

Theorem 1.1. *Let* (H1)–(H3) *hold. Then there exists a positive number* λ^* *such that* (1.1) *has a large positive solution* (u,v) *for* $\lambda > \lambda^*$.

Theorem 1.2. *Let* (H1)–(H3) *hold. Further let* $F(s) = f(s, cs)$ *and* $G(s) = g(\tilde{c}s, s)$ *for any* $c, \tilde{c} > 0$ *and assume that* f *and* g *be sufficiently smooth functions in the neighborhood of zero with* $F(0) = G(0) = 0$, $F^{(k)}(0) = 0 = G^{(l)}(0)$ *for* $k = 1, 2, \dots [p-1]$, $l = 1, 2, \dots [q-1]$ *where* $[s]$ *denotes the integer part of* s. *Then* (1.1) *has at least two positive solutions provided* λ *is large.*

This paper extends the recent work in [1], where the authors study such systems with weaker coupling, namely systems of the form,

$$\begin{gathered}
-\Delta_p u = \lambda_1 \alpha(v) + \mu_1 \delta(u) \quad \text{in } \Omega \\
-\Delta_q v = \lambda_2 \beta(u) + \mu_2 \gamma(v) \quad \text{in } \Omega \\
u = 0 = v \quad \text{on } \partial\Omega
\end{gathered} \tag{1.2}$$

where $\lambda_1, \lambda_2, \mu_1$ and μ_2 are non-negative parameters, with the following conditions:

(C1) $\alpha, \beta, \delta, \gamma \in C^1(0,\infty) \cap C[0,\infty)$ be monotone functions such that
$$\lim_{x\to\infty} \alpha(x) = \lim_{x\to\infty} \beta(x) = \lim_{x\to\infty} \delta(x) = \lim_{x\to\infty} \gamma(x) = \infty.$$

(C2) $\lim_{x\to\infty} \dfrac{\alpha(M[\beta(x)]^{1/q-1})}{x^{p-1}} = 0$ for every $M > 0$.

(C3) $\lim_{x\to\infty} \dfrac{\delta(x)}{x^{p-1}} = \lim_{x\to\infty} \dfrac{\gamma(x)}{x^{q-1}} = 0$.

In [1], authors establish an existence result for the system (1.2) when $\lambda_1 + \mu_1$ and $\lambda_2 + \mu_2$ are large. In addition, for the case when $f(0) = h(0) = g(0) = \gamma(0) = 0$, authors discuss a multiplicity result for $\lambda_1 + \mu_1$ and $\lambda_2 + \mu_2$ large. Here we extend this study to classes of systems with much stronger coupling. Our approach is based on the method of sub-and supersolutions (see e.g. [4]). In Section 2, we will prove Theorem 1.1, in Section 3, we will prove Theorem 1.2 and in Section 4, we discuss some examples with strong coupling.

2. Proof of Theorem 1.1

We extend $f(u,v)$ and $g(u,v)$ for all $(u,v) \in \mathbb{R}^2$ smoothly such that there exists a constant $k_0 > 0$ such that $f(u,v), g(u,v) \geq -k_0$ for all $(u,v) \in \mathbb{R}^2$. We shall establish Theorem 1.1 by constructing a positive weak subsolution $(\psi_1, \psi_2) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ and a supersolution $(z_1, z_2) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ of (1.1) such that $\psi_i \leq z_i$ for $i = 1, 2$. That is, ψ_i, z_i satisfies

$(\psi_1,\psi_2)=(0,0)=(z_1,z_2)$ on $\partial\Omega$,

$$\begin{gathered}\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla\xi\,dx\le\lambda\int_\Omega f(\psi_1,\psi_2)\xi\,dx,\\ \int_\Omega |\nabla\psi_2|^{p-2}\nabla\psi_2\cdot\nabla\xi\,dx\le\lambda\int_\Omega g(\psi_1,\psi_2)\xi\,dx,\\ \int_\Omega |\nabla z_1|^{p-2}\nabla z_1\cdot\nabla\xi\,dx\ge\lambda\int_\Omega f(z_1,z_2)\xi\,dx,\\ \int_\Omega |\nabla z_2|^{p-2}\nabla z_2\cdot\nabla\xi\,dx\ge\lambda\int_\Omega g(z_1,z_2)\xi\,dx\end{gathered}$$

for all $\xi\in W:=\big\{\eta\in C_0^\infty(\Omega):\eta\ge 0\text{ in }\Omega\big\}$.

Let $\lambda_1^{(r)}$ the first eigenvalue of $-\Delta_r$ with Dirichlet boundary conditions and ϕ_r the corresponding eigenfunction with $\phi_r>0;\Omega$ and $\|\phi_r\|_\infty=1$ for $r=p,q$. Let $m,\delta>0$ be such that $|\nabla\phi_r|^r-\lambda_1^{(r)}\phi_r^r\ge m$ on $\overline{\Omega}_\delta=\{x\in\Omega|d(x,\partial\Omega)\le\delta\}$ for $r=p,q$. (This is possible since $|\nabla\phi_r|\ne 0$ on $\partial\Omega$ while $\phi_r=0$ on $\partial\Omega$ for $r=p,q$). We shall verify that

$$(\psi_1,\psi_2):=\Big(\big[\frac{\lambda k_0}{m}\big]^{1/p-1}\big(\frac{p-1}{p}\big)\phi_p^{p/p-1},\big[\frac{\lambda k_0}{m}\big]^{1/q-1}\big(\frac{q-1}{q}\big)\phi_q^{q/q-1}\Big),$$

is a subsolution of (1.1) for λ large. Let $\xi\in W$. Then

$$\begin{aligned}\int_\Omega|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla\xi\,dx&=\Big(\frac{\lambda k_0}{m}\Big)\int_\Omega\phi_p|\nabla\phi_p|^{p-2}\nabla\phi_p\cdot\nabla\xi\,dx\\ &=\Big(\frac{\lambda k_0}{m}\Big)\Big\{\int_\Omega|\nabla\phi_p|^{p-2}\nabla\phi_p\cdot\nabla(\phi_p\xi)\,dx-\int_\Omega|\nabla\phi_p|^p\xi\,dx\Big\}\\ &=\Big(\frac{\lambda k_0}{m}\Big)\Big\{\int_\Omega[\lambda_1^{(p)}\phi_p^p-|\nabla\phi_p|^p]\xi\,dx\Big\}.\end{aligned}$$

Similarly

$$\int_\Omega|\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla\xi\,dx=\Big(\frac{\lambda k_0}{m}\Big)\Big\{\int_\Omega[\lambda_1^{(q)}\phi_q^q-|\nabla\phi_q|^q]\xi\,dx\Big\}.$$

Now on $\overline{\Omega}_\delta$ we have $|\nabla\phi_r|^r-\lambda_1^{(s)}\phi_r^r\ge m$ for $r=p,q$. Which implies that

$$\begin{gathered}\frac{k_0}{m}\Big(\lambda_1^{(p)}\phi_p^p-|\nabla\phi_p|^p\Big)-f(\psi_1,\psi_2)\le 0,\\ \frac{k_0}{m}\Big(\lambda_1^{(q)}\phi_q^q-|\nabla\phi_q|^q\Big)-g(\psi_1,\psi_2)\le 0.\end{gathered}$$

Next on $\Omega-\overline{\Omega}_\delta$ we have $\phi_p\ge\mu$, $\phi_q\ge\mu$ for some $\mu>0$, and therefore for λ large

$$\begin{gathered}f(\psi_1,\psi_2)\ge\frac{k_0}{m}\lambda_1^{(p)}\ge\frac{k_0}{m}\lambda_1^{(p)}\phi_p^p-|\nabla\phi_p|^p,\\ g(\psi_1,\psi_2)\ge\frac{k_0}{m}\lambda_1^{(q)}\ge\frac{k_0}{m}\lambda_1^{(q)}\phi_q^q-|\nabla\phi_q|^q.\end{gathered}$$

Hence

$$\begin{gathered}\int_\Omega|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla\xi\,dx\le\lambda\int_\Omega f(\psi_1,\psi_2)\xi\,dx,\\ \int_\Omega|\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla\xi\,dx\le\lambda\int_\Omega g(\psi_1,\psi_2)\xi\,dx;\end{gathered}$$

i.e., (ψ_1, ψ_2) is a subsolution of (1.1) for λ large.

Next let e_r be the solution of $-\Delta_r e_r = 1$ in Ω, $e_r = 0$ on $\partial\Omega$ for $r = p, q$. Let $(z_1, z_2) := \Big(\frac{c}{\mu_p}\lambda^{1/p-1}e_p, [g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]^{1/q-1}\lambda^{1/q-1}e_q\Big)$ where $\mu_r = \|e_r\|_\infty$; $r = p, q$. Then

$$
\begin{aligned}
\int_\Omega |\nabla z_1|^{p-2}\nabla z_1 \cdot \nabla\xi\, dx &= \lambda\big(\frac{c}{\mu_p}\big)^{p-1}\int_\Omega |\nabla e_p|^{p-2}\nabla e_p \cdot \nabla\xi\, dx \\
&= \frac{1}{(\mu_p)^{p-1}}(c\lambda^{1/p-1})^{p-1}\int_\Omega \xi\, dx.
\end{aligned}
$$

By (H2) we can choose c large enough so that

$$
\begin{aligned}
&\frac{1}{(\mu_p)^{p-1}}(c\lambda^{1/p-1})^{p-1}\int_\Omega \xi\, dx \\
&\geq \lambda\int_\Omega f(c\lambda^{1/p-1}, [g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]^{1/q-1}\lambda^{1/q-1}\mu_q)\xi\, dx \\
&\geq \lambda\int_\Omega f(c\lambda^{1/p-1}\frac{e_p}{\mu_p}, [g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]^{1/q-1}\lambda^{1/q-1}e_q)\xi\, dx \\
&= \lambda\int_\Omega f(z_1, z_2)\xi\, dx.
\end{aligned}
$$

Next

$$
\begin{aligned}
\int_\Omega |\nabla z_2|^{q-2}\nabla z_2 \cdot \nabla\xi\, dx &= \lambda[g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]\int_\Omega |\nabla e_q|^{q-2}\nabla e_q \cdot \nabla\xi\, dx \\
&= \lambda[g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]\int_\Omega \xi\, dx
\end{aligned}
$$

By (H3) choose c large so that $\frac{1}{\lambda^{1/q-1}}\mu_q \geq \frac{[g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]^{1/q-1}}{c\lambda^{1/p-1}}$, then

$$
\begin{aligned}
&\lambda[g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]\int_\Omega \xi\, dx \\
&\geq \lambda\int_\Omega g\big(c\lambda^{1/p-1}, [g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]^{1/q-1}\lambda^{1/q-1}\mu_q\big)\xi\, dx \\
&\geq \lambda\int_\Omega g\big(c\lambda^{1/p-1}\frac{e_p}{\mu_p}, [g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]^{1/q-1}\lambda^{1/q-1}e_q\big)\xi\, dx \\
&= \lambda\int_\Omega g(z_1, z_2)\xi\, dx;
\end{aligned}
$$

i.e., (z_1, z_2) is a supersolution of (1.1) with $z_i \geq \psi_i$ for c large, $i = 1, 2$. (Note $|\nabla e_r| \neq 0; \partial\Omega$ for $r = p, q$).

Thus, there exists a solution (u, v) of (1.1) with $\psi_1 \leq u \leq z_1, \psi_2 \leq v \leq z_2$. This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

To prove Theorem 1.2, we will construct a subsolution (ψ_1, ψ_2), a strict supersolution (ζ_1, ζ_2), a strict subsolution (w_1, w_2), and a supersolution (z_1, z_2) for (1.1) such that $(\psi_1, \psi_2) \leq (\zeta_1, \zeta_2) \leq (z_1, z_2)$, $(\psi_1, \psi_2) \leq (w_1, w_2) \leq (z_1, z_2)$, and

$(w_1, w_2) \not\leq (\zeta_1, \zeta_2)$. Then (1.1) has at least three distinct solutions (u_i, v_i), $i = 1, 2, 3$, such that $(u_1, v_1) \in [(\psi_1, \psi_2), (\zeta_1, \zeta_2)], (u_2, v_2) \in [(w_1, w_2), (z_1, z_2)]$, and

$$(u_3, v_3) \in [(\psi_1, \psi_2), (z_1, z_2)] \setminus \big([(\psi_1, \psi_2), (\zeta_1, \zeta_2)] \cup [(w_1, w_2), (z_1, z_2)]\big).$$

We first note that $(\psi_1, \psi_2) = (0, 0)$ is a solution (hence a subsolution). As in Section 2, we can always construct a large supersolution (z_1, z_2). We next consider

$$\begin{gathered} -\Delta_p w_1 = \lambda \tilde{f}(w_1, w_2) \quad \text{in } \Omega \\ -\Delta_q w_2 = \lambda \tilde{g}(w_1, w_2) \quad \text{in } \Omega \\ w_1 = 0 = w_2 \quad \text{on } \partial\Omega \end{gathered} \tag{3.1}$$

where $\tilde{f}(u, v) = f(u, v) - 1$ and $\tilde{g}(u, v) = g(u, v) - 1$. Then by Theorem 1.1, (3.1) has a positive solution (w_1, w_2) when λ is large. Clearly this (w_1, w_2) is a strict subsolution of (1.1). Finally we construct the strict supersolution (ζ_1, ζ_2).

To do so, we let ϕ_p, ϕ_q as described in Section 2. We note that there exists positive constants c_1 and c_2 such that

$$\phi_p \leq c_1 \phi_q \quad \text{and} \quad \phi_q \leq c_2 \phi_p. \tag{3.2}$$

Let $(\zeta_1, \zeta_2) = (\epsilon\phi_p, \epsilon\phi_q)$ where $\epsilon > 0$. Let $H_p(s) := \lambda_1^{(p)} s^{p-1} - \lambda f(s, c_2 s)$ and $H_q(s) := \lambda_1^{(q)} s^{q-1} - \lambda g(c_1 s, s)$. Observe that $H_p(0) = H_q(0) = 0$, $H_p^{(k)}(0) = 0 = H_q^{(l)}(0)$ for $k = 1, 2, \dots [p-2]$ and $l = 1, 2, \dots [q-2]$. $H_p^{(p-1)}(0) > 0$ and $H_q^{(q-1)}(0) > 0$ if p, q are integers, while $\lim_{r\to 0} H^{([p])}(r) = +\infty = \lim_{r\to 0} H^{([q])}(r)$ if p, q are not integers. Thus there exists θ such that $H_p(s) > 0$ and $H_q(s) > 0$ for $s \in (0, \theta]$. Hence for $0 < \epsilon \leq \theta$ we have

$$\begin{aligned} \lambda_1^{(p)}(\zeta_1)^{p-1} = \lambda_1^{(p)}(\epsilon\phi_p)^{p-1} &> \lambda f(\epsilon\phi_p, c_2\epsilon\phi_p) \\ &\geq \lambda f(\epsilon\phi_p, \epsilon\phi_q) \\ &= \lambda f(\zeta_1, \zeta_2) \quad x \in \Omega, \end{aligned} \tag{3.3}$$

and similarly we get

$$\begin{aligned} \lambda_1^{(q)}(\zeta_2)^{q-1} = \lambda_1^{(q)}(\epsilon\phi_q)^{q-1} &> \lambda g(c_1\epsilon\phi_q, \epsilon\phi_q) \\ &\geq \lambda g(\epsilon\phi_p, \epsilon\phi_q) \\ &= \lambda g(\zeta_1, \zeta_2), \quad x \in \Omega. \end{aligned} \tag{3.4}$$

Using the inequalities (3.3) and (3.4) we have,

$$\begin{aligned} \int_\Omega |\nabla\zeta_1|^{p-2}\nabla\zeta_1 \cdot \nabla\xi \, dx = \epsilon^{p-1} \int_\Omega |\nabla\phi_p|^{p-2}\nabla\phi_p \cdot \nabla\xi \\ = \int_\Omega \lambda_1^{(p)}(\epsilon\phi_p)^{p-1}\xi \, dx \\ > \lambda \int_\Omega f(\zeta_1, \zeta_2)\xi \, dx. \end{aligned}$$

Similarly we have

$$\int_\Omega |\nabla\zeta_2|^{q-2}\nabla\zeta_2 \cdot \nabla\xi \, dx > \lambda \int_\Omega g(\zeta_1, \zeta_2)\xi \, dx$$

Thus (ζ_1, ζ_2) is a strict supersolution. Here we can choose ϵ small so that $(w_1, w_2) \not\leq (\zeta_1, \zeta_2)$.

Hence there exists solutions $(u_1, v_1) \in [(\psi_1, \psi_2), (\zeta_1, \zeta_2)]$, $(u_2, v_2) \in [(w_1, w_2), (z_1, z_2)]$, and $(u_3, v_3) \in [(\psi_1, \psi_2), (z_1, z_2)] \setminus \big([(\psi_1, \psi_2), (\zeta_1, \zeta_2)] \cup [(w_1, w_2), (z_1, z_2)]\big)$. Since $(\psi_1, \psi_2) \equiv (0, 0)$ is a solution it may turn out that $(u_1, v_1) \equiv (\psi_1, \psi_2) \equiv (0, 0)$. In any case we have two positive solutions (u_2, v_2) and (u_3, v_3). Hence Theorem 1.2 holds.

Remark 3.1. Note that in the construction of the supersolution (ζ_1, ζ_2) we require the conditions at zero on F and G only for the constants $c = c_2$ and $\tilde{c} = c_1$.

4. Examples

Example 4.1. Consider the problem

$$\begin{gathered} -\Delta_p u = \lambda[v^\alpha + (uv)^\beta - 1] \quad \text{in } \Omega \\ -\Delta_q v = \lambda[u^\sigma + (uv)^{\gamma/2} - 1] \quad \text{in } \Omega \\ u = 0 = v \quad \text{on } \partial\Omega \end{gathered} \tag{4.1}$$

where $\alpha, \beta, \sigma, \gamma$ are positive parameters. Then it is easy to see that (4.1) satisfies the hypotheses of Theorem 1.1 if $\max\{\sigma, \gamma\}\frac{\alpha}{q-1} < p - 1$, $(\max\{\sigma, \gamma\}\frac{1}{q-1} + 1)\beta < p - 1$ and $\max\{\sigma, \gamma\} < q - 1$.

Example 4.2. Let

$$h(x) = \begin{cases} x^\alpha; & x \le 1 \\ \frac{\alpha}{\sigma}x^\sigma + (1 - \frac{\alpha}{\sigma}); & x > 1, \end{cases} \quad \text{and} \quad \gamma(x) = \begin{cases} x^\mu; & x \le 1 \\ \frac{\mu}{\delta}x^\delta + (1 - \frac{\mu}{\delta}); & x > 1, \end{cases}$$

where $\alpha, \sigma, \mu, \delta$ are positive parameters. Here we assume $\alpha > p-1$ if p is an integer, $\alpha > [p]$ if p is not an integer, $\mu > q - 1$ if q is an integer and $\mu > [q]$ if q is not an integer.

Consider the problem

$$\begin{gathered} -\Delta u = \lambda[1 + u^\beta]h(v) \quad \text{in } \Omega \\ -\Delta v = \lambda\gamma(u) \quad \text{in } \Omega \\ u = 0 = v \quad \text{on } \partial\Omega \end{gathered} \tag{4.2}$$

where $0 \le \beta < p - 1$. Then it is easy to see that (4.2) satisfies the hypotheses of Theorem 1.2 if $\delta\sigma < [p - 1 - \beta](q - 1)$ and $\delta < q - 1$.

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