Sixth Mississippi State Conference on Differential Equations and Computational Simulations, *Electronic Journal of Differential Equations*, Conference 15 (2007), pp. 127–139.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

A NON-RESONANT GENERALIZED MULTI-POINT BOUNDARY-VALUE PROBLEM OF DIRICHELET TYPE INVOLVING A P-LAPLACIAN TYPE OPERATOR

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ABSTRACT. We study the existence of solutions for the generalized multi-point boundary-value problem

$$
\begin{gathered}
(\phi(x'))' = f(t,x,x') + e \quad 0<t<1,\\
x(0)=\sum_{i=1}^{m-2} a_i x(\xi_i), \quad x(1)=\sum_{j=1}^{n-2} b_j x(\tau_j),
\end{gathered}
$$

in the non-resonance case. Our methods consist in using topological degree and some a priori estimates.

1. Introduction

Let ϕ be an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ satisfying $\phi(0)=0$, $f:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a function satisfying Carathéodory conditions and $e:[0,1]\to\mathbb{R}$ be a function in $L^1[0,1]$. Let $\xi_i,\tau_j\in(0,1)$, a_i, $b_j\in\mathbb{R}$, $i=1,2,\dots,m-2$, $j=1,2,\dots,n-2$, $0<\xi_1<\xi_2<\dots<\xi_{m-2}<1$, $0<\tau_1<\tau_2<\dots<\tau_{n-2}<1$ be given. We study the problem of existence of solutions for the generalized multi-point boundary-value problem

$$
\begin{gathered}
(\phi(x'))' = f(t,x,x') + e, \quad 0<t<1,\\
x(0)=\sum_{i=1}^{m-2} a_i x(\xi_i), \quad x(1)=\sum_{j=1}^{n-2} b_j x(\tau_j),
\end{gathered}
\tag{1.1}
$$

in the non-resonance case. We say that this problem is non-resonant if the associated problem:

$$
\begin{gathered}
(\phi(x'))' = 0, \quad 0<t<1,\\
x(0)=\sum_{i=1}^{m-2} a_i x(\xi_i), \quad x(1)=\sum_{j=1}^{n-2} b_j x(\tau_j),
\end{gathered}
\tag{1.2}
$$

2000 *Mathematics Subject Classification.* 34B10, 34B15, 34L30, 34L90.
Key words and phrases. Generalized multi-point boundary value problems; non-resonance; p-Laplace type operator; a priori estimates; topological degree.

Published February 28, 2007.

has the trivial solution as its only solution. This is the case, (see Proposition 2.1 below), if

$$\Big(\sum_{i=1}^{m-2} a_i\xi_i\Big)\Big(1-\sum_{j=1}^{n-2} b_j\Big) \neq \Big(1-\sum_{i=1}^{m-2} a_i\Big)\Big(\sum_{j=1}^{n-2} b_j\tau_j - 1\Big).$$

This problem was studied by Gupta, Ntouyas, and Tsamatos in [20] and by the author in [16] when the homeomorphism ϕ from $\mathbb{R}$ onto $\mathbb{R}$ is the identity homeomorphism, i.e for second order ordinary differential equations. The study of multi-point boundary value problems for second order ordinary differential equations was initiated by Il'in and Moiseev in [22, 23] motivated by the works of Bitsadze and Samarskii on nonlocal linear elliptic boundary value problems, [2, 3, 4] and has been the subject of many papers, see for example, [5, 6, 11, 12, 13, 14, 15, 17, 18, 19, 21, 24, 27, 28]. More recently multipoint boundary value problems involving a p-Lalacian type operator or the more general operator $-(\phi(x'))'$ has been studied in [1, 7, 8, 9, 10, 25] to mention a few.

We present in Section 2 some a priori estimates for functions $x(t)$ that satisfy the boundary conditions in (1.1). Our a priori estimates are sharper versions of the corresponding estimates in [16] and explicitly utilize the non-resonance condition for the boundary value problem (1.1). In section 3, we present an existence theorem for the boundary value problem (1.1) using degree theory.

2. A Priori Estimates

We shall assume throughout that ϕ is an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ satisfying $\phi(0) = 0$. We shall also assume that the homeomorphism ϕ satisfies the following conditions:

(a) For any constant $M > 0$,

$$\limsup_{z\to\infty} \frac{\phi(Mz)}{\phi(z)} \equiv \alpha(M) < \infty. \tag{2.1}$$

(b) For any σ, $0 \le \sigma < 1$,

$$\widetilde{\alpha}(\sigma) \equiv \limsup_{z\to\infty} \frac{\phi(\sigma z)}{\phi(z)} < 1. \tag{2.2}$$

Proposition 2.1. *The boundary-value problem* (1.2) *has only the trivial solution if and only if*

$$\Big(\sum_{i=1}^{m-2} a_i\xi_i\Big)\Big(1-\sum_{j=1}^{n-2} b_j\Big) \neq \Big(1-\sum_{i=1}^{m-2} a_i\Big)\Big(\sum_{j=1}^{n-2} b_j\tau_j - 1\Big). \tag{2.3}$$

Proof. It is obvious that $x(t) = At + B$, $t \in [0, 1]$, A, $B \in \mathbb{R}$, is a general solution for the differential equation

$$(\phi(x'))' = 0, \quad 0 < t < 1,$$

in (1.2). If, now, $x(t) = At + B$, $t \in [0, 1]$, A, $B \in \mathbb{R}$, is a solution to the boundary value problem (1.2) then we must have

$$B = \sum_{i=1}^{m-2} a_i(A\xi_i + B), \quad A + B = \sum_{j=1}^{n-2} b_j(A\tau_j + B).$$

In other words A, B must satisfy the system of equations

$$\begin{aligned} A(\sum_{i=1}^{m-2} a_i\xi_i) + B(\sum_{i=1}^{m-2} a_i - 1) = 0, \\ A(\sum_{j=1}^{n-2} b_j\tau_j - 1) + B(\sum_{j=1}^{n-2} b_j - 1) = 0. \end{aligned} \tag{2.4}$$

Now, the system of equations (2.4) has $A = 0$, $B = 0$ as the only solution if and only if

$$\det \begin{pmatrix} \sum_{i=1}^{m-2} a_i\xi_i & \sum_{i=1}^{m-2} a_i - 1 \\ \sum_{j=1}^{n-2} b_j\tau_j - 1 & \sum_{j=1}^{n-2} b_j - 1 \end{pmatrix} \neq 0,$$

or

$$\Big(\sum_{i=1}^{m-2} a_i\xi_i\Big)\Big(\sum_{j=1}^{n-2} b_j - 1\Big) - \Big(\sum_{i=1}^{m-2} a_i - 1\Big)\Big(\sum_{j=1}^{n-2} b_j\tau_j - 1\Big) \neq 0. \tag{2.5}$$

It is now obvious that (2.5) is equivalent to (2.3). Hence the boundary value problem (1.2) has only the trivial solution if and only if the condition (2.3) holds. This completes the proof of the Proposition. $\square$

We shall assume in the following that $\xi_i, \tau_j \in (0,1)$, $a_i, b_j \in \mathbb{R}$, $i = 1, 2, \dots, m-2$, $j = 1, 2, \dots, n-2$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \dots < \tau_{n-2} < 1$ satisfy the condition (2.3). We observe that when condition (2.3) holds at least one of $1 - \sum_{i=1}^{m-2} a_i$, $1 - \sum_{j=1}^{n-2} b_j$ is non-zero. Now, for $a \in R$, we set $a^+ = \max(a, 0)$, $a^- = \max(-a, 0)$ so that $a = a^+ - a^-$ and $|a| = a^+ + a^-$. Next, in case $1 - \sum_{i=1}^{m-2} a_i \neq 0$, we notice that

$$\sigma_1 \equiv \min\Big\{\frac{\sum_{i=1}^{m-2} a_i^+}{1 + \sum_{i=1}^{m-2} a_i^-}, \frac{1 + \sum_{i=1}^{m-2} a_i^-}{\sum_{i=1}^{m-2} a_i^+}\Big\} \in [0, 1)$$

is well-defined. Similarly, if $1 - \sum_{j=1}^{n-2} b_j \neq 0$, we see that

$$\sigma_2 \equiv \min\{\frac{\sum_{j=1}^{n-2} b_j^+}{1 + \sum_{j=1}^{n-2} b_j^-}, \frac{1 + \sum_{j=1}^{n-2} b_j^-}{\sum_{j=1}^{n-2} b_j^+}\} \in [0, 1)$$

is well-defined. Accordingly, let us define

$$\sigma_1 \equiv \begin{cases} \min\{\frac{\sum_{i=1}^{m-2} a_i^+}{1+\sum_{i=1}^{m-2} a_i^-}, \frac{1+\sum_{i=1}^{m-2} a_i^-}{\sum_{i=1}^{m-2} a_i^+}\} \in [0, 1) & \text{if } 1 - \sum_{i=1}^{m-2} a_i \neq 0, \\ 1 & \text{if } 1 - \sum_{i=1}^{m-2} a_i = 0; \end{cases} \tag{2.6}$$

and

$$\sigma_2 \equiv \begin{cases} \min\{\frac{\sum_{j=1}^{n-2} b_j^+}{1+\sum_{j=1}^{n-2} b_j^-}, \frac{1+\sum_{j=1}^{n-2} b_j^-}{\sum_{j=1}^{n-2} b_j^+}\} \in [0, 1) & \text{if } 1 - \sum_{j=1}^{n-2} b_j \neq 0, \\ 1 & \text{if } 1 - \sum_{j=1}^{n-2} b_j = 0. \end{cases} \tag{2.7}$$

The a priori estimate obtained in the following proposition is a sharpening of the a priori estimate of Lemma 2 of [16]. We repeat the details given in Lemma 2 of [16] for the sake of completeness.

Proposition 2.2. *Let ξ_i,$\tau_j \in (0,1)$, a_i, $b_j \in \mathbb{R}$, $i = 1,2,\dots,m-2$, $j = 1,2,\dots,n-2$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \dots < \tau_{n-2} < 1$, with $(\sum_{i=1}^{m-2} a_i\xi_i)(1-\sum_{j=1}^{n-2} b_j) \neq (1-\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j\tau_j - 1)$ be given. Also let the function $x(t)$ be such that $x(t)$, $x'(t)$ be absolutely continuous on $[0,1]$ and $x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i)$, $x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j)$. Then*

$$\|x\|_\infty \leq M\|x'\|_\infty, \tag{2.8}$$

where

$$\begin{aligned}M = \min\Big\{&\frac{1}{|\sum_{i=1}^{m-2} a_i|}(\sum_{i=1}^{m-2} |a_i|\lambda_i + \frac{\sum_{i=1}^{m-2}|a_i\xi_i|}{|1-\sum_{i=1}^{m-2} a_i|}),\\ &\frac{1}{|\sum_{j=1}^{n-2} b_j|}(\sum_{j=1}^{n-2} |b_j|\mu_j + \frac{\sum_{j=1}^{n-2}|b_j(1-\tau_j)|}{|1-\sum_{j=1}^{n-2} b_j|}), 1 + \frac{\sum_{i=1}^{m-2}|a_i\xi_i|}{|1-\sum_{i=1}^{m-2} a_i|},\\ &1 + \frac{\sum_{j=1}^{n-2}|b_j(1-\tau_j)|}{|1-\sum_{j=1}^{n-2} b_j|}, \frac{1}{1-\sigma_1}, \frac{1}{1-\sigma_2}\Big\}\end{aligned}$$

with $\lambda_i = \max(\xi_i, 1-\xi_i)$ for $i = 1,2\dots,m-2$, $\mu_j = \max(\tau_j, 1-\tau_j)$ for $j = 1,2,\dots,n-2$, σ_1 as defined in (2.6) and σ_2 as defined in (2.7).

Proof. We first observe that at least one of $(1-\sum_{i=1}^{m-2} a_i)$, $(1-\sum_{j=1}^{n-2} b_j)$ is non-zero, in view of our assumption

$$\Big(\sum_{i=1}^{m-2} a_i\xi_i\Big)\Big(1-\sum_{j=1}^{n-2} b_j\Big) \neq \Big(1-\sum_{i=1}^{m-2} a_i\Big)\Big(\sum_{j=1}^{n-2} b_j\tau_j - 1\Big).$$

Accordingly, $M < \infty$. Next, we see from $x(\xi_i) - x(0) = \int_0^{\xi_i} x'(s)ds$ for $i = 1,2,\dots,m-2$ and the assumption that $x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i)$, that

$$(1-\sum_{i=1}^{m-2} a_i)x(0) = \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} x'(s)ds.$$

It then follows that

$$|x(0)| \leq \frac{\sum_{i=1}^{m-2}|a_i\xi_i|}{|1-\sum_{i=1}^{m-2} a_i|}\|x'\|_\infty. \tag{2.9}$$

Also, since $x(t) = x(\xi_i) + \int_{\xi_i}^t x'(s)ds$, we see that

$$(\sum_{i=1}^{m-2} a_i)x(t) = \sum_{i=1}^{m-2} a_i x(\xi_i) + \sum_{i=1}^{m-2} a_i \int_{\xi_i}^t x'(s)ds = x(0) + \sum_{i=1}^{m-2} a_i \int_{\xi_i}^t x'(s)ds.$$

We, now, use (2.9) to get

$$\begin{aligned}|\sum_{i=1}^{m-2} a_i||x(t)| &\leq |x(0)| + \sum_{i=1}^{m-2} |a_i|| \int_{\xi_i}^t x'(s)ds|\\ &\leq \Big(\frac{\sum_{i=1}^{m-2}|a_i\xi_i|}{|1-\sum_{i=1}^{m-2} a_i|} + \sum_{i=1}^{m-2}\lambda_i|a_i|\Big)\|x'\|_\infty.\end{aligned}$$

It is now immediate that

$$\|x\|_\infty \leq \frac{1}{|\sum_{i=1}^{m-2} a_i|}\Big(\frac{\sum_{i=1}^{m-2}|a_i\xi_i|}{|1-\sum_{i=1}^{m-2} a_i|}+\sum_{i=1}^{m-2}\lambda_i|a_i|\Big)\|x'\|_\infty. \tag{2.10}$$

Similarly, starting from $x(1)-x(\tau_j)=\int_{\tau_j}^1 x'(s)ds$ and proceeding, as above, we obtain the estimate

$$\|x\|_\infty \leq \frac{1}{|\sum_{j=1}^{n-2} b_j|}\Big(\frac{\sum_{j=1}^{n-2}|b_j(1-\tau_j)|}{|1-\sum_{j=1}^{n-2} b_j|}+\sum_{j=1}^{n-2}\mu_j|b_j|\Big)\|x'\|_\infty. \tag{2.11}$$

If we next use the equation $x(t)=x(0)+\int_0^t x'(s)ds$ and the estimate (2.9) we obtain

$$\|x\|_\infty \leq \Big(\frac{\sum_{i=1}^{m-2}|a_i\xi_i|}{|1-\sum_{i=1}^{m-2} a_i|}+1\Big)\|x'\|_\infty. \tag{2.12}$$

Similarly, starting from the equation $x(t)=x(1)-\int_t^1 x'(s)ds$, we obtain the estimate

$$\|x\|_\infty \leq (\frac{\sum_{j=1}^{n-2}|b_j(1-\tau_j)|}{|1-\sum_{j=1}^{n-2} b_j|}+1)\|x'\|_\infty. \tag{2.13}$$

Next, since $x(0)=\sum_{i=1}^{m-2} a_i x(\xi_i)$ we see that

$$x(0)+\sum_{i=1}^{m-2} a_i^- x(\xi_i)=\sum_{i=1}^{m-2} a_i^+ x(\xi_i).$$

It follows that there must exist χ_1, χ_2 in $[0,1]$ such that

$$(1+\sum_{i=1}^{m-2} a_i^-)x(\chi_1)=(\sum_{i=1}^{m-2} a_i^+)x(\chi_2). \tag{2.14}$$

If, now, one of $x(\chi_1)$, $x(\chi_2)$ is zero, we see using one of the two equations

$$x(t)=x(\chi_k)+\int_{\tau_k}^t x'(s)ds, k=1,2;\ t\in[0,1] \tag{2.15}$$

that

$$\|x\|_\infty \leq \|x'\|_\infty. \tag{2.16}$$

If both $x(\chi_1)$, $x(\chi_2)$ are non-zero and $1-\sum_{i=1}^{m-2} a_i \neq 0$, so that $1+\sum_{i=1}^{m-2} a_i^- \neq \sum_{i=1}^{m-2} a_i^+$, it is easy to see from (2.14) that $x(\chi_1)\neq x(\chi_2)$. It then follows easily from (2.14) and (2.15) that

$$\|x\|_\infty \leq \frac{1}{1-\sigma_1}\|x'\|_\infty, \tag{2.17}$$

where

$$\sigma_1=\min\{\frac{\sum_{i=1}^{m-2} a_i^+}{1+\sum_{i=1}^{m-2} a_i^-},\frac{1+\sum_{i=1}^{m-2} a_i^-}{\sum_{i=1}^{m-2} a_i^+}\}\in[0,1).$$

Similarly, we see from $x(1)=\sum_{j=1}^{n-2} b_j x(\tau_j)$ that either (2.16) holds or

$$\|x\|_\infty \leq \frac{1}{1-\sigma_2}\|x'\|_\infty, \tag{2.18}$$

where

$$\sigma_2 = \min\{\frac{\sum_{j=1}^{n-2} b_j^+}{1+\sum_{j=1}^{n-2} b_j^-}, \frac{1+\sum_{j=1}^{n-2} b_j^-}{\sum_{j=1}^{n-2} b_j^+}\} \in [0,1).$$

The proposition is now immediate from (2.10), (2.11), (2.12), (2.13), (2.16), (2.17) and (2.18) and the definitions of σ_1, σ_2 as given in (2.6), (2.7). $\square$

The following lemma is needed in the next proposition.

Lemma 2.3. *Let us set*

$$\begin{aligned} A = {} & [(1-\sum_{i=1}^{m-2} a_i)(1-\sum_{j=1}^{n-2} b_j)]^+ + \sum_{j=1}^{n-2}[b_j(1-\tau_j)(1-\sum_{i=1}^{m-2} a_i)]^+ \\ & + \sum_{i=1}^{m-2}[a_i\xi_i(1-\sum_{j=1}^{n-2} b_j)]^+ \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} B = {} & [(1-\sum_{i=1}^{m-2} a_i)(1-\sum_{j=1}^{n-2} b_j)]^- + \sum_{j=1}^{n-2}[b_j(1-\tau_j)(1-\sum_{i=1}^{m-2} a_i)]^- \\ & + \sum_{i=1}^{m-2}[a_i\xi_i(1-\sum_{j=1}^{n-2} b_j)]^-. \end{aligned} \tag{2.20}$$

Then $A \neq B$, when the non-resonance assumption (2.3) *holds.*

Proof. We note that

$$\begin{aligned} & A - B \\ & = (1-\sum_{i=1}^{m-2} a_i)(1-\sum_{j=1}^{n-2} b_j) + \sum_{j=1}^{n-2} b_j(1-\tau_j)(1-\sum_{i=1}^{m-2} a_i) + \sum_{i=1}^{m-2} a_i\xi_i(1-\sum_{j=1}^{n-2} b_j) \\ & = 1-\sum_{i=1}^{m-2} a_i - (1-\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j) + (\sum_{j=1}^{n-2} b_j)(1-\sum_{i=1}^{m-2} a_i) \\ & \quad - (1-\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j\tau_j) + (\sum_{i=1}^{m-2} a_i\xi_i)(1-\sum_{j=1}^{n-2} b_j) \\ & = 1-\sum_{i=1}^{m-2} a_i - (1-\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j\tau_j) + (\sum_{i=1}^{m-2} a_i\xi_i)(1-\sum_{j=1}^{n-2} b_j) \\ & = (\sum_{i=1}^{m-2} a_i\xi_i)(1-\sum_{j=1}^{n-2} b_j) - (1-\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j\tau_j - 1) \neq 0, \end{aligned}$$

in view of the non-resonance assumption (2.3). Hence $A \neq B$. This completes the proof of the lemma. $\square$

Let us define

$$\sigma^* = \min\{\frac{A}{B}, \frac{B}{A}\} \in [0,1), \tag{2.21}$$

where A, B are as defined in Lemma 2.3. Accordingly, we see that

$$\widetilde{\alpha}(\sigma^*) = \limsup_{z\to\infty} \frac{\phi(\sigma^* z)}{\phi(z)} < 1,$$

in view of our assumption (2.2). Let $\varepsilon > 0$ be such that $\widetilde{\alpha}(\sigma^*) + \varepsilon < 1$ and the constant C_ε be such that

$$\phi(\sigma^* z) \le (\widetilde{\alpha}(\sigma^*) + \varepsilon)\phi(z) + C_\varepsilon, \quad \text{for every } z \in \mathbb{R}. \tag{2.22}$$

Proposition 2.4. *Let $\xi_i,\tau_j \in (0,1)$, a_i, $b_j \in \mathbb{R}$, $i = 1,2,\dots,m-2$, $j = 1,2,\dots,n-2$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \dots < \tau_{n-2} < 1$, with $(\sum_{i=1}^{m-2} a_i\xi_i)(1-\sum_{j=1}^{n-2} b_j) \ne (1-\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j\tau_j - 1)$ be given. Also let the function $x(t)$ be such that $x(t)$, $x'(t)$ be absolutely continuous on $[0,1]$ with $(\phi(x'))' \in L^1(0,1)$ and $x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i)$, $x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j)$. Then*

$$\|\phi(x')\|_\infty \le \frac{1}{1-\widetilde{\alpha}(\sigma^*)-\varepsilon}\|(\phi(x'))'\|_{L^1(0,1)} + \frac{C_\varepsilon}{1-\widetilde{\alpha}(\sigma^*)-\varepsilon}, \tag{2.23}$$

where ε and C_ε are as in (2.22).

Proof. For $i = 1,2,\dots,m-2$ we see using mean value theorem that there exist χ_i in $[0,1]$ such that

$$x(\xi_i) - x(0) = \xi_i x'(\chi_i).$$

It then follows using $x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i)$ that

$$(1-\sum_{i=1}^{m-2} a_i)x(0) = \sum_{i=1}^{m-2} a_i\xi_i x'(\chi_i). \tag{2.24}$$

Again, for $j = 1,2,\dots,n-2$ we see using mean value theorem that there exist λ_j in $[0,1]$ such that

$$x(1) - x(\tau_j) = (1-\tau_j)x'(\lambda_j),$$

and we see using $x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j)$ that

$$(\sum_{j=1}^{n-2} b_j - 1)x(1) = \sum_{j=1}^{n-2} b_j(1-\tau_j)x'(\lambda_j). \tag{2.25}$$

Also, we see that there exists a $\lambda \in [0,1]$ such that

$$x(1) - x(0) = x'(\lambda). \tag{2.26}$$

Now, we see from equations (2.24), (2.25), (2.26) that

$$\begin{aligned}
&(1-\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j - 1)x'(\lambda)\\
&= (1-\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j - 1)(x(1)-x(0))\\
&= (1-\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j(1-\tau_j)x'(\lambda_j)) - (\sum_{j=1}^{n-2} b_j - 1)(\sum_{i=1}^{m-2} a_i\xi_i x'(\chi_i)).
\end{aligned}$$

It follows that

$$
\begin{aligned}
&(1-\sum_{i=1}^{m-2} a_i)(1-\sum_{j=1}^{n-2} b_j)x'(\lambda)+\sum_{j=1}^{n-2} b_j(1-\tau_j)(1-\sum_{i=1}^{m-2} a_i)x'(\lambda_j)\\
&+\sum_{i=1}^{m-2} a_i\xi_i(1-\sum_{j=1}^{n-2} b_j)x'(\chi_i)=0.
\end{aligned}
$$

Using, next, the intermediate value theorem we see that there exist υ_1, υ_2 in $[0,1]$ such that

$$
Ax'(\upsilon_1)-Bx'(\upsilon_2)=0, \tag{2.27}
$$

where A, B are as defined in (2.19), (2.20). Suppose, now, one of $x'(\upsilon_1)$, $x'(\upsilon_2)$ is zero. We then see from one of the following equations

$$
\phi(x'(t))=\phi(x'(\upsilon_k))+\int_{\upsilon_k}^{t}(\phi(x'))'(s)ds, \quad k=1,2;\ t\in[0,1] \tag{2.28}
$$

that

$$
\|\phi(x')\|_\infty\le\|(\phi(x'))'\|_{L^1(0,1)}. \tag{2.29}
$$

Let us, next, suppose that both $x'(\upsilon_1)$, $x'(\upsilon_2)$ are non-zero. Since, now, $A\neq B$, in view of Lemma 2.3 we see from equation (2.27) that $x'(\upsilon_1)\neq x'(\upsilon_2)$. We now use the equations

$$
\begin{aligned}
\phi(x'(t))&=\phi(x'(\upsilon_1))+\int_{\upsilon_k}^{t}(\phi(x'))'(s)ds=\phi(\frac{B}{A}x'(\upsilon_2))+\int_{\upsilon_k}^{t}(\phi(x'))'(s)ds,\\
\phi(x'(t))&=\phi(x'(\upsilon_2))+\int_{\upsilon_k}^{t}(\phi(x'))'(s)ds=\phi(\frac{A}{B}x'(\upsilon_1))+\int_{\upsilon_k}^{t}(\phi(x'))'(s)ds,
\end{aligned}
$$

along with the definition of σ^*, as given in (2.21), (2.22) and the estimate (2.29) to obtain the estimate (2.23). This completes the proof of the proposition. □

3. Existence Theorem

Let ϕ be an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ satisfying $\phi(0)=0$, $f:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a function satisfying Carathéodory conditions and $e:[0,1]\to\mathbb{R}$ be a function in $L^1[0,1]$. Let ξ_i, $\tau_j\in(0,1)$, a_i, $b_j\in\mathbb{R}$, $i=1,2,\dots,m-2$, $j=1,2,\dots,n-2$, $0<\xi_1<\xi_2<\dots<\xi_{m-2}<1$, $0<\tau_1<\tau_2<\dots<\tau_{n-2}<1$ with $(\sum_{i=1}^{m-2}a_i\xi_i)(1-\sum_{j=1}^{n-2}b_j)\neq(1-\sum_{i=1}^{m-2}a_i)(\sum_{j=1}^{n-2}b_j\tau_j-1)$.

Theorem 3.1. *Let $f:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a function satisfying Carathéodory's conditions such that there exist non-negative functions $d_1(t)$, $d_2(t)$, and $r(t)$ in $L^1(0,1)$ such that*

$$
|f(t,u,v)|\le d_1(t)\phi(|u|)+d_2(t)\phi(|v|)+r(t),
$$

for a. e. $t\in[0,1]$ and all u, $v\in\mathbb{R}$. Suppose, further,

$$
\alpha(M)\|d_1\|_{L^1(0,1)}+\|d_2\|_{L^1(0,1)}<1-\widetilde{\alpha}(\sigma^*) \tag{3.1}
$$

where M is as defined in Proposition 2.2, $\alpha(M)$ is as defined in (2.1), σ^* and $\widetilde{\alpha}(\sigma^*)$ are as defined in (2.21), (2.22). Then, for every given function $e(t)\in L^1[0,1]$, the boundary value problem (1.1) has at least one solution $x(t)\in C^1[0,1]$.

Proof. We consider the family of boundary-value problems

$$\begin{gathered}(\phi(x'))' = \lambda f(t,x,x') + \lambda e, 0 < t < 1, \lambda \in [0,1] \\ x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i),\ \ x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j).\end{gathered} \tag{3.2}$$

Also, we define an operator $\Psi : C^1[0,1] \times [0,1] \to C^1[0,1]$ by setting for $(x,\lambda) \in C^1[0,1] \times [0,1]$

$$\begin{aligned}\Psi(x,\lambda)(t) = x(0) + \int_0^t \phi^{-1}(\phi(x'(0)) + \lambda \int_0^s (f(\tau, x(\tau), x'(\tau)) + e(\tau))d\tau)ds \\ + (x(0) - \sum_{i=1}^{m-2} a_i x(\xi_i)) + t(x(1) - \sum_{j=1}^{n-2} b_j x(\tau_j))\end{aligned} \tag{3.3}$$

Let us, suppose that $x(t) \in C^1[0,1]$ is a solution to the operator equation, for some $\lambda \in [0,1]$,

$$\begin{aligned} x &= \Psi(x,\lambda) \\ &= x(0) + \int_0^t \phi^{-1}(\phi(x'(0)) + \lambda \int_0^s (f(\tau, x(\tau), x'(\tau)) + e(\tau))d\tau)ds \\ &\quad + (x(0) - \sum_{i=1}^{m-2} a_i x(\xi_i)) + t(x(1) - \sum_{j=1}^{n-2} b_j x(\tau_j))\end{aligned} \tag{3.4}$$

Evaluating this equation at $t = 0$ we see that $x(t)$ satisfies the boundary condition

$$x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i).$$

Next, we differentiate the equation (3.4) with respect to t to get

$$x'(t) = \phi^{-1}(\phi(x'(0)) + \lambda \int_0^t (f(\tau, x(\tau), x'(\tau)) + e(\tau))d\tau) + x(1) - \sum_{j=1}^{n-2} b_j x(\tau_j). \tag{3.5}$$

Evaluating, now, the equation (3.5) at $t = 0$ we see that $x(t)$ satisfies the boundary condition

$$x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j),$$

and on differentiating the equation (3.5) with respect to t we get

$$(\phi(x'))' = \lambda f(t,x,x') + \lambda e, \quad 0 < t < 1,\ \lambda \in [0,1].$$

Thus we see that if $x(t) \in C^1[0,1]$ is a solution to the operator equation $x = \Psi(x,\lambda)$ for some $\lambda \in [0,1]$ then $x(t)$ is a solution to the boundary value problems (3.2) for the corresponding $\lambda \in [0,1]$. Conversely, it is easy to see that if $x(t) \in C^1[0,1]$ is a solution to the boundary value problems (3.2) for some $\lambda \in [0,1]$ then $x(t) \in C^1[0,1]$ is a solution to the operator equation $x = \Psi(x,\lambda)$ for the corresponding $\lambda \in [0,1]$.

Next, it is easy to show, following standard arguments, that $\Psi : C^1[0,1] \times [0,1] \to C^1[0,1]$ is a completely continuous operator.

We shall next show that there is a constant $R > 0$, independent of $\lambda \in [0,1]$, such that if $x(t) \in C^1[0,1]$ is a solution to (3.4), equivalently to the boundary value problems (3.2), for some $\lambda \in [0,1]$ then $\|x\|_{C^1[0,1]} < R$.

We note first that if $x(t) \in C^1[0,1]$ satisfies

$$x = \Psi(x, 0), \tag{3.6}$$

then $x(t) = 0$ for all $t \in [0,1]$. Indeed, from the definition of Ψ or from the boundary value problem (3.2), it follows that $x(t) = x(0) + x'(0)t$. It then follows from the two boundary conditions in (3.2) and the non-resonance assumption (2.3) that $x(0) = x'(0) = 0$, implying $x(t) = 0$ for all $t \in [0,1]$.

We shall assume, in the following, that $\lambda \in (0,1]$. We shall also assume that σ^*, as defined in (2.21) is positive, since the proof for the case $\sigma^* = 0$ is simpler. Let us choose $\varepsilon > 0$ such that $\widetilde{\alpha}(\sigma^*) + \varepsilon < 1$ and

$$(\alpha(M) + \varepsilon)\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)} < 1 - \widetilde{\alpha}(\sigma^*) - \varepsilon, \tag{3.7}$$

which is possible to do, in view of our assumption (3.1). Here M is as defined in Propostion 2.2 and $\alpha(M)$ is as defined in (2.1) so that for the $\varepsilon > 0$, chosen above, there exists a constant $C^1_\varepsilon > 0$ such that

$$\phi(Mz) \le (\alpha(M) + \varepsilon)\phi(z) + C^1_\varepsilon, \quad \text{for every } z \in \mathbb{R}. \tag{3.8}$$

Also, from Proposition 2.4 we see that there is a constant $C^2_\varepsilon > 0$, for the chosen $\varepsilon > 0$, such that

$$\phi(\|x'\|_\infty) \le \frac{1}{1 - \widetilde{\alpha}(\sigma^*) - \varepsilon}\|(\phi(x'))'\|_{L^1(0,1)} + C^2_\varepsilon. \tag{3.9}$$

We, now, see from the equation in (3.2), using our assumptions on the function f, Proposition 2.2, and estimates (3.8), (3.9) that

$$\begin{aligned}
&\|(\phi(x'))'\|_{L^1(0,1)} \\
&\le \phi(\|x\|_\infty)\|d_1\|_{L^1(0,1)} + \phi(\|x'\|_\infty)\|d_2\|_{L^1(0,1)} + \|r\|_{L^1(0,1)} + \|e\|_{L^1(0,1)} \\
&\le \phi(M\|x'\|_\infty)\|d_1\|_{L^1(0,1)} + \phi(\|x'\|_\infty)\|d_2\|_{L^1(0,1)} + \|r\|_{L^1(0,1)} + \|e\|_{L^1(0,1)} \\
&\le ((\alpha(M) + \varepsilon)\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)})\phi(\|x'\|_\infty) + \|r\|_{L^1(0,1)} + \|e\|_{L^1(0,1)} \\
&\quad + C^1_\varepsilon\|d_1\|_{L^1(0,1)} \\
&\le \frac{(\alpha(M) + \varepsilon)\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)}}{1 - \widetilde{\alpha}(\sigma^*) - \varepsilon}\|(\phi(x'))'\|_{L^1(0,1)} + C\varepsilon,
\end{aligned}$$

where $C\varepsilon = \|r\|_{L^1(0,1)} + \|e\|_{L^1(0,1)} + C^1_\varepsilon\|d_1\|_{L^1(0,1)} + C^2_\varepsilon[(\alpha(M) + \varepsilon)\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)}]$. It, now, follows from (3.7) that there exists a constant R_0, independent of $\lambda \in [0,1]$, such that if $x(t) \in C^1[0,1]$ is a solution to the boundary value problems (3.2) for some $\lambda \in [0,1]$ then

$$\|(\phi(x'))'\|_{L^1(0,1)} \le R_0.$$

This combined with (3.9) and (2.8) give that there exists a constant $R > 0$ such that

$$\|x\|_{C^1[0,1]} < R.$$

This then implies that $\deg_{LS}(I - \Psi(\cdot, \lambda), B(0,R), 0)$ is well-defined for all $\lambda \in [0,1]$, where $B(0,R)$ is the ball with center 0 and radius R in $C^1[0,R]$.

Let, now, X denote the two-dimensional subspace of $C^1[0,1]$ given by

$$X = \{A + Bt \mid \text{for } A, B \in \mathbb{R}\}. \tag{3.10}$$

Let us define the isomorphism $i:\mathbb{R}^2\to X$ by

$$i\begin{pmatrix}A\\B\end{pmatrix}=i_{\begin{pmatrix}A\\B\end{pmatrix}}\in X,\quad\text{for }\begin{pmatrix}A\\B\end{pmatrix}\in\mathbb{R}^2,\tag{3.11}$$

where

$$i_{\begin{pmatrix}A\\B\end{pmatrix}}(t)=A+Bt,\quad\text{for }t\in[0,1].\tag{3.12}$$

Also, we define a 2×2 matrix

$$\mathbb{A}=\begin{pmatrix}-(1-\sum_{i=1}^{m-2}a_i) & \sum_{i=1}^{m-2}a_i\xi_i\\ -(1-\sum_{j=1}^{n-2}b_j) & -(1-\sum_{j=1}^{n-2}b_j\tau_j)\end{pmatrix}.\tag{3.13}$$

We note that

$$\det\mathbb{A}=(1-\sum_{i=1}^{m-2}a_i)(1-\sum_{j=1}^{n-2}b_j\tau_j)+(\sum_{i=1}^{m-2}a_i\xi_i)(1-\sum_{j=1}^{n-2}b_j)\neq 0,$$

in view of the non-resonance assumption (2.3).

Next, we define a function $G:\mathbb{R}^2\to\mathbb{R}^2$ by setting

$$\begin{aligned}G\begin{pmatrix}A\\B\end{pmatrix}&=\mathbb{A}\cdot\begin{pmatrix}A\\B\end{pmatrix}\\&=\begin{pmatrix}-A(1-\sum_{i=1}^{m-2}a_i)+B(\sum_{i=1}^{m-2}a_i\xi_i)\\ -A(1-\sum_{j=1}^{n-2}b_j)-B(1-\sum_{j=1}^{n-2}b_j\tau_j)\end{pmatrix}\quad\text{for }\begin{pmatrix}A\\B\end{pmatrix}\in\mathbb{R}^2.\end{aligned}\tag{3.14}$$

We note that for $v(t)=A+Bt\in X$ we have

$$(I-\Psi(\cdot,0))(v)=i_{G\begin{pmatrix}A\\B\end{pmatrix}}$$

and it follows that

$$G=i^{-1}\circ((I-\Psi(\cdot,0))|_X\circ i.$$

Now, we see from the homotopy invariance property of the Leray-Schauder degree that

$$\begin{aligned}\deg_{LS}(I-\Psi(\cdot,1),B(0,R),0)&=\deg_{LS}(I-\Psi(\cdot,0),B(0,R),0)\\&=\deg_B(I-\Psi(\cdot,0)|_X,X\cap B(0,R),0)\\&=\deg_B(G,\mathbb{B}(0,R),0),\end{aligned}$$

where $\mathbb{B}(0,R)$ denotes the ball of radius R in $\mathbb{R}^2$ with center at the origin. Finally, we have that

$$\deg_B(G,\mathbb{B}(0,R),0)=\begin{cases}1, & \text{if }\det\mathbb{A}>0\\ -1, & \text{if }\det\mathbb{A}<0.\end{cases}$$

Accordingly, we see from the non-resonance assumption (2.3) i.e.

$$\det\mathbb{A}=(1-\sum_{i=1}^{m-2}a_i)(1-\sum_{j=1}^{n-2}b_j\tau_j)+(\sum_{i=1}^{m-2}a_i\xi_i)(1-\sum_{j=1}^{n-2}b_j)\neq 0$$

that $\deg_{LS}(I-\Psi(\cdot,1),B(0,R),0)\neq 0$ and there is $x(t)\in B(0,R)\subset C^1[0,1]$ that satisfies

$$x=\Psi(x,1),$$

equivalently $x(t)$ is a solution to the boundary value (1.1). This completes the proof of the theorem. □

References

[1] Bai, Cuan-zhi and Fang, Jin-xuan; *Existence of multiple positive solutions for nonlinear m-point boundary value problems*, Applied Mathematics and Computation, 140 (2003), 297-305.

[2] Bitsadze A. V., *On the theory of nonlocal boundary value problems*, Soviet Math. Dokl. 30(1984), No.1, p. 8-10.

[3] Bitsadze A. V., *On a class of conditionally solvable nonlocal boundary value problems for harmonic functions*, Soviet Math. Dokl. 31(1985), No. 1, p. 91-94.

[4] Bitsadze A. V. and Samarskiĭ A. A.; *On some simple generalizations of linear elliptic boundary problems*, Soviet Math. Dokl. 10(1969), No. 2, p. 398-400.

[5] Feng W. and Webb J. R. L.; *Solvability of three-point boundary value problems at resonance*, Nonlinear Analysis T.M.A. 30(1997) 3227-3238 .

[6] Feng W. and Webb J.R.L., *Solvability of m-point boundary value problems with nonlinear growth*, J. Math. Anal. Appl. 212 (1997) 467–480.

[7] Garcia-Huidobro M., Gupta C. P. and Manasevich R., *Solvability for a Non-linear Three-Point Boundary Value problem with p-Laplacian-Like Operator at Resonance*, Abstract Analysis and Applications, Vol. 6, No. 4, (2001) pp. 191-213.

[8] Garcia-Huidobro M., Gupta C. P. and Manasevich R., *An m-point boundary value problem of Neumann type for a p-Laplacian like operator*, Nonlinear Analysis, 56(2004) 1071-1089.

[9] Garcia-Huidobro M., Gupta C. P. and Manasevich R., *A Dirichelet-Neumann m-Point bvp with a p-Laplacian Like operator*, Nonlinear Analysis, 62(2005) 1067-1089.

[10] Garcia-Huidobro M. and Manasevich R., *A three point boundary value problem containing the operator* $(\phi(u'))'$, Discrete and Continuous Dynamical Systems, Proceedings of the fourth international conference on dynamical systems and differential equations, Wilmington, (2003), 313-319.

[11] Gupta C. P., *Solvability of a three-point boundary value problem for a second order ordinary differential equation*, Journal of Mathematical Analysis and Applications, 168(1992) 540-551.

[12] Gupta C. P., *A note on a second order three-point boundary value problem*, Journal of Mathematical Analysis and Applications, 186(1994) 277-281.

[13] Gupta C. P., *A second order m-point boundary value problem at resonance*, Nonlinear Analysis T.M.A., 24(1995) 1483-1489.

[14] Gupta C. P. , *Existence theorems for a second order m-point boundary value problem at resonance*, International Jour. Math. & Math. Sci. 18(1995) 705-710.

[15] Chaitan P. Gupta, *Solvability of a multi-point boundary value problem at resonance*, Results in Math., 28(1995) 270-276.

[16] Gupta C. P., *A generalized Multi-Point Boundary Value Problem for Second Order Ordinary Differential Equations*, Applied Mathematics and Computation 89(1998), p. 133-146.

[17] Gupta C. P., Ntouyas S., Tsamatos P.Ch., *On an m-point boundary value problem for second order ordinary differential equations*, Nonlinear Analysis T.M.A., 23(1994) 1427-1436.

[18] Gupta C. P., Ntouyas S., Tsamatos P.Ch., *Solvability of an m-point boundary value problem for second order ordinary differential equations*, Journal of Mathematical Analysis and Applications, 189(1995) 575-584.

[19] Gupta C. P., Ntouyas S. K., and Tsamatos P. Ch., *Existence results for m-point boundary value problems*, Diff. Equations and Dynamical Systems, 2(1994), No. 4, p. 289-298.

[20] Gupta C. P., Ntouyas S. K., and Tsamatos P. Ch., *Existence results for multi-point boundary value problems for second order ordinary differential equations.* Bulletin Greek Math. Soc. 43 (2000) pp. 105-123.

[21] Gupta C.P., and Trofimchuk, S., *Solvability of multi point boundary value problem of Neumann type*, Abstract Analysis and Applications, 4(1999) 71-81.

[22] Il'in V. A. and Moiseev E. I., *Nonlocal boundary value problem of the first kind for a Sturm Liouville operator in its differential and difference aspects*, Differential Equations, 23(1987) 803-810.

[23] Il'in V. A. and Moiseev E. I., *Nonlocal boundary value problem of the second kind for a Sturm Liouville operator*, Differential Equations, 23(1987) 979-987.

[24] Liu B., *Solvability of multi point boundary value problem at resonance (IV)*, Applied Mathematics and Computations,143(2003) 275-299.

[25] Liu, Y. and Ge, W., *Multiple positive solutions to a three-point boundary value problem with p-Laplacian*, Journal of Mathematical Analysis and Applications, 277(2003) 293-302.

[26] J. Mawhin, *Topological degree methods in nonlinear boundary value problems,* in NSF-CBMS Regional Conference Series in Math. No. 40, Amer. Math. Soc., Providence, RI, 1979.

[27] Sedziwy, S., *Multipoint boundary value problems for a second order differential equations*, Journal of Mathematical Analysis and Applications, 236(1999) 384-398.

[28] Thompson, H. B. and Tisdell, C., *Three-point boundary value problems for second-order ordinary differential equations*, Math. Comput. Modelling, 34(2001), no. 3-4, 311–318.

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