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EXISTENCE OF NONCONTINUABLE SOLUTIONS

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Abstract. This paper presents necessary and sufficient conditions for an n-th order differential equation to have a non-continuable solution with finite limits of its derivatives up to the orders $n-2$ at the right-hand end point of the definition interval.

1. Introduction

Consider the n-th order differential equation

$$y^{(n)} = f\big(t, y, y', \dots, y^{(n-2)}\big) g\big(y^{(n-1)}\big) \tag{1.1}$$

where $n \geq 2$, $f \in C^0(R_+ \times R^{n-1})$, $g \in C^0(R), R_+ = [0, \infty)$, $R = (-\infty, \infty)$ and $M > 0$ exists such that

$$g(x) > 0 \quad \text{for } |x| \geq M. \tag{1.2}$$

This inequality will be assumed throughout the paper.

So we study equations for which g is nonzero in neighbourhoods of ∞ and $-\infty$; this case can be easily transformed into (1.1) and (1.2).

A solution y of (1.1) defined on $[T, \tau) \subset R_+$ is called noncontinuable if $\tau < \infty$ and y cannot be defined at $t = \tau$. Sometimes such solutions are called singular of the second kind [1, 3, 10]. A noncontinuable solution y is called nonoscillatory if $y \neq 0$ in a left neighbourhood of τ.

Sufficient conditions for the existence of noncontinuable solutions for the Cauchy problem can be found in [10]. For $f(t, x_1, \dots, x_{n-1}) \equiv r(t)|x_1|^\lambda \times \operatorname{sgn} x_1, r \neq 0$ in [3]. For $n = 2$ in [2, 4, 8]. Sufficient conditions for the nonexistence of noncontinuable solutions of (1.1) and of its special cases be found in [5, 6, 7, 10].

Jaroš and Kusano [9] investigated the differential equation

$$y'' = r(t)|y|^\sigma |y'|^\lambda \operatorname{sgn} y \tag{1.3}$$

with $\sigma > 0, r < 0$ on R_+. They proved that there exists a noncontinuable solution y of (1.3) fulfilling $\lim_{t \to \tau_-} y(t) \in [0, \infty), \lim_{t \to \tau_-} y'(t) = -\infty$ if, and only if $\lambda > 2$; they call it a black hole solution.

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In [1], a problem is formulated for (1.1): To find conditions under which (1.1) has a noncontinuable solution y fulfilling the conditions

$$\begin{gathered} \tau \in (0,\infty), \quad c_i \in R, \quad \lim_{t\to\tau_-} y(t) = c_i \quad i = 0,1,2,\dots,n-2, \\ \lim_{t\to\tau_-} |y^{(n-1)}(t)| = \infty \end{gathered} \tag{1.4}$$

and y is defined in a left neighbourhood of τ.

Note that (1.4) is a boundary-value problem and a solution y fulfilling (1.4) is nonoscillatory. The obtained results are summed up in the following theorem.

Theorem 1.1 ([1]). *Let $\tau \in (0,\infty)$, $f(t,x_1,\dots,x_{n-1})x_1 \neq 0$ for $x_1 \neq 0$ and $g(x) \geq 0$ for $x \in R$.*

- (i) *If $M_1 \in (0,\infty)$ is such that $g(x) \leq x^2$ for $|x| \geq M_1$, then* (1.1) *has no solution y fulfilling* (1.4)*.*
- (ii) *Let $\tau \in (0,\infty)$, $c_0 \neq 0$, $\lambda > 2$, $M_1 \in (0,\infty)$ and $g(x) \geq |x|^\lambda$ for $|x| \geq M_1$, then* (1.1) *has a solution y fulfilling* (1.4) *that is defined in a left neighbourhood of τ.*

In the present paper, we generalize Theorem 1.1 and the necessary and sufficient condition for the existence of a noncontinuable solution y fulfilling (1.4) will be stated if $f(\tau,c_0,\dots,c_{n-2}) \neq 0$. Sufficient conditions for the existence of a noncontinuable solution y fulfilling (1.4) are given in case $f(\tau,c_0,\dots,c_{n-2}) = 0$.

Notation. Let $\int_M^\infty \frac{d\sigma}{g(\sigma)} < \infty$. Then we put

$$F(z) = \int_z^\infty \frac{d\sigma}{g(\sigma)}, z \geq M$$

and $F^{-1} : (0, F(M)] \to [M,\infty)$ denotes the inverse function to F. Similarly, if $\int_{-\infty}^{-M} \frac{d\sigma}{g(\sigma)} < \infty$, put

$$G(z) = \int_{-\infty}^z \frac{d\sigma}{g(\sigma)}, z \leq -M$$

and $G^{-1} : (0, G(-M)] \to (-\infty,-M]$ is the inverse function to G.

The next lemma follows from the definitions of F and G.

Lemma 1.2.
- (i) *Let $\int_M^\infty \frac{d\sigma}{g(\sigma)} < \infty$. Then functions F and F^{-1} are decreasing, $\lim_{z\to\infty} F(z) = 0$ and $\lim_{z\to0_+} F^{-1}(z) = \infty$.*
- (ii) *Let $\int_{-\infty}^{-M} \frac{d\sigma}{g(\sigma)} < \infty$. Then functions G and G^{-1} are increasing, $G > 0, G^{-1} < 0, \lim_{z\to-\infty} G(z) = 0$ and $\lim_{z\to0_+} G^{-1}(z) = -\infty$.*

Denote by $[[a]]$ the entire part of the number a.

2. Main results

The next theorem gives conditions for the nonexistence of a solution y fulfilling (1.4).

Theorem 2.1. *Let the following two assumptions hold.*

- (i) *Let either*

$$\int_M^\infty \frac{d\sigma}{g(\sigma)} = \infty \tag{2.1}$$

or

$$\int_M^\infty \frac{d\sigma}{g(\sigma)} < \infty \quad \text{and} \quad \int_0^{F(M)} F^{-1}(\sigma)d\sigma = \infty; \tag{2.2}$$

(ii) *Let either*

$$\int_{-\infty}^{-M} \frac{d\sigma}{g(\sigma)} = \infty \tag{2.3}$$

or

$$\int_{-\infty}^{-M} \frac{d\sigma}{g(\sigma)} < \infty \quad \text{and} \quad \int_0^{G(-M)} |G^{-1}(\sigma)|d\sigma = \infty. \tag{2.4}$$

Then (1.1) *has no noncontinuable solution y fulfilling* (1.4) *that is defined in a left neighbourhood of τ.*

Proof. Suppose, contrarily, that y is a noncontinuable solution of (1.1) fulfilling (1.4) that is defined on $[T, \tau) \subset R_+$. Furthermore, suppose that $\lim_{t\to\tau_-} y^{(n-1)}(t) = \infty$; the opposite case, if $\lim_{t\to\tau_-} y^{(n-1)}(t) = -\infty$, can be studied similarly using (2.3) and (2.4). From this, $T_1 \in [T, \tau)$ and $M_1 > 0$ exist such that

$$f\big(t, y(t), \dots, y^{(n-2)}(t)\big) \le M_1, \quad y^{(n-1)}(t) \ge M \quad \text{for } t \in [T_1, \tau).$$

Hence, the integration of (1.1) and (1.2) yields

$$\begin{aligned}\int_{y^{(n-1)}(t)}^\infty \frac{d\sigma}{g(\sigma)} &= \int_t^\tau \frac{y^{(n)}(\sigma)d\sigma}{g(y^{(n-1)}(\sigma))} \\ &= \int_t^\tau f\big(\sigma, y(\sigma), \dots, y^{(n-2)}(\sigma)\big)d\sigma \\ &\le M_1(\tau - t) \le M_1\tau, \quad t \in [T_1, \tau).\end{aligned} \tag{2.5}$$

It follows from this that (2.1) is not valid and hence (2.2) holds.

Let $T_2 \in [T_1, \tau)$ be such that $\tau - T \le F(M)M_1^{-1}$. From this and from (2.5)

$$F\big(y^{(n-1)}(t)\big) \le M_1(\tau - t) \in (0, F(M)] \quad \text{for } t \in [T_2, \tau);$$

hence, Lemma 1.2 yields

$$y^{(n-1)}(t) \ge F^{-1}\left(M_1(\tau - t)\right), \quad t \in [T_2, \tau)$$

and an integration on $[T_2, \tau)$ and (2.2) yield

$$\begin{aligned}\infty > c_{n-2} - y^{(n-2)}(T_2) &= y^{(n-2)}(\tau) - y^{(n-2)}(T_2) = \int_{T_2}^\tau y^{(n-1)}(\sigma)d\sigma \\ &\ge \int_{T_2}^\tau F^{-1}(M_1(\tau - s))\, ds = \frac{1}{M_1}\int_0^{M_1(\tau - T_2)} F^{-1}(\sigma)d\sigma = \infty.\end{aligned}$$

This contradiction proves that a noncontinuable solution y fulfilling (1.4) does not exist. □

The following theorem formulates necessary and sufficient conditions for the existence of a solution y fulfilling (1.4) in case $f(\tau, c_0, \dots, c_{n-2}) \ne 0$.

Theorem 2.2. *Let $\tau > 0$ and $c_i \in R$, $i = 0, 1, \dots, n-2$ be such that*

$$f(\tau, c_0, c_1, \dots, c_{n-2}) \ne 0. \tag{2.6}$$

Then (1.1) *has a noncontinuable solution* y *fulfilling* (1.4) *if and only if one of the following two conditions holds:*

$$\int_M^\infty \frac{d\sigma}{g(\sigma)} < \infty \quad \text{and} \quad \int_0^{F(M)} F^{-1}(\sigma)\, d\sigma < \infty; \tag{2.7}$$

$$\int_{-\infty}^{-M} \frac{d\sigma}{g(\sigma)} < \infty \quad \text{and} \quad \int_0^{G(-M)} |G^{-1}(\sigma)|\, d\sigma < \infty. \tag{2.8}$$

In this case y *is defined in a left neighbourhood of* τ.

Moreover, let $f(0, c_0, c_1, \dots, c_{n-2}) \neq 0$ *and either* (2.7) *or* (2.8) *holds. Then there exists* $\tau_0 > 0$ *such that for every* $0 < \tau \leq \tau_0$*, a noncontinuable solution* y *fulfilling* (1.4) *exists and is defined on* $[0, \tau)$.

Proof. Necessity: This follows from Theorem 2.1.

Sufficiency: We prove the statement in case $f(\tau, c_0, \dots, c_{n-2}) > 0$; the opposite case can be studied similarly. There exist $N > 0$ and $\bar{\tau} \in [0, \tau)$ such that

$$f(t, x_1, \dots, x_{n-1}) > 0 \quad \text{for } t \in [\bar{\tau}, \tau], |x_i - c_{i-1}| \leq N, i = 1, 2, \dots, n-1. \tag{2.9}$$

From this, positive constants M_1 and M_2 exist such that

$$\begin{gathered} 0 < M_1 = \min\{f(t, x_1, \dots, x_{n-1}) : t \in [\bar{\tau}, \tau],\ |x_i - c_{i-1}| \leq N, i = 1, 2, \dots, n-1\}, \\ \infty > M_2 = \max\{f(t, x_1, \dots, x_{n-1}) : t \in [\bar{\tau}, \tau],\ |x_i - c_{i-1}| \leq N, i = 1, 2, \dots, n-1\}. \end{gathered} \tag{2.10}$$

Consider the auxiliary problem

$$\begin{gathered} y^{(n)} = f\left(t, \chi_0(y), \chi_1(y'), \dots, \chi_{n-2}(y^{(n-2)})\right) g(y^{(n-1)}), \\ y^{(i)}(\tau) = c_i, i = 0, 1, \dots, n-2,\ y^{(n-1)}(\tau) = k, \end{gathered} \tag{2.11}$$

where $k \in \{k_0, k_0 + 1, \dots\}, k_0 \geq [[2M]]$,

$$\chi_i(s) = \begin{cases} s & \text{for } |s - c_i| \leq N \\ c_i + N & \text{for } s > c_i + N \\ c_i - N & \text{for } s < c_i - N, \end{cases} \tag{2.12}$$

where $i = 0, 1, \dots, n-2$. Furthermore, let $J = [T, \tau) \subset [\bar{\tau}, \tau)$ be such that $0 < \tau - T \leq 1$,

$$(\tau - T) \sum_{j=1}^{n-2} |c_j| + \frac{1}{M_1} \int_0^{M_1(\tau - T)} F^{-1}(z)\, dz \leq N, \tag{2.13}$$

and

$$M_2(\tau - T) < \int_M^{2M} \frac{ds}{g(s)}; \tag{2.14}$$

this choice is possible due to the second inequality in (2.7), (1.2) and Lemma 1.2; it does not depend on k.

Denote by y_k a solution of (2.11) and by J_k the intersection of its maximal definition interval and $[T, \tau)$. We prove that

$$y_k^{(n-1)}(t) > M \quad \text{for} \quad t \in J_k. \tag{2.15}$$

As $k \geq k_0 \geq [[2M]]$, (2.11) yields (2.15) is valid in a left neighbourhood of τ. Suppose, contrarily, that $T_1 \in J_k$ exists such that $y_k^{(n-1)}(T_1) = M$ and $y_k^{(n-1)}(t) >$

M on $(T_1,\tau]$. Then (1.2), (2.10), (2.11) and (2.12) yield $y_k^{(n)}(t)>0$ on $[T_1,\tau]$ and
$$y_k^{(n)}(t)\le M_2\, g(y_k^{(n-1)}(t)),\ g(y_k^{(n-1)}(t))>0,\ t\in[T_1,\tau].$$
From this, and an integration on $[T_1,\tau]$, we obtain
$$\int_M^{2M}\frac{ds}{g(s)}\le\int_M^k\frac{ds}{g(s)}\le M_2(\tau-T_1)\le M_2(\tau-T).$$
The contradiction with (2.14) proves that (2.15) holds. According to (2.10), (2.11) and (2.12), we have
$$y_k^{(n)}(t)\ge M_1\, g(y_k^{(n-1)}(t)),\quad t\in J_k$$
and an integration on $[t,\tau]$, (1.2), and (2.15) yield
$$F(y_k^{(n-1)}(t))\ge\int_{y_k^{(n-1)}(t)}^k\frac{ds}{g(s)}\ge M_1(\tau-t),\quad t\in J_k.$$
As, according to (2.14), $M_1(\tau-T)\le M_2(\tau-T)<F(M)$, we have $M_1(\tau-t)\in(0,f(M)]$ and
$$y_k^{(n-1)}(t)\le F^{-1}(M_1(\tau-t)),\quad t\in J_k. \tag{2.16}$$
From this and from Lemma 1.2, $y_k^{(n-1)}$ is bounded on J_k and hence $J_k=[T,\tau]$; moreover, J_k is defined on the same interval $[T,\tau]$ for every $k=k_0,k_0+1,\dots$.

We estimate the functions $y_k^{(i)}$. Taylor's formula, $T-\tau\le 1$, (2.13), and (2.16) yield
$$\begin{aligned}\big|y_k^{(i)}(t)-c_i\big|&\le\sum_{j=i+1}^{n-2}\frac{|c_j|}{(j-i)!}(\tau-t)^{j-i}+\Big|\int_\tau^t\frac{|(t-s)^{n-i-2}|}{(n-i-2)!}F^{-1}(M_1(\tau-s))\,ds\Big|\\&\le(\tau-T)\sum_{j=1}^{n-2}|c_j|+\Big|\int_\tau^t F^{-1}(M_1(\tau-s))\,ds\Big|\\&\le(\tau-T)\sum_{j=1}^{n-2}|c_j|+\frac{1}{M_1}\int_0^{M_1(\tau-T)}F^{-1}(z)\,dz\\&\le N,\quad t\in[T,\tau],\ i=0,1,\dots,n-2.\end{aligned}\tag{2.17}$$
From this and from (2.12), y_k is a solution of (1.1), as well.

As the estimations (2.15), (2.16) and (2.17) and the definition interval of y_k do not depend on k, then according to the Arzel-Ascoli Theorem (see [3, Lemma 10.2]) there exists a subsequence of $\{y_k\}_{k=k_0}^\infty$ that converges locally uniformly to a solution y of (1.1) on $[T,\tau)$ together with all derivatives up to the order $n-1$. Evidently conditions (1.4) hold for $i=0,1,\dots,n-2$ and we prove $\lim_{t\to\tau_-}y^{(n-1)}(t)=\infty$. As $y^{(n-1)}$ is increasing on $[T,\tau)$, there exists a limit as $t\to\tau_-$. Suppose, contrarily, that $\lim_{t\to\tau_-}y^{(n-1)}(t)=Q<\infty$. Then Lemma 1.2 yields the existence of $T_2\in[T,\tau)$ such that
$$Q<F^{-1}(M_2(\tau-T_2)),\quad M(\tau-T_2)\le F(M). \tag{2.18}$$
Moreover, there exists a subsequence of $\{y_k^{(n-1)}\}_{k=k_0}^\infty$, we denote it $\{y_k^{(n-1)}\}_{k=k_0}^\infty$ for simplicity, that converges to $y^{(n-1)}$ on $[T,T_2]$. From this, $\bar k$ exists such that
$$y_k^{(n-1)}(T_2)\le 2Q\quad\text{for } k=\bar k,\bar k+1,\dots.$$

According to (1.1) and (2.17), $y_k^{(n)}(t) \le M_2 g(y_k^{(n-1)}(t))$, we obtain, by integration on $[T_2, \tau)$,

$$M_2(\tau - T_2) \ge \int_{y_k^{(n-1)}(T_2)}^{k} \frac{ds}{g(s)} \ge \int_{2Q}^{k} \frac{ds}{g(s)},\ k \ge \bar{k}.$$

Thus,

$$M_2(\tau - T_2) \ge \int_{2Q}^{\infty} \frac{ds}{g(s)} = F(2Q), \tag{2.19}$$

so $2Q \ge F^{-1}(M_2(\tau - T_2))$, which contradicts (2.18). Hence, $\lim_{t\to\tau_-} y(t) = \infty$. Let $f(0, c_0, c_1, \dots, c_{n-2}) > 0$. Then there exist $N > 0$ and $\bar\tau_0 \le 1$ such that

$$f(t, x_1, \dots, x_{n-1}) > 0 \quad \text{for } t \in [0, \bar\tau_0],\ [x_i - c_{i-1}| \le N,\ i = 1, \dots, n-1.$$

Define

$$M_1 = \min\{f(t, x_1, \dots, x_{n-1}) : t \in [0, \bar\tau_0], |x_i - c_{i-1}| \le N, i = 1, 2, \dots, n-1\},$$
$$M_2 = \max\{f(t, x_1, \dots, x_{n-1}) : t \in [0, \bar\tau_0], |x_i - c_{i-1}| \le N, i = 1, 2, \dots, n-1\}.$$

Constants N, M_1 and M_2 are given by (2.9) and (2.10), but for $[0, \bar\tau_0]$ instead of $[\bar\tau, \tau]$. Let $0 < \tau_0 \le \bar\tau_0$ be a number such that (2.13) and (2.14) hold with $T = 0$ and $\tau = \tau_0$. It is clear that (2.13) and (2.14) are valid for $\tau \le \tau_0$ and $T = 0$ and a noncontinuable solution y fulfilling (1.4) exists according to the first part of the proof, and it is defined on $[0, \tau)$. □

Next, we prove a comparison theorem.

Theorem 2.3. *Let $\tau > 0$ and $c_i \in R_+$, be such that $f(\tau, c_0, \dots, c_{n-2}) \ne 0$. Let $f_1 \in C^0(R_+ \times R^{n-1})$, $f_1(\tau, c_0, \dots, c_{n-1}) \ne 0$ and let $\bar g \in C^0(R)$ exist such that*

$$\bar g(x) \ge g(x) > 0 \quad \text{for } |x| \ge M. \tag{2.20}$$

(i) *If* (1.1) *has a solution fulfilling* (1.4)*, then the equation*

$$y^{(n)} = f_1\big(t, y, \dots, y^{(n-2)}\big)\bar g\big(y^{(n-1)}\big) \tag{2.21}$$

has the same property.

(ii) *If* (2.21) *has no solution fulfilling* (1.4)*, then* (1.1) *has the same property.*

Proof. (i) According to Theorem 2.2 either (2.7) or (2.8) holds. Suppose that (2.7) is valid; if (2.8) holds the proof is similar. Then (2.20) yields

$$\int_M^z \frac{d\sigma}{\bar g(\sigma)} \le \int_M^z \frac{d\sigma}{g(\sigma)},\ z \le M. \tag{2.22}$$

According to (2.7) and (2.22), $\int_M^\infty \frac{d\sigma}{\bar g(\sigma)} < \infty$. Denote $F_1(z) = \int_z^\infty \frac{d\sigma}{\bar g(\sigma)}$, $z \ge M$ and let F_1^{-1} be the inverse function to F_1. As $F_1(z) \le F(z), z \ge M$, and as F and F_1 are nonincreasing, then $F_1^{-1}(z) \le F^{-1}(z), z \ge F_1(M)$ and, hence, (2.7) yields

$$\int_0^{F_1(M)} F_1^{-1}(\sigma) d\sigma \le \int_0^{F_1(M)} F^{-1}(\sigma)\, ds < \infty. \tag{2.23}$$

Hence, Theorem 2.2 applied to (2.21) proves that it has a noncontinuable solution y fulfilling (1.4).

(ii) Suppose, contrarily, that (1.1) has a solution y fulfilling (1.4). Then Theorem 2.2 yields either (2.7) or (2.8) holds. Suppose that (2.7) holds. (2.21) has no solution fulfilling (1.4); hence according to Theorem 2.2 (i) (applied to (2.21)) either

$$\int_M^\infty \frac{d\sigma}{\bar{g}(\sigma)} = \infty \tag{2.24}$$

or

$$\int_M^\infty \frac{d\sigma}{\bar{g}(\sigma)} = \infty \quad \text{and} \quad \int_0^{F_1(M)} F_1^{-1}(\sigma)\, d\sigma = \infty. \tag{2.25}$$

As (2.7) and (2.20) yield (2.22), (2.24) is in a contradiction with (2.7) and (2.22). As (2.7) yields (2.23), the inequality (2.23) contradicts (2.25).

If (2.8) holds, the proof is similar. □

Example. Consider problem (1.1), (1.2) with $g(x) = |x|^\lambda$ for $|x| \ge M$, $\lambda \in R$. Let $\tau > 0, c_i, i = 0, \dots, n-2$ be such that $f(\tau, c_0, \dots, c_{n-2}) \ne 0$. Then, according to Theorem 2.2, (1.1) has a noncontinuable solution y fulfilling (1.4) if and only if $\lambda > 2$.

Remark. Theorem 1.1 (ii) follows from Theorem 2.3 and the Example.

Let us turn our attention to the case when (2.6) does not hold.

Theorem 2.4. *Let $\beta \in \{-1, 1\}$, $\delta > 0$, $\varepsilon > 0$, $\tau \in (0, \infty)$, $\alpha \in \{-1, 1\}$, $s \in \{0, 1, \dots, n-2\}$ and $c_i \in R, i = 0, 1, \dots, n-2$ be such that $\tau > \varepsilon$,*

$$\lambda > \delta(n - s - 2) + 2, \tag{2.26}$$

$$c_s = 0, (-1)^{i-s} \beta\, c_i \ge 0 \quad \text{for } i = s+1, \dots, n-2, \tag{2.27}$$

$$n - s + \frac{1-\alpha}{2} \quad \text{be odd}, \tag{2.28}$$

$$g(x) \ge |x|^\lambda \quad \text{for } \beta x \ge M. \tag{2.29}$$

Let, moreover, a positive function r exist such that

$$\begin{gathered} \alpha f(t, x_1, \dots, x_{n-1}) \operatorname{sgn} x_{s+1} \ge r(t)|x_{s+1}|^\delta \\ \text{for } t \in [\tau - \varepsilon, \tau] \cap R_+,\ |x_i - c_{i-1}| \le \varepsilon,\ i = 1, 2, \dots, n-1. \end{gathered} \tag{2.30}$$

Then there exists a solution y of (1.1) fulfilling (1.4) that is defined in a left neighbourhood of τ.

Proof. Let $\alpha = 1$ and $\beta = 1$; thus $n - s$ is odd. For the other cases the proof is similar. Note that (2.30) and $c_s = 0$ yield $f(\tau, c_0, \dots, c_{n-2}) = 0$. Consider problem (2.11) and (2.12) with $N = \varepsilon$ and $\bar{\tau} = \max(0, \tau - \varepsilon)$. Put

$$M_1 = ((n-s-1)!)^{-\delta} \min_{t \in [\bar{\tau}, \tau]} r(t) > 0,$$

$$\delta_1 = \frac{\delta(n-s-1)+1}{\lambda + \delta - 1},$$

$$M_2 = \max\{|f(t, x_1, \dots, x_{n-1})| : t \in [\bar{\tau}, \tau],\, |x_i - c_{i-1}| \le \varepsilon, i = 1, 2, \dots, n-1\},$$

$$M_3 = \Big(\frac{M_1(\lambda + \delta - 1)}{\delta(n-s-1)+1}\Big)^{-1/(\lambda+\delta-1)},$$

$$M_4 = (\lambda - 1)\varepsilon^\delta \min_{t \in [0, \tau]} r(t),$$

and $M_5 = M_4^{-1/(\lambda-1)}$. Note that due to (2.26), $\delta_1 \in (0, 1)$.

Furthermore, let $J = [T, \tau) \subset [\bar{\tau}, \tau)$ be such that $0 < \tau - T \le 1$,

$$(\tau - T)\sum_{j=0}^{n-2}|c_j| + \frac{M_3}{1-\delta_1}(\tau - T)^{1-\delta_1} + \frac{\lambda-1}{\lambda-2}M_5(\tau-T)^{\frac{\lambda-2}{\lambda-1}} \le \varepsilon, \tag{2.31}$$

and

$$M_2(\tau - T) < \int_M^{2M}\frac{ds}{g(s)}.$$

Denote by y_k a solution of (2.11) and by J_k the intersection of its maximal definition interval and $[T, \tau]$. We prove, similarly as in the proof of Theorem 2.2, (see (2.15)) that

$$y_k^{(n-1)}(t) > M \quad \text{for } t \in J_k; \tag{2.32}$$

hence (2.27), (2.30) and (2.32) yield

$$c_{s+1} \le 0, \quad c_{s+2} \ge 0, \dots, c_{n-2} \le 0, \tag{2.33}$$

$$(-1)^{j-s}y_k^{(j)}(t) > 0 \quad \text{for } j = s+1, s+2, \dots, n-2,$$
$$\operatorname{sgn} y_k^{(s)}(t) = 1, \quad t \in J_k - \{\tau\}.$$

From this, (2.11), (2.12) and (2.30),

$$y_k^{(n)}(t) \ge 0 \quad \text{and} \quad y_k^{(n-1)} \text{ is nondecreasing on } J_k. \tag{2.34}$$

The Taylor formula at $t = \tau$, (2.33), (2.34), and $n - s$ being odd yield

$$\begin{aligned} y_k^{(s)}(t) &= \sum_{j=s}^{n-2} c_j\frac{(t-\tau)^{j-s}}{(j-s)!} + \int_\tau^t \frac{(t-\sigma)^{n-s-2}}{(n-s-2)!}\,y_k^{(n-1)}(\sigma)\,d\sigma \\ &\ge \int_\tau^t \frac{(t-\sigma)^{n-s-2}}{(n-s-2)!}\,y_k^{(n-1)}(\sigma)\,d\sigma \\ &\ge \frac{(\tau-t)^{n-s-1}}{(n-s-1)!}\,y_k^{(n-1)}(t), \quad t \in J_k. \end{aligned}$$

Let $T^* \in [T, \tau)$ be a number such that

$$0 \le y_k^{(s)}(T) \le \varepsilon \quad \text{for } t \in [T^*, \tau),$$

and, if $T^* > T$,

$$y_k^{(s)}(T) > \varepsilon \quad \text{for } t \in [T, T^*);$$

this choice is possible due to (2.34).

Let $T^* > T$ and $t \in [T, T^*)$. Then (2.11), (2.12), (2.29), (2.30) and (2.32) yield

$$y_k^{(n)}(t) \ge r(t)\ \varepsilon^\delta\big(y_k^{(n-1)}(t)\big)^\lambda,$$

and since $\lambda > 1$, an integration on $[t, T^*]$ shows

$$\big(y_k^{(n-1)}(t)\big)^{1-\lambda} \ge \big(y_k^{(n-1)}(t)\big)^{1-\lambda} - \big(y_k^{(n-1)}(T^*)\big)^{1-\lambda} \ge M_4(T^* - t)$$

and

$$y_k^{(n-1)}(t) \le M_5(T^* - t)^{-\frac{1}{\lambda-1}},\ t \in [T, T^*). \tag{2.35}$$

Similarly, for $t \in [T^*, \tau)$, we have

$$y_k^{(n)}(t) \ge r(t)(y_k^{(s)}(t))^\delta(y_k^{(n-1)}(t))^\lambda \ge M_1(\tau - t)^{\delta(n-s-1)}(y_k^{(n-1)}(t))^{\lambda+\delta}. \tag{2.36}$$

Hence, as $\lambda+\delta>1$, an integration on $[t,\tau]$ yields

$$y_k^{(n-1)}(t)\le M_3(\tau-t)^{-\delta_1},\quad t\in J_k, k=k_0,k_{0+1},\dots \tag{2.37}$$

From this and from (2.32) and (2.35) we have $J_k=[T,\tau]$. Moreover, as $\tau-T\le 1$ and $\delta_1<1$, Taylor's theorem, (2.35), (2.31) and (2.37) yield

$$\begin{aligned}
\big|y_k^{(i)}(T)-c_i\big| &\le \sum_{j=i+1}^{n-2}\frac{|c_j|}{(j-i)!}(\tau-T)^{j-i}+\Big|\int_\tau^{T^*}\frac{(T-\sigma)^{n-i-2}}{(n-i-2)!}\,y_k^{(n-1)}(\sigma)\,d\sigma\Big|\\
&\quad+\Big|\int_{T^*}^{T}\frac{(T-\sigma)^{n-i-2}}{(n-i-2)!}y_k^{(n-1)}(\sigma)\,d\sigma\Big|\\
&\le(\tau-T)\sum_{j=0}^{n-2}|c_j|+\frac{M_3}{1-\delta_1}(\tau-T)^{1-\delta_1}+\frac{\lambda-1}{\lambda-2}M_5(\tau-T)^{\frac{\lambda-2}{\lambda-1}}\\
&\le\varepsilon,\quad i=0,1,\dots,n-2.
\end{aligned}$$

From this and from (2.12) and (2.34), $\chi_i(y^{(i)}(t))=y^{(i)}(t), t\in[T,\tau]$ and y_k is the solution of (1.1) fulfilling $y_k^{(i)}(\tau)=c_i, i=0,1,\dots,n-2$ and $y_k^{(n-1)}=k$. Moreover, as $\chi_s(y^{(s)}(t))=y^{(s)}(t)$, the estimations (2.36) and (2.37) holds on $[T,\tau)$. The statement of the theorem follows from this and from the Arzèl-Ascoli Theorem similarly as in the proof of Theorem 2.2; when prooving $\lim_{t\to\tau_-}y(t)=\infty$, T_2 has to be defined such that $M_2(\tau-T_2)<\int_{2Q}^\infty\frac{ds}{g(s)}$ (this is possible due to (1.2)) and the inequality in (2.19) is in contradiction with the choice of T_2. □

The following Corollary shows that conditions (2.26) and (2.28) cannot be weakened.

Corollary 2.5. *Let $c_i=0$, $i=0,1,\dots,n-2$, $\delta>0$, $s\in\{0,1,\dots,n-2\}$, $\alpha\in\{-1,1\}$, $\tau\in(0,\infty)$, $r\in C^0(R_+)$ and $r>0$ on $[0,\tau]$. Then the equation*

$$y^{(n)}=\alpha r\,(t)|y^{(s)}|^\delta|y^{(n-1)}|^\lambda\ \operatorname{sgn}y^{(s)} \tag{2.38}$$

has a noncontinuable solution y fulfilling (1.4) *if and only if*

$$\lambda>\delta(n-s-2)+2\quad\textit{and}\quad n-s+\frac{1-\alpha}{2}\quad\textit{is odd.} \tag{2.39}$$

Proof. If (2.39) holds the statement follows from Theorem 2.4. Let (2.39) be not valid. Let, contrarily, (2.38) have a solution y fulfilling (1.4) defined on $[\bar\tau,\tau)\subset R_+$. Suppose, for simplicity, that $\alpha=1$ and $\lim_{t\to\tau_-}y^{(n-1)}(t)=\infty$. In the other cases the proof is similar.

As $c_i=0$ for $i=0,1,\dots,n-2$, there exists $t_0\in[\bar\tau,\tau)$ such that

$$\begin{gathered}
(-1)^{i-s+\beta}y^{(i)}(t)>0,\quad i=s,s+1,\dots,n-2,\\
y^{(n-1)}(t)\ge 1,\quad y^{(n)}(t)>0\quad\text{on } J=[t_0,\tau),
\end{gathered}\tag{2.40}$$

where $\beta=0$ $(\beta=1)$ if $n-s$ is odd (is even).

Let $n-s$ be even. Then (2.40) yields $y^{(s)}(t)<0$ on J and according to (2.38) $y^{(n)}(t)<0$ on J which contradicts (2.40).

Let $n-s$ be odd and $\lambda\le\delta(n-s-2)+2$. From this, from (1.4), (2.40) and Taylor's theorem, we get

$$0<y^{(s)}(t)=\int_\tau^t\frac{(t-\sigma)^{n-s-2}}{(n-s-2)!}y^{(n-1)}(\sigma)\,d\sigma\le(\tau-t)^{n-s-2}|y^{(n-2)}(t)|, \tag{2.41}$$

with $t \in J$. Furthermore, using (2.40), we have

$$|y^{n-2}(t)| = \int_t^\tau y^{(n-1)}(s)\,ds \ge y^{(n-1)}(t)(\tau - t), \quad t \in J.$$

From this and from (2.41)

$$0 < y^{(s)}(t)[y^{(n-1)}(t)]^{n-s-2} \le |y^{(n-2)}(t)|^{n-s-1} \le |y^{(n-2)}(t_0)|^{n-s-1} = M_1, \quad t \in J.$$

Thus, from $\lambda \le \delta(n-s-2)+2$ and from $y^{(n-1)}(t) \ge 1$ (see (2.40)), we have

$$\begin{aligned}
\infty = \log \frac{y^{(n-1)}(\tau)}{y^{(n-1)}(t_0)} &= \int_{t_0}^\tau \frac{y^{(n)}(\sigma)}{y^{(n-1)}(\sigma)}\,d\sigma \\
&= \int_{t_0}^\tau r(\sigma)(y^{(s)}(\sigma))^\delta [y^{(n-1)}(\sigma)]^{\lambda-1}\,ds \\
&\le M_1^\delta \int_{t_0}^\tau r(\sigma)[y^{(n-1)}(\sigma)]^{\lambda-1-\delta(n-s-2)} d\sigma \\
&\le M_1^\delta \int_{t_0}^\tau r(\sigma) y^{(n-1)}(\sigma)\,d\sigma \\
&\le M_1^\delta \max_{z\in[T,\tau]} r(z)\,|y^{(n-2)}(t_0)| < \infty.
\end{aligned}$$

This contradiction proves the statement. □

Remark. Theorem 2.4 is proved in [1] in case $s = 0$ and under further assumptions.

Remark. Let the assumptions of Theorems 2.4 hold with the exception of (2.26). If $\lambda \le 2$, Theorem 2.1 with $g = x^\lambda$ and Theorem 2.3 (ii) yield (1.1) has no solution y fulfilling (1.4). So there is a problem what happens if $2 < \lambda < \delta(n-s-2)+2$. In special cases, see e.g. Corollary 2.5, no noncontinuable solution y fulfilling (1.4) exists.

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