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REVISITING THE METHOD OF SUB- AND SUPERSOLUTIONS FOR NONLINEAR ELLIPTIC PROBLEMS

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Abstract. We discuss some of the historical highlights of the theory of sub-supersolutions for boundary value problems for nonlinear elliptic equations and variational inequalities.

1. Introduction

Sub- and supersolutions (upper- and lower solutions) have played an important role in the study of nonlinear boundary value problems for elliptic partial differential equations for a long time. While some of the underlying principles are already present in the Perron process for obtaining harmonic functions satisfying (in a generalized sense) given boundary data (see, e.g. [16]), Scorza-Dragoni's paper [41] was one of the earliest works, where the existence of an ordered pair of solutions of differential inequalities was used to establish the existence of a solution of a given boundary value problem for a nonlinear second order ordinary differential equation. This was followed by some fundamental work of Nagumo [34, 35] which inspired much work on such problems subject to Dirichlet boundary conditions for both ordinary and partial differential equations during the decade of the sixties (see, e.g. [1, 2, 20, 38, 39]). Using Cesari's method, Knobloch [21] introduced the sub-supersolution method to the study of periodic boundary value problems for nonlinear second order ordinary differential equations. Using somewhat different techniques, similar problems were subsequently studied in [37, 32], among others.

In all of the above cited papers sub- and supersolutions are assumed to be smooth solutions of differential inequalities (i.e., solutions in a classical sense); such smooth sub- and supersolutions were also used to study Dirichlet and/or Neumann boundary value problems for semilinear elliptic problems in [3, 36], for general (nonlinear) boundary value problems in [14, 15, 33], and also for systems of nonlinear ordinary differential equations in [5, 18, 22].

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Deuel and Hess [12] and Hess [19] were the first to formulate concepts of weak sub- and supersolutions and obtained existence results for weak solutions of semilinear elliptic Dirichlet problems. This subject was subsequently continued by several authors (see, e.g., [6, 8, 24, 25, 26, 27, 28, 29, 30, 40]).

In some sense the weak approach gives a unifying way of treating such problems which is the theme of this paper. Further we shall discuss a formulation of weak sub- and supersolutions which allows, by proper choices of certain convex sets involved, for existence results for all the types of boundary conditions considered heretofore. The development follows that given in the recent papers [26, 27, 40].

We remark here that certain existence results are also possible in the presence of non ordered pairs of sub- and supersolutions and refer to [9, 11, 17] for some such results.

There are three recent monographs which cover this theory in considerable depth and provide a wealth of applications; they are [7], [10], and [13].

2. Sub- and supersolutions

2.1. **General remarks.** As said, we are interested here in sub-supersolution results for boundary value problems with second order principal operators and general boundary conditions, where the problems may or may not contain obstacles or constraints. We shall, following [26, 27], give the weak (variational) formulation of the problem, and deduce that the boundary conditions (or at least parts of them) may usually be encoded into the set of test (admissible) functions.

With this in mind one can show that in several cases (covering those that have been studied in the literature), by formulating the problem as a variational inequality, even if it is a smooth equation, simple, unified, and general definitions of sub- and supersolutions are possible. These concepts of sub- and supersolutions extend the classical definitions for equations subject to Dirichlet, Neumann, Robin, or No-Flux (periodic boundary conditions for the one space dimensional problem) boundary conditions (see e.g. [39]). The definitions are motived by the recent definitions of sub-supersolutions for variational inequalities in [26, 29, 30]. Another byproduct of the unified approach is that one can demonstrate the existence of solutions and extremal solutions between sub- and supersolutions and other properties of the solution sets when sub- and supersolutions exist.

Since the problems considered are variational inequalities with a closed convex constraint set K, by suitably chosing K one may even deduce sub-and supersolution results for finite difference equations. This topic appears worth pursuing.

2.2. **Definitions.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary and $W^{1,p}(\Omega)$ be the usual first-order Sobolev space with the norm

$$\|u\| = \|u\|_{W^{1,p}(\Omega)} = \Big(\|u\|^p_{L^p(\Omega)} + \||\nabla u|\|^p_{L^p(\Omega)}\Big)^{1/p}, \quad u \in W^{1,p}(\Omega). \tag{2.1}$$

Assume that K is a closed, convex subset of $W^{1,p}(\Omega)$, and consider the following variational inequality on K:

$$\begin{gathered}\int_\Omega A(x,\nabla u)\cdot(\nabla v - \nabla u)dx + \int_\Omega f(x,u)(v-u)dx \\ + \int_{\partial\Omega} g(x,u)(v-u)\,dS \ge 0, \quad \forall v \in K \\ u \in K.\end{gathered} \tag{2.2}$$

To simplify the notation we use u and v instead of $u|_{\partial\Omega}$ and $v|_{\partial\Omega}$ for the trace of u and v on $\partial\Omega$ in the surface integral in (2.2). In the variational inequality (2.2), A is an elliptic operator, f is the lower order term, and g is a boundary term.

As remarked above, different boundary conditions require different definitions of sub- and supersolutions. As a consequence, separate arguments and calculations are needed to study the existence and properties of solutions between sub- and supersolutions. In what follows, we show that common, unified definitions of sub- and supersolutions may be given for various types of boundary conditions (including unilateral constraints). Thus a common, comprehensive general existence theorem is possible for many different types of boundary value problems. The discussion to follow is motivated by and also generalizes the work in the papers [26, 29, 30].

We first give the assumptions on the principal operator:

$$A : \Omega \times \mathbb{R}^N \to \mathbb{R}$$

is a Carathéodory function satisfying the growth condition

$$|A(x, \xi)| \le a_1(x) + b_1|\xi|^{p-1}, \quad \text{for a.e. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^N, \tag{2.3}$$

with $p \in [1, \infty)$ (fixed), $a_1 \in L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$, and $b_1 > 0$. Moreover, A is monotone, i.e.,

$$(A(x, \xi) - A(x, \xi')) \cdot (\xi - \xi') \ge 0, \quad \text{for a.e. } x \in \Omega, \text{ all } \xi, \xi' \in \mathbb{R}^N, \tag{2.4}$$

and A is coercive in the following sense: There exist $a_2 \in L^1(\Omega)$ and $b_2 > 0$ such that

$$A(x, \xi) \cdot \xi \ge b_2|\xi|^p - a_2(x), \quad \text{for a.e. } x \in \Omega, \text{ all } \xi \in \mathbb{R}^N. \tag{2.5}$$

We also suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions subject to certain growth conditions to be specified later.

We shall use the standard notation $u \wedge v = \min\{u, v\}$, $u \vee v = \max\{u, v\}$, $U * V = \{u * v : u \in U, v \in V\}$, and $u * V = \{u\} * V$, where $u, v \in W^{1,p}(\Omega)$, $U, V \subset W^{1,p}(\Omega)$ and $* \in \{\wedge, \vee\}$.

The following are the definitions of sub- and supersolutions of (2.2).

Definition 2.1. A function $\underline{u} \in W^{1,p}(\Omega)$ is called a subsolution of (2.2) if the following conditions are satisfied:

$$f(\cdot, \underline{u}) \in L^q(\Omega), \quad g(\cdot, \underline{u}) \in L^{\tilde{q}}(\partial\Omega), \tag{2.6}$$

where $q \in (1, p^*)$ and $\tilde{q} \in (1, \tilde{p}^*)$,

$$\underline{u} \vee K \subset K, \tag{2.7}$$

and

$$\int_\Omega A(x, \nabla \underline{u}) \cdot \nabla(v - \underline{u}) dx + \int_\Omega f(x, \underline{u})(v - \underline{u}) dx + \int_{\partial\Omega} g(x, \underline{u})(v - \underline{u})\, dS \ge 0, \tag{2.8}$$

for all $v \in \underline{u} \wedge K$.

Here, p^* is the Sobolev conjugate exponent of p

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } N > p \text{ and } N > 1 \\ \infty & \text{if } N \le p \text{ or } N = 1 \end{cases}$$

and

$$\tilde{p}^* = \begin{cases} \frac{(N-1)p}{N-p} & \text{if } N > p \text{ and } N > 1 \\ \infty & \text{if } N \le p \text{ or } N = 1. \end{cases}$$

We have a similar definition for supersolutions of (2.2).

Definition 2.2. A function $\overline{u} \in W^{1,p}(\Omega)$ is called a supersolution of (2.2) if the following conditions are satisfied:

$$f(\cdot, \overline{u}) \in L^q(\Omega), \quad g(\cdot, \overline{u}) \in L^{\tilde{q}}(\partial\Omega), \tag{2.9}$$

where $q \in (1, p^*)$ and $\tilde{q} \in (1, \tilde{p}^*)$,

$$\overline{u} \wedge K \subset K, \tag{2.10}$$

and

$$\int_\Omega A(x, \nabla \overline{u}) \cdot \nabla(v - \overline{u}) dx + \int_\Omega f(x, \overline{u})(v - \overline{u}) dx + \int_{\partial\Omega} g(x, \overline{u})(v - \overline{u})\, dS \geq 0, \tag{2.11}$$

for all $v \in \overline{u} \vee K$.

The following is the main general existence theorem; its proof is patterned after the arguments used in [25, 26, 29] and may be found in [27].

Theorem 2.3. *Assume there exists a pair of sub- and supersolution of (2.2) such that $\underline{u} \leq \overline{u}$ and that f and g satisfy the following growth conditions between $\underline{u}$ and $\overline{u}$:*

$$|f(x, u)| \leq a_3(x), \quad |g(\xi, v)| \leq \tilde{a}_3(\xi), \tag{2.12}$$

for almost all $x \in \Omega$, $\xi \in \partial\Omega$, all $u \in [\underline{u}(x), \overline{u}(x)]$, $v \in [\underline{u}(\xi), \overline{u}(\xi)]$, where $a_3 \in L^{q'}(\Omega)$, $\tilde{a}_3 \in L^{\tilde{q}'}(\partial\Omega)$, $q \in (1, p^*)$, $\tilde{q} \in (1, \tilde{p}^*)$, and p^*, $\tilde{p}^*$ are defined as in Definition 2.1.

Then, there exists a solution u of (2.2) such that $\underline{u} \leq u \leq \overline{u}$.

The above result has the the following generalization. The proof again follows ideas already used in [24, 25] and is given in [27].

Theorem 2.4. *Assume that $\underline{u}_1, \dots, \underline{u}_k$ (resp. $\overline{u}_1, \dots, \overline{u}_m$) are subsolutions (resp. supersolutions) of (2.2) such that*

$$\underline{u}_0 := \max\{\underline{u}_1, \dots, \underline{u}_k\} \leq \min\{\overline{u}_1, \dots, \overline{u}_m\} =: \overline{u}_0, \tag{2.13}$$

and that f and g have the following growth conditions between the sub- and supersolutions:

$$|f(x, u)| \leq a_3(x), \quad |g(\xi, v)| \leq \tilde{a}_3(\xi), \tag{2.14}$$

for a.a. $x \in \Omega$, $\xi \in \partial\Omega$, all $u \in [\min\{\underline{u}_1(x), \dots, \underline{u}_k(x)\}, \max\{\overline{u}_1(x), \dots, \overline{u}_m(x)\}]$, all $v \in [\min\{\underline{u}_1(\xi), \dots, \underline{u}_k(\xi)\}, \max\{\overline{u}_1(\xi), \dots, \overline{u}_m(\xi)\}]$, where where a_3 and $\tilde{a}_3$ are as in Theorem 2.3.

Then, there exists a solution u of (2.2) such that $\underline{u}_0 \leq u \leq \overline{u}_0$.

Remark 2.5. (a) The above theorem suggests more general definitions of sub- and supersolutions. Namely: An element $\alpha \in W^{1,p}(\Omega)$ is a subsolution if it is the supremum of a finite number of functions each of which is a subsolution satisfying Definition 2.1 and an element $\beta \in W^{1,p}(\Omega)$ is a supersolution if it is the infimum of a finite number of supersolutions each of which is a supersolution satisfying Definition 2.2. In this case the set of subsolutions is closed with respect to the operation $\vee$ and the set of supersolutions is closed with respect to the operation $\wedge$, and, of course, Theorem 2.4 is simply a restatement of Theorem 2.3. Thus, if we let $\mathcal{S}$ be the set of solutions of (2.2) between $\underline{u}_0$ and $\overline{u}_0$. Theorem 2.4 means that $\mathcal{S} \neq \emptyset$ and under the above assumptions, one can prove (cf. [25, 30]) that $\mathcal{S}$ is compact and directed. As a consequence, $\mathcal{S}$ has greatest (the supremum of all subsolutions) and smallest (the infimum of all supersolutions) elements with respect to the standard ordering, which are the extremal solutions of (2.2) between $\underline{u}_0$ and $\overline{u}_0$. Such results also have a long history and likely go back to [1], see also [8, 24, 39].

(b) If only a subsolution (or a supersolution) of (2.2) exists and f and g satisfy certain one-sided growth conditions then we can also show the existence of solutions of (2.2) above the subsolution (or below the supersolution). We can also show the existence of a minimal solution above that subsolution (or a maximal solution below that supersolution) (see e.g. [30]).

(c) The question of the structure of the set of all solutions which lie between a given pair of sub-and supersolutions has been addressed in the literature often with the latest results given in [31]. This paper's bibliography also provides a fairly complete list of references for this problem.

3. Some examples

3.1. **Problems subject to Dirichlet boundary conditions.**

Consider the boundary-value problem

$$-\operatorname{div}[A(x,\nabla u)] + f(x,u) = 0 \quad \text{in } \Omega, \tag{3.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{3.2}$$

the variational form of which is the inequality (2.2) with $g = 0$ and

$$K = W_0^{1,p}(\Omega), \tag{3.3}$$

which is equivalent to the variational equality:

$$\int_\Omega A(x,\nabla u)\cdot\nabla v dx + \int_\Omega f(x,u) v dx = 0, \quad \forall v \in W_0^{1,p}(\Omega)$$
$$u \in W_0^{1,p}(\Omega).$$

In this case, for assumption (2.7) (respectively (2.10)) to be fulfilled, we need that

$$\underline{u} \le 0 \quad \text{on } \partial\Omega \quad (\text{respectively } \overline{u} \ge 0 \text{ on } \partial\Omega). \tag{3.4}$$

Concerning condition (2.8), it can be checked that the set $\{v - \underline{u} : v \in \underline{u} \wedge W_0^{1,p}(\Omega)\}$ is dense in the negative cone of $W_0^{1,p}(\Omega)$:

$$W_-^{1,p}(\Omega) := \{w \in W_0^{1,p}(\Omega) : w \le 0 \quad \text{a.e. on } \Omega\}.$$

Therefore, condition (2.8), in this particular case, becomes the following condition

$$\int_\Omega A(x,\nabla \underline{u})\cdot\nabla v dx + \int_\Omega f(x,\underline{u}) v dx \ge 0, \quad \forall v \in W_0^{1,p}(\Omega),\ v \le 0 \text{ on } \Omega. \tag{3.5}$$

In view of (3.4) and (3.5), we re-obtain the classical concept of sub- and supersolution for equations with homogeneous Dirichlet boundary condition (cf. e.g. [19, 8, 24, 25]).

For problems with nonhomogeneous Dirichlet conditions, we have equation (3.1) together with

$$u = h \quad \text{on } \partial\Omega, \tag{3.6}$$

instead of (3.2), where $h \in W^{1-\frac{1}{p},p}(\partial\Omega)$ is the trace of a function in $W^{1,p}(\Omega)$, still denoted by h, for simplicity. In this case, problem (3.1)–(3.6) is, in the variational form, the inequality (2.2) with $g = 0$ and

$$K = \{h\} \oplus W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : u = h,\ \text{on } \partial\Omega\}.$$

The condition $\underline{u} \vee K \subset K$ is satisfied if and only if $\underline{u}$ satisfies the boundary condition $\underline{u} \le h$ a.e. on $\partial\Omega$. The set

$$\{v - \underline{u} : v \in \underline{u} \wedge [\{h\} \oplus W_0^{1,p}(\Omega)]\} = \{w - (\underline{u} - h) : w \in (\underline{u} - h) \wedge W_0^{1,p}(\Omega)\}$$

is dense in the negative cone $W_-^{1,p}(\Omega)$ (because $\underline{u} - h \le 0$ on $\partial\Omega$). Condition (2.8) is again equivalent to (3.5).

3.2. Problems with Neumann and Robin boundary conditions. In the case where $K = W^{1,p}(\Omega)$, (2.2) reduces to the variational equality

$$\begin{gathered}\int_\Omega A(x, \nabla u) \cdot \nabla v dx + \int_\Omega f(x, u) v dx + \int_{\partial\Omega} g(x, u) v \, dS = 0, \quad \forall v \in W^{1,p}(\Omega) \\ u \in W^{1,p}(\Omega),\end{gathered} \tag{3.7}$$

which is the weak form of the boundary value problem

$$\begin{gathered}-\operatorname{div}[A(x, \nabla u)] + f(x, u) = 0 \quad \text{in } \Omega \\ A(x, \nabla u) \cdot n = -g(x, u) \quad \text{on } \partial\Omega.\end{gathered}$$

When $g = 0$ on $\partial\Omega$, we have a homogeneous Neumann boundary condition. Otherwise, one has a nonhomogeneous Neumann boundary condition which also may depend on u. It is clear that condition (2.7) always holds. Also, for any $\underline{u}$ in $W^{1,p}(\Omega)$, we have $\underline{u} \wedge W^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega) : v \le \underline{u} \text{ a.e. on } \Omega\}$. Therefore, (2.8) is equivalent to the inequality

$$\int_\Omega A(x, \nabla \underline{u}) \cdot \nabla w dx + \int_\Omega f(x, \underline{u}) w dx + \int_{\partial\Omega} g(x, \underline{u}) w \, dS \ge 0,$$

for all $w \in W^{1,p}(\Omega)$ such that $w \le 0$ a.e. on Ω, which is, in its turn, equivalent to

$$\begin{gathered}\int_\Omega A(x, \nabla \underline{u}) \cdot \nabla w dx + \int_\Omega f(x, \underline{u}) w dx + \int_{\partial\Omega} g(x, \underline{u}) w \, dS \le 0, \\ \forall w \in W^{1,p}(\Omega), \ w \ge 0 \quad \text{a.e. on } \Omega.\end{gathered} \tag{3.8}$$

We have a similar condition for supersolutions of (3.7). These concepts of sub- and supersolutions here coincide with the classical ones for sub- and supersolutions in Neumann problems. Our definitions here also cover the cases where the Neumann conditions also depend on u. In fact, when $g(x, u) = au$, we have a Robin boundary condition.

3.3. Mixed boundary conditions. By choosing $K = \{u \in W^{1,p}(\Omega) : u = h \text{ on } \Gamma\}$, where Γ is a measurable subset of $\partial\Omega$, we have the equation (3.1) with a mixed boundary condition consisting of a Dirichlet condition on Γ and a Neumann/Robin condition on $\partial\Omega \setminus \Gamma$.

3.4. Obstacle problems. Let K be a convex subset of $W_0^{1,p}(\Omega)$. The inequality (2.2), in this case, formulates problems with unilateral constraints (such as obstacle problems) and homogeneous Dirichlet boundary conditions, which were discussed in [29]. Many results in that paper are particular cases of those discussed here. In fact, the general definitions of sub- supersolutions presented here are motivated in part by the concepts and arguments in [29].

3.5. **Zero flux and periodic problems.** Let us consider the choice

$$K = \{u \in W^{1,p}(\Omega) : u = \text{const on } \partial\Omega\}.$$

For $u \in W^{1,p}(\Omega)$, we note that $u \vee K \subset K$ (resp. $u \wedge K \subset K$) if and only if $u \in K$. In fact, it is clear that if $u \in K$, then $u \vee K, u \wedge K \subset K$. Conversely, assume that $u \vee K \subset K$. For any constant function c, we have $u \vee c = \max\{u, c\} =$ constant on $\partial\Omega$. Therefore, either $u \le c$ a.e. on $\partial\Omega$ or $u \ge c$ a.e. on $\partial\Omega$ (with respect to the Hausdorff measure). Since this is true for any $c \in \mathbb{R}$, we must have $u =$ constant on $\partial\Omega$, that is, $u \in K$. For $\underline{u} \in K$, we have

$$\begin{aligned} \underline{u} \wedge K &= \{v \in K : v \le \underline{u} \text{ a.e. on } \Omega\} \\ &= \{\underline{u} - w : w \in K, w \ge 0 \text{ a.e. on } \Omega\}. \end{aligned}$$

Therefore, inequality (2.8) is equivalent to

$$\int_\Omega A(x, \nabla \underline{u}) \cdot \nabla v dx + \int_\Omega f(x, \underline{u}) v dx + \int_{\partial\Omega} g(x, \underline{u}) v \, dS \le 0,$$

for all $v \in K$ such that $v \ge 0$ on Ω. One has a similar equivalence for supersolutions. Note that in this particular case, the definitions for sub- and supersolutions here reduce to those in [26]. We note that in the case that $g \equiv 0$ the problem considered here is the boundary value problem

$$\begin{gathered} -\operatorname{div} A(x, \nabla u) + f(x, u) = 0 \\ u|_{\partial\Omega} = \text{constant}, \quad \int_{\partial\Omega} A(x, \nabla u) \cdot n \, dS = 0, \end{gathered}$$

where the constant boundary data are not specified. This problem in dimension $N = 1$ (the periodic boundary value problem) was first studied by sub- and supersolution methods by Knobloch [21]. (See also [4], [40], where free boundary problems of this type are studied.)

3.6. **Unilateral problems.** For another example, let us consider the boundary value problem consisting of (3.1) and the following unilateral boundary condition on the boundary:

$$\begin{gathered} u \ge \psi, \\ A(x, \nabla u) \cdot n \ge 0, \\ (u - \psi)[A(x, \nabla u) \cdot n] = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.9}$$

(ψ is a measurable function on $\partial\Omega$) which occur in problems with semi-permeable media. The problem can be formulated as the variational inequality (2.2) with $g = 0$ and

$$K = \{u \in W^{1,p}(\Omega) : u \ge \psi \text{ a.e. on } \partial\Omega\}.$$

It is worth noting that in this case, there is a non-symmetry concerning conditions (2.7) and (2.10) in the definitions of sub- and supersolutions. In fact, it is easy to see that for $\underline{u}, \overline{u} \in W^{1,p}(\Omega)$, $\underline{u}$ always satisfies (2.7), while (2.10) holds if and only if $\overline{u} \ge \psi$ a.e. on $\partial\Omega$, that is $\overline{u} \in K$.

Some Remarks

Problems for which the domain Ω is unbounded may be tackled in a similar way by using classical approaches (see e.g. [24]).

The above definitions and approach may be extended in a straightforward manner to problems with lower terms depending also on the gradient of u, i.e., $f = f(x, u, \nabla u)$. We can also extend them to problems with locally Lipschitz constraints together with convex constraints (variational hemivariational inequalities) such as those considered, for example, in [6] and the references therein.

For some recent applications of the sub- supersolution method in the study of zero flux problems we refer to [4] and Ambrosetti-Prodi type problems for the p-Laplacian to [23].

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