Sixth Mississippi State Conference on Differential Equations and Computational Simulations, *Electronic Journal of Differential Equations*, Conference 15 (2007), pp. 229–238.
ISSN: 1072-6691. URL: http://ejde.mathMississippi State Universitytxstate.edu or http://ejde.math.unt.edu
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GLOBAL ATTRACTIVITY IN A NONLINEAR DIFFERENCE EQUATION

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ABSTRACT. In this paper, we study the asymptotic behavior of positive solutions of the nonlinear difference equation

$$x_{n+1} = x_n f(x_{n-k}),$$

where $f : [0,\infty) \to (0,\infty)$ is a unimodal function, and k is a nonnegative integer. Sufficient conditions for the positive equilibrium to be a global attractor of all positive solutions are established. Our results can be applied to to some difference equations derived from mathematical biology.

1. INTRODUCTION

Our aim in this paper is to study the global attractivity of the difference equation

$$x_{n+1} = x_n f(x_{n-k}), \quad n = 0, 1, \dots \tag{1.1}$$

where $k \in \{0, 1, 2, \dots\}$, and $f : [0,\infty) \to (0,\infty)$ is a unimodal function; i.e, f is first increasing, and then decreasing. Global attractivity of (1.1) under different assumptions on f has been studied by several authors, see, for example, Kocic and Ladas [5], Qian [10] and Graef and Qian [2]. However, few results can be found under the assumption that f is a unimodal function. Besides its theoretic interest, our motivation to study (1.1) comes from the following observation. Consider the delay logistic equation

$$x'(t) = x(t)[a + bx(t-\tau) - cx^2(t-\tau)], \quad t \ge 0, \tag{1.2}$$

where

$$a, c, \tau \in (0,\infty) \quad \text{and} \quad b \in (-\infty,\infty). \tag{1.3}$$

This equation is a model of single species with a quadratic per-capita growth rate (see [1] for details). One may see [4] and [7] also for some extensions of (1.2). The following difference equation with piecewise constant arguments

$$x'(t) = x(t)[a + bx([t-k]) - cx^2([t-k])], \quad t \ge 0, \tag{1.4}$$

2000 *Mathematics Subject Classification.* 39A10.
Key words and phrases. Nonlinear difference equation; global attractor; unimodal function; positive equilibrium.

Published February 28, 2007.

where $[\cdot]$ denotes the greatest integer function, may be viewed as a semidiscretization of (1.2). By an argument similar to that in [3, Section 8.2], one can see that (1.4) leads to the following equation

$$x_{n+1} = x_n e^{a+bx_{n-k}-cx_{n-k}^2}, \quad n = 0, 1, \dots \tag{1.5}$$

whose oscillatory and stability properties completely characterize the same properties for (1.4), and so this leads us to study (1.5), which is a special case of (1.1).

In the following, we only consider the positive solutions of (1.1). If we let

$$x_{-k}, x_{-k+1}, \dots, x_0 \tag{1.6}$$

be $k+1$ given nonnegative numbers with $x_0 > 0$, and $\bar{x}$ be the unique solution of $f(x) = 1$, then (1.1) has a unique positive solution with initial condition (1.6). Clearly, $\bar{x}$ is the only positive equilibrium. In the following section, we establish some sufficient conditions such that $\bar{x}$ attracts all the positive solutions of (1.1). Then, in Section 3, we apply our results about (1.1) to (1.5) to establish some sufficient conditions for the global attractivity of the positive equilibrium of (1.5).

2. Global attractivity of (1.1)

Consider the difference equation

$$x_{n+1} = g(x_n), \quad n = 1, 2, \dots, \tag{2.1}$$

where $g \in C[R, R]$. Let G be any set in R. We say that V is a Liapunov function for (2.1) on G if

(i) V is continuous on G, and
(ii) $\dot{V}(x) = V(g(x)) - V(x) \le 0$ for all $x \in G$.

The following lemma on the asymptotic behavior of (2.1) is taken from [8] and will be needed later.

Lemma 2.1. *Let G be a bounded open positively invariant set. If*

(i) *V is a Liapunov function for (2.1) on G,*
(ii) *$M \subset G$, where M is the largest invariant set of $E = \{x : \dot{V}(x) = 0, x \in \bar{G}\}$,*
(iii) *V is constant on M.*

Then M is globally asymptotically stable relative to G.

The following theorem is our main result in this section.

Theorem 2.2. *Let $f : [0, \infty) \to (0, \infty)$ satisfy the following assumptions:*

(i) *f is a unimodal function obtaining its maximum at $x = x^*$;*
(ii) *$\bar{x} > x^*$ is the unique equilibrium point;*
(iii) *$[\ln f(x)]'' \le 0$ on (x^*, m_0);*
(iv) *$\bar{x} f(m_0)^{k+1} \ge x^*$;*
(v) *$f(m_0) \ge \frac{1}{f(x^*)}$,*

where $m_0 = \bar{x} f(x^*)^{k+1}$. Then $\bar{x}$ is the global attractor of all positive solutions of (1.1).

Proof. First, we show that every non-oscillatory (about $\bar{x}$) solution $\{x_n\}$ tends to $\bar{x}$. We assume that $x_n \ge \bar{x}$ eventually. The proof for the case that $x_n \le \bar{x}$ eventually is similar, and so is omitted. By (1.1),

$$\frac{x_{n+1}}{x_n} = f(x_{n-k}).$$

Since for large n, $x_{n-k} \geq \bar{x}$, and $\bar{x} > x^*$, we have $\frac{x_{n+1}}{x_n} \leq f(\bar{x}) = 1$. Hence, $\{x_n\}$ is non-increasing and $\lim_{n\to\infty} x_n = l$ exists. Obviously, $l = \bar{x}$.

Next, we show that every oscillatory solution tends to $\bar{x}$. Suppose that $\{x_n\}$ is an oscillatory solution of (1.1). We say that x_s is a local maximum of $\{x_n\}$, if

$$x_s \geq \bar{x}, \quad x_s \geq x_{s-1}, \quad x_s \geq x_{s+1}.$$

Similarly, we say that x_r is a local minimum of $\{x_n\}$, if

$$x_r \leq \bar{x}, \quad x_r \leq x_{r-1}, \quad x_r \leq x_{r+1}.$$

By (1.1),

$$x_s = x_{s-1} f(x_{s-k-1}) \geq x_{s-1},$$

so we have $f(x_{s-k-1}) \geq 1$, and hence $x_{s-k-1} \leq \bar{x}$. Thus,

$$x_s = x_{s-k-1} \prod_{i=s-k}^{s} f(x_{i-k-1}) \leq \bar{x}[f(x^*)]^{k+1} = m_0.$$

So, m_0 is an upper bound of $\{x_n\}$. By a similar argument, we have

$$x_r \geq \bar{x}[f(m_0)]^{k+1} = m_1.$$

Thus, there exists some $N_0 > 0$, such that

$$m_1 \leq x_n \leq m_0, \text{ for } n \geq N_0.$$

Notice that under assumption (iv), $m_1 = \bar{x} f(m_0)^{k+1} \geq x^*$, by induction, we can prove that

$$m_{2s+1} \leq x_n \leq m_{2s}, \quad \text{for } n \geq N_s$$

where $\{m_s\}$ is defined by

$$\begin{gathered} m_{s+1} = \bar{x}[f(m_s)]^{k+1} \\ m_0 = \bar{x}[f(x^*)]^{k+1}. \end{gathered} \tag{2.2}$$

To prove x_n tends to $\bar{x}$, it suffices to show that m_s tends to $\bar{x}$. Let $G = (0, m_0)$. Clearly, G is a positively invariant set of IVP (2.2). Define

$$V(x) = \big(\ln \frac{x}{\bar{x}}\big)^2, \quad x \in G.$$

Then

$$\dot{V}(x) = [(k+1)\ln f(x)]^2 - \big(\ln \frac{x}{\bar{x}}\big)^2.$$

Let

$$g(x) = (k+1)\ln f(x).$$

To get $\dot{V}(x) < 0$ on G for $x \neq \bar{x}$, we need $|g(x)| < |\ln \frac{x}{\bar{x}}|$, which is equivalent to

$$\begin{aligned} g(x) &< -\ln \frac{x}{\bar{x}} = \ln \frac{\bar{x}}{x} \quad \text{for } x < \bar{x}; \\ g(x) &> -\ln \frac{x}{\bar{x}} = \ln \frac{\bar{x}}{x} \quad \text{for } x > \bar{x}. \end{aligned} \tag{2.3}$$

Let $h(x) = \ln \frac{\bar{x}}{x}$. Observe that

$$g'(x) = \frac{(k+1)f'(x)}{f(x)} < 0, \quad g''(x) = (k+1)(\ln f(x))'' \leq 0 \quad \text{on } (x^*, m_0); \tag{2.4}$$

and

$$\big(h(x)\big)' = -\frac{1}{x} < 0, \quad \big(h(x)\big)'' = \frac{1}{x^2} > 0 \quad \text{for } x > 0. \tag{2.5}$$

So, g and h look as in the following graph

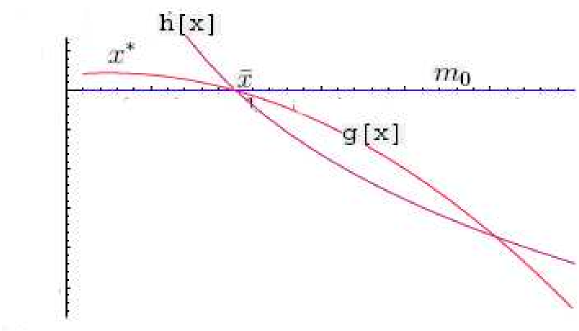

FIGURE 1. $g(x^*) = \max g(x)$, $g(\bar{x}) = h(\bar{x})$, by the concavity of f and g, $h(x) > g(x)$ for $x < \bar{x}$, $h(x) < g(x)$ for $x > \bar{x}$

We first show that (2.3) holds on $(\bar{x}, m_0)$. Since $g(\bar{x}) = \ln \frac{\bar{x}}{\bar{x}} = 0$, by the monotonicity and concavity of g and $\ln \frac{\bar{x}}{x}$, it's enough to show that $g(m_0) \geq \ln(\bar{x}/m_0)$; i.e,

$$[f(m_0)]^{k+1} \geq \frac{\bar{x}}{m_0} = \frac{\bar{x}}{\bar{x} f(x^*)^{k+1}} = \frac{1}{f(x^*)^{k+1}}.$$

From (v), we can see that (2.3) holds immediately on $(\bar{x}, m_0)$. Furthermore, we have $g'(x) > \left(\ln \frac{\bar{x}}{x}\right)'$ at $x = \bar{x}$. So, for $x \in (\bar{x}, x^*)$, again by (2.4) and (2.5), we know that $g(x) > \ln \frac{\bar{x}}{x}$. For $x \leq x^*$, since f is increasing and $\ln \frac{\bar{x}}{x}$ is decreasing, (2.3) is satisfied automatically. Thus, we have

$$\dot{V}(x) < 0 \quad \text{for } x \in G \quad \text{and} \quad x \neq \bar{x},$$

and

$$E = \{x : \dot{V}(x) = 0,\ x \in G\} = \{\bar{x}\}.$$

Hence, by Lemma 2.1, $\bar{x}$ is a global attractor relative to G, and so every solution $\{m_s\}$ of IVP (2.2) tends to $\bar{x}$. Then, it follows that $\{x_n\}$ tends to $\bar{x}$. The proof of Theorem 2.2 is complete. $\square$

We can get a linearized stability result by using the following lemma.

Lemma 2.3 ([9]). *Assume that $q \in \mathbf{R}$ and $k \in \{0, 1, 2, \dots\}$. Then the delay difference equation*

$$x_{n+1} - x_n + q x_{n-k} = 0, \quad n = 0, 1, \dots \tag{2.6}$$

is asymptotically stable if and only if

$$0 < q < 2 \cos \frac{k\pi}{2k+1}. \tag{2.7}$$

Corollary 2.4. *If the assumptions of Theorem 2.2 hold, and*

$$-f'(\bar{x})\bar{x} < 2 \cos \frac{k\pi}{2k+1}, \tag{2.8}$$

then (1.1) is globally asymptotically stable.

Proof. Let $y_n = \ln x_n$, (1.1) becomes

$$y_{n+1} - y_n - \ln f(e^{y_{n-k}}) = 0, \quad n = 0, 1, \dots \tag{2.9}$$

The linearized form of (2.9) is

$$y_{n+1} - y_n - f'(\bar{x})\bar{x}y_{n-k} = 0, \quad n = 0, 1, \dots \tag{2.10}$$

Let $p = -f'(\bar{x})\bar{x}$. We have $p > 0$ since $f'(\bar{x}) < 0$. By (2.7), we have that (2.10) is stable if (2.8) holds. Hence, (1.1) is locally stable. Then combining with the global attractivity from Theorem 2.2, we get the global stability result. $\square$

Although we can not prove it now, we believe that if the conditions in Theorem 2.2 hold, then (2.8) holds, and so the conditions in Theorem 2.2 imply the global stability of (1.1). In Section 3, we will show that this is true for (1.5).

3. Global attractivity of (1.5)

In this section, we apply our results in Section 2 to establish some sufficient conditions for the global attarctivity of (1.5). Two cases of (1.5) with $b \le 0$ and $b > 0$ are considered. For (1.5), $\bar{x} = \frac{b+\sqrt{b^2+4ac}}{2c}$ is the only positive equilibrium.

Theorem 3.1. *Assume that $b \le 0$, and*

$$(k+1)a \le \ln \frac{b + \sqrt{b^2 + 8ac}}{b + \sqrt{b^2 + 4ac}}. \tag{3.1}$$

Then $\bar{x}$ is a global attractor of all positive solutions of (1.5).

Proof. To apply Theorem 2.2, we need to show that all assumptions of Theorem 2.2 are satisfied. Here, $f(x) = \exp\big(a + bx - cx^2\big)$, and clearly, $f(x)$ is decreasing on $(0, \infty)$, obtaining its maximum at $x^* = 0$. Note that

$$m_0 = \bar{x}f(0)^{k+1} = \bar{x}e^{(k+1)a}, \tag{3.2}$$

and it is easy to see that assumption (ii), (iii) and (iv) of Theorem 2.2 are satisfied. Now, wee only need to check (v), which is

$$e^{a+bm_0-cm_0^2} \ge \frac{1}{e^a} = e^{-a},$$

that is,

$$a + bm_0 - cm_0^2 \ge -a. \tag{3.3}$$

Let $g(x) = cx^2 - bx - 2a$. Then $x_1 = \frac{b+\sqrt{b^2+8ac}}{2c}$ is the only positive solution of $g(x) = 0$. Since g is increasing on $(0, \infty)$, (3.3) is equivalent to $m_0 \le x_1$, which is (3.1). Thus, all assumptions of Theorem 2.2 are satisfied, and so $\bar{x}$ is a global attractor of (1.5). $\square$

The following result is a consequence of the above theorem and Corollary 2.4.

Corollary 3.2. *If $b \le 0$ and* (3.1) *hold, then* (1.5) *is globally asymptotically stable.*

Proof. By (2.8) in Corollary 2.4, we need

$$-f'(\bar{x})\bar{x} = \frac{\sqrt{b^2 + 4ac}(b + \sqrt{b^2 + 4ac})}{2c} < 2\cos\frac{k\pi}{2k+1}. \tag{3.4}$$

Now, we claim that (3.1) implies (3.4). First, we want to simplify the expression. Let $A = (k+1)a$ and $C = c/(k+1)$. From (3.1), it is easy to see that $A \leq \ln 2$, and (3.4) can be written as

$$\frac{\sqrt{b^2+4AC}(b+\sqrt{b^2+4AC})}{2C} < 2(k+1)\cos\frac{k\pi}{2k+1}. \tag{3.5}$$

Next, we let

$$B = \frac{|b|}{\sqrt{C}} = -\frac{b}{\sqrt{C}} > 0.$$

Then (3.5) can be written as

$$\frac{\sqrt{B^2+4A}(-B+\sqrt{B^2+4A})}{2} < 2(k+1)\cos\frac{k\pi}{2k+1}.$$

After simplification, this becomes

$$\frac{\sqrt{B^2+4A}}{B+\sqrt{B^2+4A}}A < (k+1)\cos\frac{k\pi}{2k+1}. \tag{3.6}$$

Since $A \leq \ln 2$, the left hand side of Inequality (3.6) is less than $\ln 2$. On the other hand, the right hand side of (3.6) can be written as

$$(k+1)\sin\frac{\pi}{4(k+\frac{1}{2})}.$$

If we use $s = k + \frac{1}{2}$, then the right hand side of Inequality (3.6) is

$$g(s) = (s+\frac{1}{2})\sin\frac{\pi}{4s}, \quad s \geq \frac{1}{2}.$$

We claim that g is a decreasing function for $s \geq \frac{1}{2}$. Observe that

$$g'(s) = \sin\frac{\pi}{4s} - \Big(\frac{\pi}{4s} + \frac{\pi}{8s^2}\Big)\cos\frac{\pi}{4s},$$

and notice that $\frac{\pi}{4s} \geq \sin\frac{\pi}{4s}$ for $s \geq \frac{1}{2}$,

$$\begin{aligned}
g''(s) &= \frac{\pi}{4s^3}\cos\frac{\pi}{4s} - \Big(\frac{\pi^2}{16s^3} + \frac{\pi^2}{32s^4}\Big)\sin\frac{\pi}{4s} \\
&\geq \frac{\pi}{4s^3}\Big[1 - \Big(\frac{\pi}{4s}\Big)^2\Big] - \Big(\frac{\pi^2}{16s^3} + \frac{\pi^2}{32s^4}\Big)\frac{\pi}{4s} \\
&= \frac{\pi}{4s^3}\Big(1 - \frac{\pi^2}{16s^2} - \frac{\pi^2}{16s} - \frac{\pi^2}{32s^2}\Big) \\
&\geq \frac{\pi}{4s^3}\Big(1 - \frac{\pi^2}{12}\Big) > 0.
\end{aligned}$$

So g' is increasing, $\max_{s\geq\frac{1}{2}} g'(s) = \lim_{s\to\infty} g'(s) = 0$, and g is decreasing for $s \geq \frac{1}{2}$. We get

$$\min_{s\geq\frac{1}{2}} g(s) = \lim_{s\to\infty}(s+\frac{1}{2})\sin\frac{\pi}{4s} = \frac{\pi}{4},$$

which is greater than $\ln 2$. Thus (3.6) holds, and therefore (3.4) holds. The proof is complete. $\square$

Example 3.3. Consider the difference equation

$$x_{n+1} = x_n e^{0.1 - x_{n-1} - x_{n-1}^2}.$$

Here $k = 1$, $a = 0.1$, $b = -1$, $c = 1$, and

$$(k+1)a = .2 < \ln \frac{b + \sqrt{b^2 + 8ac}}{b + \sqrt{b^2 + 4ac}} \approx 0.623.$$

Hence, by Corollary 3.2, $\bar{x}$ is globally asymptotically stable.

Theorem 3.4. *Assume $b > 0$ and*

$$\frac{(k+1)D}{4c} \le \ln \frac{b + \sqrt{2D}}{b + \sqrt{D}}, \tag{3.7}$$

where $D = b^2 + 4ac$. Then all positive solutions of (1.5) *tend to $\bar{x}$.*

Proof. Let $f(x) = \exp\big(a + bx - cx^2\big)$. We show that f satisfies all the conditions assumed in Theorem 2.2. Clearly, f is increasing on $(0, x^*)$ and decreasing on (x^*, ∞), where $x^* = \frac{b}{2c}$, and so assumption (i) is satisfied. (ii) and (iii) are also easy to check. We see that to have $\bar{x} f(m_0)^{k+1} \ge x^*$, we need

$$(a + bm_0 - cm_0^2) \ge \frac{1}{k+1} \ln \frac{b}{b + \sqrt{D}}. \tag{3.8}$$

By a direct but tedious calculation, (3.8) is equivalent to

$$\frac{(k+1)D}{4c} \le \ln \frac{b + \sqrt{D + \frac{4c}{k+1} \ln \frac{b+\sqrt{D}}{b}}}{b + \sqrt{D}}. \tag{3.9}$$

We claim that (3.7) implies (3.9). To prove our claim, it suffices to show

$$\frac{b + \sqrt{2D}}{b + \sqrt{D}} \le \frac{b + \sqrt{D + \frac{4c}{k+1} \ln \frac{b+\sqrt{D}}{b}}}{b + \sqrt{D}},$$

which is equivalent to

$$\sqrt{2D} \le \sqrt{D + \frac{4c}{k+1} \ln \frac{b + \sqrt{D}}{b}},$$

that is

$$\frac{(k+1)D}{4c} \le \ln \frac{b + \sqrt{D}}{b}.$$

By (3.7), it is sufficient to show that

$$\ln \frac{b + \sqrt{2D}}{b + \sqrt{D}} \le \ln \frac{b + \sqrt{D}}{b}. \tag{3.10}$$

It is not difficult to see that (3.10) is equivalent to

$$\frac{\sqrt{2} - 1}{b + \sqrt{D}} \le \frac{1}{b},$$

which is obviously true. Thus, (iv) is satisfied.

To check (v), we need $\exp\big(a + bm_0 - cm_0^2\big) \ge \exp\big(-(a + bx^* - cx^{*2})\big)$; i.e,

$$a + bm_0 - cm_0^2 \ge -\frac{D}{4c}. \tag{3.11}$$

Let

$$h(x) = a + bx - cx^2 + \frac{D}{4c}. \tag{3.12}$$

We see that the positive solution of (3.12) is $x_2 = \frac{b+\sqrt{2D}}{2c}$. Since both m_0 and x_2 are larger than x^*, and on (x^*, ∞), h is decreasing, (3.11) is equivalent to $m_0 \leq x_2$, which is satisfied by (3.7). Hence, it follows by Theorem 2.2 that $\{x_n\}$ tends to $\bar{x}$. □

Corollary 3.5. *Assume that* (3.7) *holds. Then* (1.5) *is globally asymptotically stable.*

Proof. By condition (2.8) in Corollary 2.4, we need

$$-f'(\bar{x})\bar{x} = \frac{\sqrt{D}(b+\sqrt{D})}{2c} < 2\cos\frac{k\pi}{2k+1},$$

which is equivalent to

$$\frac{D}{4c} < \cos\frac{k\pi}{2k+1} - \frac{b\sqrt{D}}{4c}.$$

Then, combining this with Theorem 3.4, we know that if

$$\frac{D}{4c} \leq \min\{\frac{1}{k+1}\ln\frac{b+\sqrt{2D}}{b+\sqrt{D}}, \cos\frac{k\pi}{2k+1} - \frac{b\sqrt{D}}{4c}\} \tag{3.13}$$

holds, then (1.5) is globally asymptotically stable. Now, we show that

$$\frac{1}{k+1}\ln\frac{b+\sqrt{2D}}{b+\sqrt{D}} \leq \cos\frac{k\pi}{2k+1} - \frac{b\sqrt{D}}{4c}, \tag{3.14}$$

under the assumption (3.7). Observe that

$$\frac{b\sqrt{D}}{4c} = \frac{b}{\sqrt{D}}\frac{D}{4c} \leq \frac{b}{\sqrt{D}}\frac{1}{k+1}\ln\frac{b+\sqrt{2D}}{b+\sqrt{D}}$$

by (3.7). So it suffices to show that

$$(\frac{b}{\sqrt{D}}+1)\frac{1}{k+1}\ln\frac{b+\sqrt{2D}}{b+\sqrt{D}} \leq \cos\frac{k\pi}{2k+1}.$$

After simplification, this becomes

$$(\frac{b}{\sqrt{D}}+1)\ln\frac{b+\sqrt{2D}}{b+\sqrt{D}} \leq (k+1)\sin\frac{\pi}{4k+2}.$$

Let $\frac{b}{\sqrt{D}} = t$ $(0 < t < 1)$ and $s = k + \frac{1}{2}$ $(s \geq \frac{1}{2})$. It is sufficient to show that

$$(t+1)\ln\frac{t+\sqrt{2}}{t+1} \leq (s+\frac{1}{2})\sin\frac{\pi}{4s}. \tag{3.15}$$

If the maximum of the function of t is less than the minimum of the function of s, then we are done. Let

$$f(t) = (t+1)\ln\frac{t+\sqrt{2}}{t+1}, \quad g(s) = (s+\frac{1}{2})\sin\frac{\pi}{4s}.$$

We claim that f is increasing on $(0,1)$. To this end, observe that

$$f'(t) = \ln\frac{t+\sqrt{2}}{t+1} - \frac{\sqrt{2}-1}{t+\sqrt{2}}, \quad f''(t) = \frac{2\sqrt{2}-3}{(t+\sqrt{2})^2(t+1)} < 0.$$

Since f' is decreasing and $\min_{0<t<1} f'(t) = f'(1) \approx 0.0166 > 0$, f is increasing for $0 < t < 1$. We know from the proof of Corollary 3.2 that g is decreasing on $(\frac{1}{2}, \infty)$. It is easy to see that $\max_{0<t<1} f(t) = f(1) \approx 0.376$ and $\min_{s \geq \frac{1}{2}} g(s) = \lim_{s\to\infty} g(s) \approx 0.786$. Thus, we have $f(t) \leq g(s)$ always; i.e, (3.15) holds. The proof of Corollary 3.5 is complete. $\square$

Example 3.6. Consider the difference equation

$$x_{n+1} = x_n e^{0.01 + x_{n-3} - 20x_{n-3}^2}.$$

Here $a = .01$, $b = 1$, $c = 20$ and $k = 3$. So

$$\frac{(k+1)D}{4c} \approx .09 \leq \ln \frac{b + \sqrt{2D}}{b + \sqrt{D}} \approx .825.$$

By Corollary 3.5, $\bar{x}$ is globally asymptotically stable.

4. Remarks

Consider the difference equation

$$x_{n+1} - x_n = x_n(a + bx_{n-k} - cx_{n-k}^2), \tag{4.1}$$

where $a, c \in (0, \infty)$, $b \in (-\infty, +\infty)$, and $k \in \{0, 1, \dots\}$. Equation (4.1) may be viewed as a discrete analogue of the delay differential equation (1.3). The global attractivity of the positive equilibrium when $k = 0$ was investigated by Rodrigues [11]. Kocic and Ladas [6], posted the following research projects: Obtain a global stability result for the positive equilibrium $\bar{x}$ of (4.1) when $k \geq 1$, and obtain explicit sufficient conditions on a, b, c and k so that all solutions of (4.1) with appropriate initial condition remain positive for all $n \geq 0$. By using an argument similar to one in this paper, we can establish a sufficient condition for the positive equilibrium $\bar{x}$ to be a global attractor of all positive solutions of (4.1). Then together with the linearized stability result, we can obtain a global stability result for the positive equilibrium $\bar{x}$ of (4.1) when $k \geq 1$. However, the problem that how to obtain explicit sufficient condition so that all solutions remain positive for all $n \geq 0$ remains open.

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