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EXISTENCE, MULTIPLICITY, AND BIFURCATION IN SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Dedicated to my friend Klaus Schmitt

Abstract. We prove new non-resonance conditions for boundary value problems for two dimensional systems of ordinary differential equations. We apply these results to the existence of solutions to nonlinear problems. We then study global bifurcation for such systems of ordinary differential equations Rotation numbers are associated with solutions and are shown to be invariant along bifurcating continua. This invariance is then used to analyze the global structure of the bifurcating continua, and to demonstrate the existence of multiple solutions to some boundary value problems.

1. Introduction

The purpose of this paper is to prove some existence, bifurcation, and multiplicity results for boundary value problems for two dimensional systems of ordinary differential equations. In this respect the paper is a continuation of [12]. Consider the parameter dependent family of boundary-value problems

$$\frac{dw}{dt} = F(\lambda, w, t), \quad t \in [0, \omega] \tag{1.1}$$

$$Bw = 0 \tag{1.2}$$

where $F = (F_1, F_2) \in C(\mathbb{R}\mathbb{R}^2 \times [0, \omega], \mathbb{R}^2)$, $t \in [0, \omega]$, $w = (u, v) \in \mathbb{R}^2$, and $\lambda \in \mathbb{R}$ is a parameter. We concentrate here on $Bw := (u(0), u(\omega))$, which we will call the Dirichlet problem. Our methods work just as well with many other boundary conditions, including the periodic problem and $Bw = (u(0), v(\omega))$. The most general form we allow for the function F will usually be

$$F(\lambda, w, t) = B(\lambda, t)w + g(\lambda, w, t) \tag{1.3}$$

with $B(\lambda, t) = \lambda J + A(t)$ or $B(\lambda, t) = \lambda A(t)$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & -p(t) \\ q(t) & 0 \end{pmatrix}$$

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with $p, q \in L^\infty(0,\omega)$ and $g(\lambda, w, t) = o(|w|)$ as $|w| \to 0$ (or ∞), uniformly with respect to λ and t in compact sets.

If $w = w(t) = (u(t), v(t))$ is an ω-periodic solution of (1.1) with $w(t) \neq 0$ for all t, then the mapping $t \mapsto \frac{w(t)}{|w(t)|}$ defines a mapping from the circle S^1 into itself. If φ denotes this mapping then the Brouwer degree $\deg(\varphi)$ is defined. It is the same as the rotation number of $w(t)$ (with respect to the origin). If $\theta = \tan^{-1}(\frac{v}{u})$ then

$$\frac{d\theta}{dt} = \frac{v'u - vu'}{u^2 + v^2}$$

and the rotation number of such an ω-periodic solution is

$$\operatorname{rot}(w) = \frac{1}{2\pi}\int_0^\omega \frac{uv' - vu'}{u^2 + v^2}\,dt = \frac{1}{2\pi}\int_0^\omega \frac{F_2 u - vF_1}{u^2 + v^2}\,dt.$$

For most of our results we will use the rotation number to distinguish solutions and branches of solutions. This idea was used in [12] to study global bifurcation from zero and solution multiplicity. In [12] only problems with $B(\lambda, t) = \lambda J$ in (1.3) were considered, and bifurcation from infinity was not studied, as it is here. Rotation numbers can be assigned to solutions of non-periodic boundary value problems, such as the two already mentioned, by appropriately extending the solutions to a larger interval on which the extension is periodic and the rotation number is an integer. In the non-systems case of second order scalar Sturm-Liouville boundary value problems on an interval $[0,\omega]$, bifurcating branches can be distinguished by the number of solution nodal points in $[0,\omega[$ [8]. Our methods are based upon Leray-Schauder degree and change of degree as the parameter $\lambda \in \mathbb{R}$ varies. Krasnosel'skii [6] first used Leray-Schauder degree to prove the existence of bifurcation at eigenvalues of odd multiplicity and Rabinowitz [8] later showed global bifurcation from these eigenvalues and proved fundamental results on global structure of bifurcating continua. These ideas and results have been applied and extended by subsequent researchers in deep and ingenious ways to understand bifurcations and solution structure for nonlinear boundary value problems. The reader is referred to the fundamental paper [8] or the expositions in [10] or [2] for the fundamental ideas. The rotation numbers of solutions have been used before to analyze global solution structure for boundary value problems, see [3], [4]. The paper [1] improves some results of [12] for superlinear systems, and makes interesting use of rotation number in connection with the Capietto-Mawhin-Zanolin continuation theorem (see [4]).

In §2 we study linear systems. In §3 we apply the ideas of §2 to study existence under nonresonance conditions. In §4 we study bifurcation from a line of trivial solutions, making use of rotation number to characterize branches. In §5 we prove results on bifurcation from infinity. In §6 we obtain conditions for bifurcating branches to bend to the left or to the right. In §7 we apply the earlier results to prove a theorem on multiplicity of solutions.

In the sequel, for $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$ we let $|x| := (\sum_{i=1}^n x_i^2)^{1/2}$; with T indicating the transpose. We will sometimes omit the T, so by $w = (u, v)$ we usually mean the column vector. For $w \in C([A,B], \mathbb{R}^n)$, $\|w\| := \max_{[A,B]} |w(t)|$, and for $w \in L^p((A,B), \mathbb{R}^n)$, $1 \le p \le \infty$, let $\|w\|_p := (\int_A^B |w(t)|^p\,dt)^{1/p}$.

2. Linear systems

We begin by considering linear systems of the form

$$\begin{aligned}\frac{du}{dt} &= -p(t)v \\ \frac{dv}{dt} &= q(t)u\end{aligned} \tag{2.1}$$

for $t \in [0, \omega]$ where $p, q \in L^\infty(0, \omega)$, together with boundary conditions

$$B(u, v) = (0, 0). \tag{2.2}$$

The boundary operator in (2.2) is linear and could represent T-periodic boundary conditions, $B(u, v) = (u(\omega), v(\omega)) - (u(0), v(0))$, the boundary operator $B(u, v) = (u(0), u(\omega))$, or $Bw = (u(0), v(\omega))$, or possibly others. The admissible boundary conditions are those that allow a well defined rotation number to be associated with nontrivial solutions to (2.1), (5), (2.2). If $w = (u, v)^T$ is a nontrivial solution with $w(\omega) - w(0) = 0$ then there is an integer rotation number defined by

$$\operatorname{rot}(w) = \frac{1}{2\pi} \int_0^\omega \frac{q(t)u^2 + p(t)v^2}{u^2 + v^2}\, dt.$$

In the case of non periodic boundary conditions such as $Bw = (u(0), u(\omega)) = (0, 0)$, more care must be taken to obtain an integer rotation number. In the latter case, extend $p(t)$ and $q(t)$ respectively to functions $\widetilde{p}(t), \widetilde{q}(t)$, on $[-\omega, \omega]$, so that both are even and extend $u(t)$ to $\widetilde{u}(t)$, odd on $[-\omega, \omega]$ and $v(t)$ to $\widetilde{v}(t)$, even on $[-\omega, \omega]$. Then $\widetilde{w} = (\widetilde{u}, \widetilde{v})^T$ satisfies

$$\begin{aligned}\frac{d\widetilde{u}}{dt} &= -\widetilde{p}(t)\widetilde{v} \\ \frac{d\widetilde{v}}{dt} &= \widetilde{q}(t)\widetilde{u}\end{aligned}$$

on $[-\omega, \omega]$. Moreover $\widetilde{w}$ satisfies the periodic conditions $\widetilde{w}(\omega) - \widetilde{w}(-\omega) = 0$. We will henceforth refer to $\widetilde{w}$ as the *odd/even* extension of w. We define the rotation number of w to be the rotation number of $\widetilde{w}$:

$$\operatorname{rot}(w) := \operatorname{rot}(\widetilde{w}) = \frac{1}{2\pi} \int_{-\omega}^{\omega} \frac{\widetilde{q}(t)\widetilde{u}^2 + \widetilde{p}(t)\widetilde{v}^2}{\widetilde{u}^2 + \widetilde{v}^2}\, dt \tag{2.3}$$

is a well defined integer. Notice that the rotation number has the properties of Brouwer degree. Indeed, in the periodic case it is the same as the degree of the map from $S^1 (= [0, \omega]/\{0, \omega\}) \to S^1$ defined by $t \longmapsto w(t)/|w(t)|$, with a similar identification in the second boundary condition considered above. One may also associate a rotation number with nontrivial solutions satisfying the boundary condition $u(0) = 0$, $v(\omega) = 0$, and others.

We wish to compare the rotation numbers associated with solutions of two different systems. Let $p_j, q_j \in L^\infty([0, T], \mathbb{R})$ for $j = 1, 2$ and let $w_j = (u_j, v_j)$ $(j = 1, 2)$ be non-trivial solutions of the jth problem, so

$$\frac{du_j}{dt} = -p_j(t)v_j, \quad \frac{dv_j}{dt} = q_j(t)u_j, \quad (u_j(T), v_j(T)) = (u_j(0), v_j(0)). \tag{2.4}$$

Lemma 2.1. *Let $p_j, q_j \in L^\infty([0, T], \mathbb{R}), w_j = (u_j, v_j)$ $(j = 1, 2)$ be a non trivial solution of (2.4) for $j = 1, 2$ respectively. Suppose that we have*

$$a(t) := \max(p_1(t), q_1(t)) \le b(t) := \min(p_2(t), q_2(t)) \quad a.e. \tag{2.5}$$

Then $\operatorname{rot}(w_1) \le \operatorname{rot}(w_2)$. *If there is a set* $E \subset [0,T]$ *of positive Lebesgue measure such that strict inequality holds in either inequality* (2.5) *for* $t \in E$*, then* $\operatorname{rot}(w_1) < \operatorname{rot}(w_2)$.

Proof. We have

$$\begin{aligned}
\operatorname{rot}(w_1) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{q_1(t)u_1^2 + p_1(t)v_1^2}{u_1^2+v_1^2}\,dt \\
&\le \frac{1}{2\pi}\int_0^{2\pi} a(t)\,dt \\
&\le \frac{1}{2\pi}\int_0^{2\pi} b(t)\,dt \\
&\le \frac{1}{2\pi}\int_0^{2\pi} \frac{q_2(t)u_2^2 + p_2(t)v_2^2}{u_2^2+v_2^2} dt = \operatorname{rot}(w_2)
\end{aligned}$$

which proves the first part of the claim. If there were a set of positive measure E on which $a(t) < b(t)$ for $t \in E$, then the second integral inequality would be strict also, and this would imply that $\operatorname{rot}(w_1) < \operatorname{rot}(w_2)$. $\square$

The same conclusion holds if we impose the boundary conditions $u(0) = 0, u(\pi) = 0$ or $u(0) = 0, v(\pi) = 0$. In those cases we consider the appropriate periodic extensions of the nontrivial solutions w to define $\operatorname{rot}(w)$. For instance, in the $u(0) = 0, u(\pi) = 0$ case, we extend u to be odd on $[-\pi,\pi]$, and v to be even on $[-\pi,\pi]$, and then both to be $2\pi-$periodic; we extend p and q to be even on $[-\pi,\pi]$ and then also 2π-periodic. Then the rotation number of w is well-defined. Using this and similar arguments one may establish the conclusion of the lemma if these other boundary conditions are imposed on (2.1).

In this paper we will, for the sake of concreteness, in the main consider the boundary conditions $B(u,v) := (u(\pi), u(0)) = (0,0)$.

We now consider nonresonance conditions. Let $\mu \in \mathbb{R}$ and consider the problem

$$\begin{aligned}
\frac{du}{dt} &= -\mu v \\
\frac{dv}{dt} &= \mu u
\end{aligned} \tag{2.6}$$

with boundary conditions

$$(u(\pi), u(0)) = (0,0). \tag{2.7}$$

It is easy to check that the eigenvalues of the above system consists of the set of integers $\mathbb{Z}$, and for $n = 0$ an eigenfunction is $w_0 = (0,1)$, while for each $n \in \mathbb{Z}\backslash\{0\}$, and eigenfunction is $w_n(t) = (\sin(nt), -\cos(nt))$. The associated rotation numbers (as defined above) are $\operatorname{rot}(w_n) = n$.

We introduce a notation useful here. Suppose F and G are real valued Lebesgue measurable functions on an interval I. The notation $F(t) \lesssim G(t)$ on I (or just $F \lesssim G$ on I) will mean that $F(t) \le G(t)$ a.e. on I and there is a set of positive measure in I on which the inequality is strict. We can now state and prove the lemma

Lemma 2.2. *Consider the problem* (2.1) *with* $\omega = \pi$ *and* $p, q \in L^\infty(0,\pi)$ *and boundary conditions* $(u(\pi), u(0)) = (0,0)$*. Suppose that there is* $n \in \mathbb{Z}$ *such that* $n \lesssim p(t) \lesssim (n+1)$ *and* $n \lesssim q(t) \lesssim (n+1)$ *on* $[0,\pi]$*. Then the problem has no non-trivial solutions.*

Proof. Suppose there is a nontrivial solution $w = (u, v)^T$. Let $\widetilde{w}$ be the odd/even extension of w. Then $\mathrm{rot}(w) := \mathrm{rot}(\widetilde{w})$ is well defined and is an integer. However the rotation numbers of (odd/even extended) solutions to the system (2.6), (2.7) with $\mu = n$ and $\mu = n+1$ are n and $n+1$, respectively, and this implies $n < \mathrm{rot}(w) < n+1$, which is impossible. This proves the lemma. $\square$

It should be clear that analogous results are true with other boundary conditions, or on other intervals, with similar proofs.

We have an immediate corollary regarding the problem

$$\begin{aligned} \frac{du}{dt} + p(t)v &= f(t) \\ \frac{dv}{dt} - q(t)u &= g(t) \end{aligned} \tag{2.8}$$

$$u(0) = 0, \quad u(\pi) = 0. \tag{2.9}$$

Corollary 2.3. *Let $p, q \in L^\infty(0, \pi)$ satisfy the conditions of the lemma, and let $f, g \in L^1(0, \pi)$. Then there is a unique solution to (2.8),* (2.9)*.*

Remark 2.4. By a solution to (2.8), (2.9) we mean a pair of functions (u, v), each absolutely continuous on $[0, \pi]$, satisfying (2.9) and also satisfying (2.8) *a.e.*

Remark 2.5. The lemmas and corollary hold with appropriate modifications for many other boundary value problems, such as the periodic one.

Now suppose $p, q \in L^\infty(0, \pi)$ satisfy

$$-M \lesssim p(t), q(t) \lesssim M \text{ a.e.} \tag{2.10}$$

for some $M > 0$. Let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & p(t) \\ -q(t) & 0. \end{pmatrix}$$

We will now study parameter dependent linear systems of the forms

$$\frac{dw}{dt} + \lambda J w + A(t) w = 0 \tag{2.11}$$

and

$$\frac{dw}{dt} + \lambda A(t) w = 0. \tag{2.12}$$

with the boundary conditions (2.9), i.e.,

$$u(0) = 0, \quad u(\pi) = 0, \text{ where } w(t) = (u(t), v(t)). \tag{2.13}$$

We first analyze the problem (2.11) with boundary conditions (2.13). Extend p and q to P and Q, respectively, even on $[-\pi, \pi]$, and then to be 2π-periodic. This is equivalent to letting $P(t + \pi) := p(\pi - t)$ and $Q(t + \pi) := q(\pi - t)$for $0 \le t \le \pi$ and then extending P and Q to be 2π periodic on $\mathbb{R}$. If $w = (u, v)$ satisfies the boundary conditions, we extend u to be even with respect to 0 and 2π periodic on $\mathbb{R}$, and extend v to be odd with respect to zero and also 2π periodic. As before, we call this the *odd/even extension of* w. For each $\mu \in \mathbb{R}$ let $W_\mu = (U_\mu, V_\mu)$ be the solution to the initial-value problem

$$
\begin{aligned}
&\frac{dU}{dt} + \mu V + P(t)V = 0\\
&\frac{dV}{dt} - \mu U - Q(t)U = 0\\
&U(0) = 0, \quad V(0) = 1.
\end{aligned}
\tag{2.14}
$$

We can still define a real valued function Ψ by the equation

$$
\begin{aligned}
\Psi(\mu) &:= \frac{1}{2\pi}\int_0^{2\pi} \frac{(\mu + P(t))V_\mu^2 + (\mu + Q(t))U_\mu^2}{U_\mu^2 + V_\mu^2}\,dt\\
&= \mu + \frac{1}{2\pi}\int_0^{2\pi} \frac{P(t)V_\mu^2 + Q(t)U_\mu^2}{U_\mu^2 + V_\mu^2}\,dt.
\end{aligned}
\tag{2.15}
$$

If also $U_\mu(\pi) = 0$ then W_μ will be 2π-periodic and have an integral rotation number (this is true because if (U, V) is any solution on $[0, \pi]$ satisfying the boundary conditions, then it has a 2π periodic extension satisfying the differential equations, as was shown in Section 2. But there is only one solution satisfying the initial conditions). Conversely, if $\Psi(\mu)$ is an integer, then the change in angle of $W_\mu(t)$ with respect to the origin over the interval $0 \le t \le 2\pi$ is an integral multiple of 2π, and hence $W_\mu(t)$ must be 2π-periodic. From this and that $P(t)$ and $Q(t)$ are even and 2π- periodic one can deduce that $U(t)$ is odd and $V(t)$ even, and hence that $U(\pi) = 0$. Thus we have the following result.

Lemma 2.6. *W_μ satisfies the boundary conditions* (2.13) *if and only if* $\Psi(\mu) \in \mathbb{Z}$.

The solution W_μ to the initial value problem (2.14) varies continuously with respect to the parameter $\mu \in \mathbb{R}$ and therefore Ψ is a continuous function. It follows from (2.10) that

$$
-M < \int_0^{2\pi} \frac{P(t)V_\mu^2 + Q(t)U_\mu^2}{U_\mu^2 + V_\mu^2} < M
$$

and hence

$$
\mu - M < \Psi(\mu) < \mu + M
$$

for all $\mu \in \mathbb{R}$. Thus $\Psi(\mu) \to \pm\infty$ as $\mu \to \pm\infty$, and there is a doubly infinite sequence $\{\mu_n : n \in \mathbb{Z}\}$ such that $\Psi(\mu_n) = n$ and a nontrivial solution W_n to (2.11), (2.13) with $\lambda = \mu_n$ and $\mathrm{rot}(W_n) = n$. Notice also that since $\Psi(\mu_n) = n$ we have $\mu_n - M < n < \mu_n + M$. We do not know if there can be more than one solution μ to the equation $\Psi(\mu) = n$. We will refer to the set of all such solutions μ for $n \in \mathbb{Z}$ as the set of eigenvalues for (2.11), (2.13). If $P = Q$ then $\Psi(\mu) = \mu + \overline{P}$ where $\overline{P}$ denotes the mean value of P. In this special case $\mu_n = \overline{P} - n$ is unique. In this case we also note that $U^2(t) + V^2(t)$ is constant. It follows from the general theory of compact linear operators that the set of eigenvalues has no finite limit point. From this and the structure of Ψ it follows that there can be at most finitely many eigenvalues associated with any given rotation number.

Each eigenvalue has a one dimensional eigenspace since if it were two dimensional the eigenspace would be a basis for all solutions to (2.11), and then all solutions would have to satisfy $u(0) = 0$.

We have proven the following result.

Theorem 2.7. *Let $p, q \in L^\infty(0,\pi)$ satisfy* (4.1)*. Then the problem* (2.11)*,* (2.13) *has a doubly infinite sequence of eigenvalues $\{\mu_n : n \in \mathbb{Z}\}$. Moreover the eigenspace associated with each eigenvalue is one dimensional and if $w \neq 0$ is a function in the eigenspace for some eigenvalue μ then there is an $n\mathbb{Z}$ such that $\Psi(\mu) = n$ and $\operatorname{rot}(w) = n$, where the latter denotes the rotation number associated with w as defined earlier in* (2.3)*. There are at most finitely many eigenvalues associated with the same rotation number.*

We now consider parameter dependent linear systems of the form (2.12) with boundary conditions (2.13). We again assume $p, q \in L^\infty(0,\pi)$ and make the additional assumption that there is a $\beta \in L^\infty(0,\pi)$ and $M > 0$ such that

$$0 \lesssim \beta(t) \leq \min(p(t), q(t)) \leq M \text{ for a.a. } t \in [0,\pi]. \tag{2.16}$$

We again extend p and q to P and Q, respectively, even on $[-\pi,\pi]$, and then 2π-periodic on the real line. If $w = (u,v)$ satisfies (2.13), we make the odd/even 2π-periodic extension of w. Let W_μ be the solution to the initial-value problem

$$\begin{gathered}\frac{dU}{dt} + \mu P(t) V = 0\\ \frac{dV}{dt} - \mu Q(t) U = 0\\ U(0) = 0, \quad V(0) = 1.\end{gathered}$$

We can as before define a real valued function Ψ by the equation

$$\begin{aligned}\Psi(\mu) &:= \frac{1}{2\pi}\int_0^{2\pi} \frac{\mu P(t)V_\mu^2 + \mu Q(t)U_\mu^2}{U_\mu^2 + V_\mu^2}\,dt\\ &= \frac{\mu}{2\pi}\int_0^{2\pi} \frac{P(t)V_\mu^2 + Q(t)U_\mu^2}{U_\mu^2 + V_\mu^2}\,dt.\end{aligned}$$

The Lemma is valid here, so $\Psi(\mu) \in \mathbb{Z}$ if and only if W_μ satisfies the boundary conditions. Clearly Ψ is a continuous real valued function. Let $\overline{\beta}$ be the mean value of β over $[0, 2\pi\}$. For $\mu > 0$ we have $\Psi(\mu) \geq \mu\overline{\beta}$ and for $\mu < 0$ we have $\Psi(\mu) < \mu\overline{\beta}$. Thus the range of Ψ is the set of real numbers and for each $n \in \mathbb{Z}$ there is at least one $\mu \in \mathbb{R}$ with $\Psi(\mu) = n$. We will refer to the set of all such solutions μ for $n \in \mathbb{Z}$ as the set of eigenvalues for (2.12), (2.13). If $P = Q$ then $\Psi(\mu) = \mu\overline{P}$ where $\overline{P}$ denotes the mean value of P. In this special case $\mu_n = n/\overline{P}$ is unique. Note that in this case $U^2(t) + V^2(t)$ is constant. As in the previous theorem, there can be at most finitely many eigenvalues associated with any given rotation number.

Each eigenvalue has a one dimensional eigenspace since if it were two dimensional the eigenspace would be a basis for all solutions to (2.12), and then all solutions would have to satisfy $u(0) = 0$.

We have proven the following result.

Theorem 2.8. *Let $p, q \in L^\infty(0,\pi)$ satisfy* (2.16)*. Then the problem* (2.12)*,* (2.13) *has a doubly infinite sequence of eigenvalues $\{\mu_n : n \in \mathbb{Z}\}$. Moreover the eigenspace associated with each eigenvalue is one dimensional and if $w \neq 0$ is a function in the eigenspace for some eigenvalue μ then there is an $n \in \mathbb{Z}$ such that $\Psi(\mu) = n$ and $\operatorname{rot}(w) = n$, where the latter denotes the rotation number associated with w as defined earlier in* (2.3)*. There are at most finitely many eigenvalues associated with the same rotation number.*

3. Nonresonance and existence

We now consider nonlinear problems of the form

$$\begin{aligned} \frac{du}{dt} + p(t,u,v)v &= f(t,u,v) \\ \frac{dv}{dt} - q(t,u,v)u &= g(t,u,v) \end{aligned} \tag{3.1}$$

with the boundary conditions (2.9); that is, the conditions are:

$$u(0) = 0, \quad u(\pi) = 0.$$

We will assume in this section that p, q, f, g satisfy Carathéodory conditions. That is, we assume that for almost all $t \in [0,\pi]$ the maps $p(t,.,.)$, $q(t,.,.)$, $f(t,.,.)$, $g(t,.,.)$ are continuous on $\mathbb{R}^2$, and for each $(u,v) \in \mathbb{R}^2$, the maps $p(.,u,v)$, $q(.,u,v)$, $f(.,u,v)$, $g(.,u,v)$ are Lebesgue measurable on $[0,\pi]$. We also assume there is a function $S_1 \in L^\infty(0,\pi)$ such that $|p(t,u,v)| + |q(t,u,v)| \le S_1(t)$ for all $(u,v) \in \mathbb{R}^2$ and *a.a.* $t \in [0,\pi]$, and for each $R \ge 0$ there is a function $M_R \in L^1(0,\pi)$ such that $|f(t,u,v)| + |g(t,u,v)| \le M_R(t)$ for all $|(u,v)| \le R$ and *a.a.* $t \in [0,\pi]$.

We now can state an existence theorem.

Theorem 3.1. *Let p, q, f, g be as described above. In addition assume:*

1. *There are functions $\alpha_1, \alpha_2, \beta_1, \beta_2 \in L^\infty(0,\pi)$ and $N \in \mathbb{Z}$ such that for all $(u,v) \in \mathbb{R}^2$,*
$$\begin{gathered} N \lesssim \alpha_1(t) \le p(t,u,v) \le \beta_1(t) \lesssim N+1, \\ N \lesssim \alpha_2(t) \le q(t,u,v) \le \beta_2(t) \lesssim N+1 \end{gathered}$$
hold on $[0,\pi]$.
2. *There is a function $m \in L^1(0,\pi)$ such that for each $\varepsilon > 0$ there is an $R(\varepsilon) \ge 0$ for which the following hold a.e.:*
$$\begin{gathered} |f(t,u,v)| \le \varepsilon m(t)|(u,v)|, \\ |g(t,u,v)| \le \varepsilon m(t)|(u,v)|. \end{gathered}$$

Then there is at least one solution to (3.1)*,* (2.9)*.*

Proof. The proof uses degree theory. We sketch the argument. Define functions A_i, $i = 1,2$ by $A_i := \frac{1}{2}(\alpha_i + \beta_i)$. Then $n \lesssim A_i \lesssim n+1$ so that for each pair $k_1, k_2 \in L^1 = L^1(0,\pi)$ there is a unique solution $w = (u,v) \in C = C([0,\pi],\mathbb{R}^2)$ to the boundary problem

$$\begin{aligned} \frac{du}{dt} + A_1(t)v &= k_1(t) \\ \frac{dv}{dt} - A_2(t)u &= k_2(t) \end{aligned}$$

with boundary conditions (2.9). Let Γ denote the linear mapping $(k_1,k_2) \mapsto w = (u,v)$ from $L^1 \times L^1$ into $C \times C$. The mapping Γ is compact.

Let C_0 denote the Banach space of all pairs of continuous functions $w = (u,v)$ on $[0,\pi]$ satisfying $u(0) = 0 = u(\pi)$, with norm $\|w\| := \max_{[0,\pi]} |w(t)|$. Define a mapping $N : C_0 \to L^1 \times L^1$ by

$$N(w)(t) := \begin{pmatrix} -A_1(t)v(t) + p(t,u(t),v(t))v(t) - f(t,u(t),v(t)) \\ A_2(t)u(t) - q(t,u(t),v(t))u(t) - g(t,u(t),v(t)) \end{pmatrix}$$

for $t \in [0,\pi]$. The mapping N is continuous and maps bounded sets into bounded sets. The boundary value problem (3.1), (2.9) is equivalent to the equation

$$w + \Gamma N(w) = 0 \tag{3.2}$$

in C_0. We now apply a homotopy to (3.2) and use Leray-Schauder degree.

The mapping $\Gamma N : C_0 \to C_0$ is completely continuous: it is continuous, and maps bounded sets into relatively compact ones. We consider the parameterized family of equations:

$$w + \lambda \Gamma N(w) = 0. \tag{3.3}$$

We shall show that there is a number $R^* > 0$ such that if w is a solution to (3.3) for any $\lambda \in [0,1]$ then $\|w\| < R^*$. It will then follow that the degree $\deg_{LS}(I + \lambda\Gamma N, B(0,R^*), 0)$ is independent of $\lambda \in [0,1]$, and from this we will be able to deduce that a solution to (3.2) exists. We proceed. .

Suppose that there is no such number R^*. It then follows that there is a sequence $\{(\lambda_n, w_n)\}$ in $[0,1] \times C_0$ such that $\|w_n\| \to \infty$ and for each $n \in \mathbb{N}$ we have $w_n + \lambda_n \Gamma N w_n = 0$. Then $w_n = (u_n v_n)$ satisfies

$$\begin{aligned} \frac{du_n}{dt} + (1-\lambda_n)A_1(t)v_n + \lambda_n p_n(t)v_n &= \lambda_n f(t,u_n,v_n) \\ \frac{dv_n}{dt} - (1-\lambda_n)A_2(t)u_n - \lambda_n q_n(t)u_n &= \lambda_n g(t,u_n,v_n) \end{aligned} \tag{3.4}$$

and $u_n(\pi) = 0 = u_n(0)$, where $p_n(t) = p(t,u_n,v_n)$, and $q_n(t) = q(t,u_n,v_n)$.Let $\widetilde{w}_n = w_n/\|w_n\|$. It follows from (3.4) and the properties of the terms in the equation that there is a constant $c_1 > 0$ such that for all $n \in \mathbb{N}$, $\|\widetilde{w}_n\|_{L^1} < c_1$. From the latter inequality it follows that there is a subsequence of $\{(\lambda_n, \widetilde{w}_n)\}$ convergent in $[0,1] \times C_0$ to some $(\widetilde{\lambda}, \widetilde{W}) = (\widetilde{\lambda}, (\widetilde{u}, \widetilde{v}))$. We relabel that convergent subsequence as $\{(\lambda_n, \widetilde{w}_n)\}$. Now for all $n \in \mathbb{N}$,

$$\begin{gathered} N \lesssim \alpha_1(t) \le p_n(t) \le \beta_1(t) \lesssim N+1, \\ N \lesssim \alpha_2(t) \le q_n(t) \le \beta_2(t) \lesssim N+1, \quad \text{a.e.} \end{gathered}$$

The sequence $\{(p_n, q_n)\}$ is bounded in $L^\infty \times L^\infty$ and hence in $L^2 \times L^2$, so there is a subsequence weakly convergent in $L^2 \times L^2$ to some $(\widetilde{p}, \widetilde{q})$ which must also satisfy the inequalities

$$\begin{gathered} N \lesssim \alpha_1(t) \le \widetilde{p}(t) \le \beta_1(t) \lesssim N+1, \\ N \lesssim \alpha_2(t) \le \widetilde{q}(t) \le \beta_2(t) \lesssim N+1, \quad \text{a.e.} \end{gathered}$$

Integrating the differential equations from 0 to t and taking limits we can conclude that

$$\begin{gathered} \frac{d\widetilde{u}}{dt} + (1-\widetilde{\lambda})A_1(t)\widetilde{v} + \widetilde{\lambda}\widetilde{p}(t)\widetilde{v} = 0 \\ \frac{d\widetilde{v}}{dt} - (1-\widetilde{\lambda})A_2(t)\widetilde{u} - \widetilde{\lambda}\widetilde{q}(t)\widetilde{u} = 0 \end{gathered}$$

and $(\widetilde{u}(\pi), \widetilde{u}(0)) = (0,0)$. Moreover we must have

$$\begin{gathered} N \lesssim \alpha_1(t) \le (1-\widetilde{\lambda})A_1(t) + \widetilde{\lambda}\widetilde{p}(t) \le \beta_1(t) \lesssim N+1 \\ N \lesssim \alpha_2(t) \le (1-\widetilde{\lambda})A_2(t) + \widetilde{\lambda}\widetilde{q}(t) \le \beta_2(t) \lesssim N+1, \quad \text{a. e.} \end{gathered} \tag{3.5}$$

which implies, by Lemma 2.2, that $\widetilde{w} = (\widetilde{u}, \widetilde{v}) = (0,0)$, a contrary to $\|\widetilde{w}\| = 1$, and we have reached a contradiction. It follows that there indeed must be a number

$R^* > 0$ such that if (λ, w) is a solution of (3.3) then $\|w\| < R^*$. Therefore by the homotopy invariance of Leray-Schauder degree $\deg_{LS}(I + \lambda\Gamma N, B(0, R^*),))$ is independent of $\lambda \in [0, 1]$ and

$$d_{LS}(I + \Gamma N, B(0, R^*), 0) = \deg_{LS}(I, B(0, R^*), 0) = 1$$

and hence there is a $w^* \in B(0, R^*) \subset C_0$ satisfying $w^* + \Gamma N w^* = 0$. This proves the theorem. $\square$

Remark 3.2. Theorem 3.1 recalls other theorems, some going back as far as [5]; the paper [7] inspired many others to look seriosly at nonresonance conditions for nonlinear differential equations. Conditions of the form $\lambda_N \lesssim p(t) \lesssim \lambda_{N+1}$ have been used mainly in boundary value problems for second order ordinary and partial differential equations. This kind of condition does not seem to have been earlier used in the situation of Theorem 3.1, perhaps because the proof of Lemma 2.2 differs from proofs given for analogous lemmas in the second order case.

4. Bifurcation from zero

Suppose $p, q \in L^\infty(0, \pi)$ satisfy

$$-M \lesssim p(t), q(t) \lesssim M \text{ a.e.} \tag{4.1}$$

for some $M > 0$. Let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & p(t) \\ -q(t) & 0 \end{pmatrix}. \tag{4.2}$$

Let $g \in C(\mathbb{R} \times [0, \pi] \times \mathbb{R}^2, \mathbb{R}^2)$ (more generally, g may be a Carathéodory function) with $g(\lambda, t, 0) = 0$, and $g(\lambda, t, w) = o(w)$ We will apply the results on (2.11) and (2.12) to study bifurcation and multiplicity questions for the systems

$$\frac{dw}{dt} + \lambda J w + A(t) w = g(\lambda, t, w) \tag{4.3}$$

and

$$\frac{dw}{dt} + \lambda A(t) w = g(\lambda, t, w) \tag{4.4}$$

with the boundary conditions (2.9), i.e.,

$$u(0) = 0, \quad u(\pi) = 0, \quad \text{where } w(t) = (u(t), v(t)). \tag{4.5}$$

We now consider bifurcation from zero. Let $p, q \in L^\infty(0, \pi)$ satisfy (4.1) and let $A(t)$ be as defined in (4.2). Let $g : \mathbb{R} \times [0, \pi] \times \mathbb{R}^2 \to \mathbb{R}^2$ be a Carathéodory function. That is, for each $(\lambda, w) \in \mathbb{R} \times \mathbb{R}^2$ the map $t \mapsto g(\lambda, t, w)$ is Lebesgue measurable, and for almost all $t \in [0, \pi]$ the map $(\lambda, w) \mapsto g(\lambda, t, w)$ is continuous. Moreover, for each $r \geq 0$ there is $\alpha_r \in L^1(0, \pi)$ such that $|g(\lambda, t, w)| \leq \alpha_r(t)$ a.e. for $|\lambda| + |w| \leq r$. We also assume $g(\lambda, 0, t) = 0$ and $|g(\lambda, w, t)| = o(|w|)$ as $|w| \to 0$, uniformly with respect to λ and t in compact sets. Let $w = (u, v)^T$ and consider the boundary value problem (4.3), (4.5).

$$\begin{gathered} \frac{dw}{dt} + \lambda J w + A(t) w = g(\lambda, t, w) \\ u(0) = 0, \quad u(\pi) = 0, \end{gathered}$$

where $w = (u, v)^T$. First we write an abstract version of (4.3), (4.5). Let $Y = L^1(0, \pi)$ and X the Banach space of $\mathbb{R}^2$ valued functions $w = (u, v)$ continuous on $[0, \pi]$ with $u(0) = 0 = u(\pi)$ with norm $\|w\| = \max_{t \in [0, \pi]} |w(t)|$. Let $D = \{w \in$

$X : w' \in Y\}$. That is, functions in D satisfy the boundary conditions and are absolutely continuous. Now let $0 \le c < 1$ be a number such that the problem

$$\begin{gathered} \frac{dw}{dt} + cJw + A(t)w = 0 \\ u(0) = 0, \quad u(\pi) = 0 \end{gathered} \tag{4.6}$$

has no non-trivial solution. In this case we define $L : D \to Y$ be defined by

$$Lw := \frac{dw}{dt} + cJw + A(t)w$$

for $w \in D$. The linear operator L has a compact inverse L^{-1}. Let $G : \mathbb{R} \times X \to Y$ be defined by $G(\lambda, w) := g(\lambda, \cdot, w(\cdot))$ for $(\lambda, w) \in \mathbb{R} \times X$. The map G is continuous and take bounded sets to bounded sets. Now our problem is equivalent to the equation in X given by.

$$w + (\lambda - c)L^{-1}Jw = L^{-1}G(\lambda, w)$$

or

$$w + \mu L^{-1}Jw = L^{-1}\widetilde{G}(\mu, w) \tag{4.7}$$

where $\mu = \lambda - c$ and $\widetilde{G}(\mu, w) = G(\mu + c, w)$. Now λ^* is an eigenvalue of (2.11), (4.5) (equivalently, $\Psi(\lambda^*) \notin \mathbb{Z}$) if and only if $\mu^* = \lambda^* - c$ is a characteristic value of $L^{-1}J$. If μ is not a characteristic value of $L^{-1}J$ then the Leray-Schauder degree $\deg_{LS}(I + \mu L^{-1}J, B(r), 0)$ is defined for $r > 0$ (where $B(r) = \{w \in X : \|w\| < r\}$). Now if μ^* is a characteristic value of $L^{-1}J$ then the null space of $I + \mu^* L^{-1}J$ is one dimensional, and thus the Leray-Schauder degree $\deg_{LS}(I + \mu L^{-1}J, B(r), 0)$ changes sign as μ crosses μ^*.

We now consider the global bifurcation question for the problem (4.3), (4.5), which is equivalent to (4.7). Let σ denote the characteristic values of $L^{-1}J$, so that $\mu \in \sigma$ if and only if $\lambda = \mu + c$ is an eigenvalue of the problem (2.11), (2.9). Let $\mathcal{S}_0$ denote the set of all non-trivial solutions (μ, w) of (4.7) and let $\mathcal{S}$ denote the closure of $\mathcal{S}_0$ in $\mathbb{R} \times X$. A point $(\mu^*, 0)$ is a bifurcation point from the line of trivial solutions if every neighborhood of $(\mu^*, 0)$ contains a member of $\mathcal{S}_0$.

Theorem 4.1. *Assume $p, q \in L^\infty(0, \pi)$ satisfy (4.1) with A as in (4.2). Let $g : \mathbb{R} \times [0, \pi] \times \mathbb{R}^2 \to \mathbb{R}^2$ be a Carathéodory function as described above with $g(\lambda, t, 0) = 0$ and $g(\lambda, t, w) = o(w)$ as $|w| \to 0$ uniformly with respect to λ, t in bounded sets. Then:*

- (c1) *Each $\mu^* \in \sigma$ is a bifurcation point of (4.7), and thus $\lambda^* = \mu^* + c$ is a bifurcation point of (4.3), (4.5).*
- (c2) *For $\mu^* \in \sigma$ let $\mathcal{C}(\mu^*)$ denote the component of $\mathcal{S}$ which contains $(\mu^*, 0)$. Then $\mathcal{C}(\mu^*)$ is either unbounded in $\mathbb{R} \times X$ or $\mathcal{C}(\mu^*)$ meets another point $(\widehat{\mu}, 0)$ with $\widehat{\mu} \in \sigma \backslash \{\mu^*\}$. Moreover* rot$(w)$ *is defined and constant for all $(\mu, w) \in \mathcal{C}(\mu^*)$ for $w \neq 0$, and is the same as the rotation number associated with the eigenfunctions of (2.11), (2.9) at $\lambda^* = \mu^* + c$. Thus $\mathcal{C}(\mu^*)$ can only meet another bifurcation point $(\widehat{\mu}, 0)$ if $\widehat{\lambda} = \widehat{\mu} + c$ is associated with same same rotation number as is λ^*.*

Proof. The proof of this theorem is based upon the Rabinowitz global bifurcation theorem and the properties of Leray-Schauder degree. The details are similar to the proof in [12, Theorem 3] and will not be repeated here. $\square$

We now consider the global bifurcation question for the problem (4.4), (2.13). We assume (2.16) holds. From this and Theorem 2.8 it follows that there is a number $c > 0$ such that (2.12) has no eigenvalues in the half-open interval $(0, c]$. Let $X = C([0, \pi], \mathbb{R}^2)$, $Y = L^1(0, \pi)$, and

$$D = \{w \in X : w = (u, v) \text{ is absolutely continuous and } u(0) = u(\pi) = 0\}.$$

Define $L : D \to Y$ by $Lw := w' + cAw$. The operator L has a compact inverse. Assume g satisfies the conditions of the preceding theorem and let the nonlinear operator G also be as defined earlier. The problem (4.4), (2.13) is equivalent to

$$w + \mu L^{-1}Aw = L^{-1}\widetilde{G}(\mu, w) \tag{4.8}$$

where $\mu = \lambda - c$ and $\widetilde{G}(\mu, w) = G(\mu + c, w)$. Let σ denote the characteristic values of $L^{-1}A$, so that $\mu \in \sigma$ if and only if $\lambda = \mu + c$ is an eigenvalue of the problem (2.12), (2.9). Let $\mathcal{S}_0$ denote the set of all non-trivial solutions (μ, w) of (4.8) and let $\mathcal{S}$ denote the closure of $\mathcal{S}_0$ in $\mathbb{R} \times X$. A point $(\mu^*, 0)$ is a bifurcation point from the line of trivial solutions if every neighborhood of $(\mu^*, 0)$ contains a member of $\mathcal{S}_0$.

Theorem 4.2. *Assume $p, q \in L^\infty(0, \pi)$ satisfy* (2.16) *with A as in* (4.2)*. Let $g : \mathbb{R} \times [0, \pi] \times \mathbb{R}^2 \to \mathbb{R}^2$ be a Carathéodory function as described above with $g(\lambda, t, 0) = 0$ and $g(\lambda, t, w) = o(w)$ as $|w| \to 0$ uniformly with respect to λ, t in bounded sets. Then:*

- (c1) *Each $\mu^* \in \sigma$ is a bifurcation point of* (4.8)*, and hence $\lambda^* = \mu^* + c$ is a bifurcation point for* (4.4)*,* (2.13)*.*
- (c2) *For $\mu^* \in \sigma$ let $\mathcal{C}(\mu^*)$ denote the component of $\mathcal{S}$ which contains $(\mu^*, 0)$. Then $\mathcal{C}(\mu^*)$ is either unbounded in $\mathbb{R} \times X$ or $\mathcal{C}(\mu^*)$ meets another point $(\widehat{\mu}, 0)$ with $\widehat{\mu} \in \sigma \backslash \{\mu^*\}$. Moreover* $\operatorname{rot}(w)$ *is defined and constant for all $(\mu, w) \in \mathcal{C}(\mu^*)$ for $w \neq 0$, and is the same as the rotation number associated with the eigenfunctions of* (2.12)*,* (2.9) *at $\lambda^* = \mu^* + c$. Thus $\mathcal{C}(\mu^*)$ can only meet another bifurcation point $(\widehat{\mu}, 0)$ if $\widehat{\lambda} = \widehat{\mu} + c$ is associated with same same rotation number as is λ^*.*

Proof. The proof is based upon the Rabinowitz global bifurcation theory, making use of the properties established for (2.12) and properties of Leray-Schauder degree. See [12, Theorem 3] for a related result and proof. □

5. Bifurcation from infinity

We shall study bifurcation from infinity in systems of the form

$$\frac{dw}{dt} + \lambda A(t)w = g(\lambda, w, t), \quad t \in [0, \pi], \tag{5.1}$$

where $w = (u, v)^T$ satisfies the boundary conditions (2.9):

$$u(0) = 0 = u(\pi),$$

We assume that $A(t)$ has the form (4.2) and satisfies (2.16), and that $g : \mathbb{R} \times [0, \pi] \times \mathbb{R}^2 \to \mathbb{R}^2$ is a Carathéodory function. That is, for each $(\lambda, w) \in \mathbb{R} \times \mathbb{R}^2$ the map $t \mapsto g(\lambda, t, w)$ is Lebesgue measurable, and for almost all $t \in [0, \pi]$ the map

$(\lambda, w) \mapsto g(\lambda, t, w)$ is continuous. Moreover, for each $r \geq 0$ there is $\alpha_r \in L^1(0, \pi)$ such that $|g(\lambda, t, w)| \leq \alpha_r(t)$ a.e. for $|\lambda| + |w| \leq r$. In addition we assume

$$\lim_{|w|\to\infty} \frac{|g(\lambda, t, w)|}{|w|} = 0 \tag{5.2}$$

uniformly with respect to λ and t in bounded sets.

We will say that (λ^*, ∞) (or λ^*) is a bifurcation point at infinity if there is a sequence $\{(\lambda_n, w_n)$ of solutions to (5.1), (2.9) with $\lambda_n \to \lambda^*$ and $\|w_n\| \to \infty$ as $n \to \infty$. We apply Leray-Schauder degree to prove the existence of continua bifurcating from infinity.

Let $X = C([0, \pi], \mathbb{R}^2)$, $Y = L^1([0, \pi], \mathbb{R}^2)$ and let D be the set of all absolutely continuous $w = (u, v)^T \in X$ satisfying $u(0) = u(\pi) = 0$. Let $0 < c \leq 1$ be such that (2.12) has no eigenvalues in the interval $, c]$. $L : D \subset X \to Y$ by

$$Lw := \frac{dw}{dt} + cA(t)w$$

for $w \in D$. It is not difficult to verify that L is a bijection from $D(L)$ onto Y, and its inverse $L^{-1} : Y \to X$ is compact. Let $\widetilde{G}(\mu, w)(t) := g(\mu + c, t, w(t))$ for $\mu \in \mathbb{R}, t \in [0, \pi]$, and $w \in X$. Then $\widetilde{G}$ is a continuous mapping from X into Y, and takes bounded sets to bounded sets. The problem (5.1), (2.9) is equivalent to the abstract equation

$$w + \mu L^{-1}Aw = L^{-1}\widetilde{G}(\mu, w) \tag{5.3}$$

where $\mu = \lambda - c$.

Let $\mathcal{S}$ denote the set of solutions (μ, w) to (5.3) and adjoin to $\mathcal{S}$ the set of (μ^*, ∞) such that (μ^*, ∞) is a bifurcation point at infinity. Denote this set by $\mathcal{S}^*$. By the component $\mathcal{C}(\mu^*)$ of $\mathcal{S}^*$ containing (μ^*, ∞), we mean the union of all components of $\mathcal{S}$ which contain sequences $\{(\mu_n, w_n)\}$ with $\mu_n \to \mu^*$ and $||w_n|| \to \infty$. We now show that each $(\lambda^* + c, \infty)$, $\lambda^* \in \sigma$ (the eigenvalues of (2.12)) is a bifurcation point at infinity for (5.3). Thus each (λ^*, ∞) is a bifurcation point at infinity for (5.1), (2.9), with associated component $\mathcal{C}_{\lambda^*} = \{(\lambda^*, w) : (\lambda^* + c, w) \in \mathcal{C}(\lambda^* + c)\}$.

Theorem 5.1. *Let g satisfy* (5.2). *Then each $\mu^* \in \sigma + c$ is a point of bifurcation at infinity for* (5.3), *and thus each $\lambda^* \in \sigma$ is a point of bifurcation at infinity for* (5.1), (2.9). *Let $\mathcal{C}(\mu^*)$ denote the component of $\mathcal{S}^*$ containing (k, ∞); then $\mathcal{C}(\mu^*) - \{(\mu^*, \infty)\}$ is unbounded. Moreover at least one of the following holds:*

(C1) *The projection of $\mathcal{C}(\mu^*)$ on $\mathbb{R}$ is unbounded, or*
(C2) *$\mathcal{C}(\mu^*)$ meets another bifurcation point at infinity $(\widehat{\mu}, \infty)$, $\widehat{\mu} \neq \mu^*$.*

Remark: It may happen that there is a line of trivial solutions, say $\mathcal{L} = \{(\lambda, 0) : \lambda \in \mathbb{R}\}$. If so, and $\mathcal{C}(\mu^*)$ meets $\mathcal{L}$ at some point $(\lambda^*, 0)$, then $\mathcal{L} \subset \mathcal{C}(\mu^*)$ and the projection of $\mathcal{C}(\mu^*)$ on $\mathbb{R}$ is unbounded.

Proof of Theorem 5.1. Bifurcation from infinity in this case may be proven by converting the problem to one of bifurcation from zero by means of the inversion $y = w/\|w\|^2$ as in [9]. For $y \in X$ let

$$H(\mu, y) := \|y\|^2\widetilde{G}(\mu, y/\|y\|^2) \quad \text{for } y \neq 0,\ H(\mu, 0) := 0.$$

One can show that H is continuous and takes bounded sets to bounded sets. Additionally,

$$\lim_{\|y\|\to 0} \frac{H(\mu, y)}{\|y\|} = 0$$

with the limit uniform w.r.t μ in bounded sets. Letting $y = w/\|w\|^2$ in (5.3) converts that equation to

$$y + \mu L^{-1}Ay = L^{-1}H(\mu, y). \tag{5.4}$$

The Rabinowitz global theory for bifurcation from the line of trivial solutions $\{(\mu, 0) : \mu \in \mathbb{R}\}$ applies to (5.4). Meeting each point $(\mu^*, 0)$ with μ^* a characteristic value of $L^{-1}A$ there is a continuum $\widetilde{\mathcal{C}}(\mu^*)$ of nontrivial solutions, and $\widetilde{\mathcal{C}}(\mu^*)$ is either unbounded in $\mathbb{R} \times X$ or meets another bifurcation point $(\widehat{\mu}, 0)$, $\widehat{\mu} \neq \mu^*$. The inversion mapping $(\mu, y) \mapsto (\mu, y/\|y\|^2)$ maps each $\widetilde{\mathcal{C}}(\mu^*)$ to a continuum $\mathcal{C}(\mu^*)$ of solutions to (5.3) meeting (μ^*, ∞). The points (μ, w) in $\mathcal{C}(\mu^*)$ produce solutions $(\mu - c, w) = (\lambda, w)$ to (5.1), (2.9); these solutions form a continuum $\mathcal{C}_1(\lambda^*)$, $\lambda^* = \mu^* - c$. By examination of the solutions close to the bifurcation point (λ^*, ∞) one can show that the rotation of such solutions is that associated with the eigenfunctions of (2.12), (2.9) at $\lambda = \lambda^*$. This may not be continued on the entire continuum, since solutions may pass through the origin. The properties of the continua and the global bifurcation theorem implies the conclusions of the theorem. □

Remark 5.2. One could prove a similar bifurcation from infinity result for equation (4.3) but that will be omitted. Besides the paper [9] of Rabinowitz on bifurcation from infinity, the reader is referred to the paper [11] of Schmitt and Wang.

6. Global behavior of continua

In this section we give conditions which imply stronger conclusions regarding the global behavior of the continua $\mathcal{C}$ bifurcating from $(\lambda^*, 0)$ or (λ^*, ∞) found in the preceding sections. We show that under some conditions on the signs of the nonlinearities the $\mathcal{C}$ bend to the left or to the right of λ^*. Consider again systems of the form

$$\frac{dw}{dt} + \lambda A(t)w = g(\lambda, t, w) \tag{6.1}$$

with boundary conditions (2.9) as before, and $w = (u, v)^T$. We will assume that $p = q$, i.e.,

$$A(t) = \begin{pmatrix} 0 & p(t) \\ -p(t) & 0 \end{pmatrix}.$$

Assume g satisfies the Carathéodory conditions. We place some additional structural conditions on g:

$$g(\lambda, t, u, v) = \begin{pmatrix} -vg_1(\lambda, t, u, v) \\ ug_2(\lambda, t, u, v) \end{pmatrix}.$$

Thus our system takes the form

$$\begin{aligned} \frac{du}{dt} &= -\lambda p(t)v - vg_1(\lambda, u, v, t) \\ \frac{dv}{dt} &= \lambda p(t)u + ug_2(\lambda, u, v, t). \end{aligned} \tag{6.2}$$

Theorem 6.1. *Assume that p and $g = (-vg_1, ug_2)$ satisfy the conditions of Theorem 4.1. Also suppose that $p \geq 0$, and $g_1(\mu, u, v, t), g_2(\mu, u, v, t) \leq 0$ (≥ 0) for all $(\mu, u, v, t) \in \mathbb{R} \times \mathbb{R}^2 \times [0, \pi]$. Let $\mathcal{C} \subset \mathbb{R} \times X$ denote the continuum bifurcating from a point $(\lambda^*, 0)$. Then $\mathcal{C} \subset [\lambda^*, \infty) \times X$ ($(-\infty, \lambda^*] \times X$).*

Proof. Suppose $g_1(\mu,u,v,t), g_2(\mu,u,v,t) \le 0$ for all $(\mu,u,v,t) \in \mathbb{R}\times\mathbb{R}^2\times[0,\pi]$. Let $(\lambda,w) = (\lambda,(u,v)) \in \mathcal{C}$, with $(u,v) \neq (0,0)$, and let $P(t) = \lambda p(t) + g_1(\mu,u(t),v(t),t)$ and $Q(t) = \lambda p(t) + g_2(\mu,u(t),v(t),t)$. Then u,v satisfy

$$\begin{aligned}\frac{du}{dt} &= -P(t)v\\ \frac{dv}{dt} &= Q(t)u.\end{aligned}$$

Suppose $\lambda < \lambda^*$. Then $P(t) \le \lambda p(t) \lnsim \lambda^* p(t)$ and, $Q(t) \le \lambda p(t) \lnsim \lambda^* p(t)$. Thus if $\mathrm{rot}(w)$ is the is the rotation number of w, by Lemma 2.1 we must have $\mathrm{rot}(w) < d_1$, where d_1 is the rotation associated with solutions to

$$\begin{aligned}\frac{du}{dt} &= -\lambda^* p(t)v\\ \frac{dv}{dt} &= \lambda^* p(t)u.\end{aligned}$$

satisfying (2.9). But by Theorem 4.1 d_1 must be the the rotation number of all solutions w with $(\lambda,w) \in \mathcal{C}$. This is a contradiction and implies that $\mu \ge \lambda^*$, and thus $\mathcal{C} \subset [\lambda^*,\infty)\times X$.

If $g_1(\mu,u,v,t), g_2(\mu,u,v,t) \ge 0$ for all $(\mu,u,v,t) \in \mathbb{R}\times\mathbb{R}^2\times[0,\pi]$ then very similar arguments show that $\mathcal{C} \subset (-\infty,\lambda^*]\times X$. □

Remark 6.2. If we assume that $p = q$, and $g = (g_1,g_2)^T$ satisfy the conditions for bifurcation from infinity given in Theorem 5.1 and p, g_1,g_2 satisfy the sign conditions of the preceding theorem, then we can show that the continua bifurcating from infinity have the same containment properties. Similar results hold for the systems (4.3).

7. Multiple solutions

In [12] the author applied results on bifurcation from the line of trivial solutions together with bending directions of the bifurcating continua to prove multiplicity of solutions under appropriate conditions. In this section we apply our results on bifurcation from infinity to obtain related results. We prove the existence multiple structurally distinct solutions to boundary value problems of the form

$$\begin{aligned}\frac{du}{dt} &= -f(\lambda,t,u,v)v\\ \frac{dv}{dt} &= g(\lambda,t,u,v)u\end{aligned} \tag{7.1}$$

$$u(0) = 0 = u(\pi) \tag{7.2}$$

where $w = (u,v)$. To obtain the existence of infinitely many solutions we apply our earlier bifurcation from infinity results to the family of problems

$$\begin{aligned}\frac{du}{dt} &= -\lambda p(t) - g_1(\lambda,t,u,v)v\\ \frac{dv}{dt} &= \lambda p(t) + g_2(\lambda,t,u,v)u\end{aligned} \tag{7.3}$$

for $\lambda \in \mathbb{R}$ with boundary conditions (7.2). Let $p \in L^\infty(0,\pi)$,$p \gnsim 0$ on $[0,\pi]$ Let

$$A(t) := \begin{pmatrix} 0 & p(t)\\ -p(t) & 0\end{pmatrix}.$$

Suppose $g := (g_1, g_2)^T$ satisfy the Carathéodory conditions and

$$\lim_{|w|\to\infty} \frac{|g(\lambda, t, w)|}{|w|} = 0) \tag{7.4}$$

uniformly with respect to λ, t in compact sets. Recall that we call $\widetilde{w}$ the odd/even extension of w provided $w = (u, v) \in C([0, \pi], \mathbb{R}^2)$, $u(0) = 0 = v(0)$ and $\widetilde{w} = (\widetilde{u}, \widetilde{v})$ where $\widetilde{u}$, $\widetilde{v}$ are, respectively, the odd extension of u and the even extension of v to $[-\pi, \pi]$.

Theorem 7.1. *In addition to the above, suppose solutions to initial value problems are unique, $g(\lambda, t, 0) = 0$, $g_i(\lambda, t, w) \le 0$, for $\lambda \ge 0$ and $g_i(\lambda, t, w) \ge 0$, for $\lambda \le 0$ $(i = 1, 2)$, and*

$$\lim_{|w|\to 0} \frac{|\lambda A(t)w - g(\lambda, t, w)|}{|w|} = 0$$

uniformly with respect to (λ, t) in bounded sets. Let $\lambda_n, n \in \mathbb{Z}$ denote the nth eigenvalue of (2.12), (7.2)*, N be a positive integer. Then* (7.1), (7.2) *with $\lambda = \lambda^*$ has at least N topologically distinct solutions if $\lambda^* > \lambda_N$ or $\lambda^* < \lambda_{-N}$. Indeed, in this case there is for each $k \in \mathbb{N}$ there is a solution $w_k = (u_k, v_k)$ such that the odd/even 2π periodic extension $\widetilde{w}_k$ of w_k has rotation number k.*

Proof. The eigenvalues of the problem linearized at infinity are the λ_n, $n \in \mathbb{Z}$. By Theorem 5.1 problem (7.3), (7.2) has at each λ_n a continuum $\mathcal{C}_n$ of solutions bifurcating from (λ_n, ∞), and $\mathcal{C}_n$ is either unbounded in $\mathbb{R} \times X$, or else meets another, distinct, bifurcation point (k, ∞) $(k \neq n)$. Now solutions to initial value problems are unique and $g(\lambda, t, 0) = 0$. Therefore the rotation number of solutions near the bifurcation point must be continued to all nontrivial solutions in $\mathcal{C}_n$ Note that in this case, since $p = q$, there is exactly one eigenvalue associated with rotation number $n \in \mathbb{Z}$, and it is λ_n. Thus $\mathcal{C}_n$ cannot meet any other bifurcation point at infinity. The sign conditions on the g_i imply that for $n \ge 1$, $\mathcal{C}_n \subset [\lambda_n, \infty)$ and for $n \le -1$, $\mathcal{C}_n \subset (-\infty, \lambda_n]$ (this follows from noticing the sign conditions of Theorem 6.1 can be used discriminately, based in this case on the sign of λ). Now the only bifurcation point from the line of trivial solutions is $(\lambda, 0) = (0, 0)$. It follows that no $\mathcal{C}_n$ with $n \neq 0$ can meet the line of trivial solutions. On the other hand the projection of these $\mathcal{C}_n$ (with $n \neq 0$) on $\mathbb{R}$ must be unbounded. Thus for $n \ge 1$, $\mathcal{C}_n$ contains a nontrivial solution (μ, w) for each $\mu > \lambda_n$, and for $n \le -1$, $\mathcal{C}_n$ contains a nontrivial solution (μ, w) for each $\mu < \lambda_n$. These nontrivial solutions have rotation number n. It follows now that problem (7.3), (7.2) with $\lambda = \lambda^*$ has at least N solutions if $\lambda^* > \lambda_N$ or $\lambda^* < \lambda_{-N}$. □

example 7.2. The following system, with $p = q = 1$,

$$\begin{aligned}
\frac{du}{dt} &= -\lambda v + \frac{\lambda v}{1 + u^2 + v^2} + \frac{\lambda \sin^2(t) u^2 v}{1 + u^4 + v^4} \\
\frac{dv}{dt} &= \lambda u - \frac{\lambda u}{1 + u^2 + v^2} - \frac{\lambda \cos^2(t) v^4 u}{2 + u^6 + v^6}
\end{aligned}$$

satisfies the conditions of the preceding theorem.

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