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NON-AUTONOMOUS INHOMOGENEOUS BOUNDARY CAUCHY PROBLEMS

MOHAMMED FILALI, BELHADJ KARIM

Abstract. In this paper we prove existence and uniqueness of classical solutions for the non-autonomous inhomogeneous Cauchy problem

$$\begin{gathered}\frac{d}{dt}u(t) = A(t)u(t) + f(t), \quad 0 \le s \le t \le T,\\ L(t)u(t) = \Phi(t)u(t) + g(t), \quad 0 \le s \le t \le T,\\ u(s) = x.\end{gathered}$$

The solution to this problem is obtained by a variation of constants formula.

1. Introduction

Consider the boundary Cauchy problem

$$\begin{gathered}\frac{d}{dt}u(t) = A(t)u(t), \quad 0 \le s \le t \le T,\\ L(t)u(t) = \Phi(t)u(t), \quad 0 \le s \le t \le T,\\ u(s) = x.\end{gathered} \tag{1.1}$$

In the autonomous case ($A(t) = A$, $L(t) = L$), the Cauchy problem (1.1) was studied by Greiner [3]. The author used the perturbation of domains of infinitesimal generators to study the homogeneous boundary Cauchy problem. He has also showed the existence of classical solution of (1.1) via a variation of constants formula. In the non-autonomous case, Kellerman [5] and Lan [6] showed the existence of an evolution family $(U(t,s))_{0\le s\le t\le T}$ which provides classical solutions of homogeneous boundary Cauchy problems. Filali and Moussi [2] showed the existence and uniqueness of classical solutions to the problem

$$\begin{gathered}\frac{d}{dt}u(t) = A(t)u(t), \quad 0 \le s \le t \le T,\\ L(t)u(t) = \Phi(t)u(t) + g(t), \quad 0 \le s \le t \le T,\\ u(s) = x.\end{gathered} \tag{1.2}$$

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In this paper, we prove existence and uniqueness of classical solutions to the problem

$$\begin{gathered} \frac{d}{dt}u(t) = A(t)u(t) + f(t), \quad 0 \le s \le t \le T, \\ L(t)u(t) = \Phi(t)u(t) + g(t), \quad 0 \le s \le t \le T, \\ u(s) = x. \end{gathered} \tag{1.3}$$

Our technique consists on transforming (1.3) into an ordinary Cauchy problem and giving an equivalence between the two problems. The solution is explicitly given by a variation of constants formula.

2. Evolution Family

Definition 2.1. A family of bounded linear operators $(U(t,s))_{0\le s\le t\le T}$ on X is an evolution family if
(a) $U(t,r)U(r,s) = U(t,s)$ and $U(t,t) = Id$ for all $0 \le s \le r \le t \le T$; and
(b) the mapping $(t,s) \to U(t,s)x$ is continuous on $\triangle$, for all $x \in X$ with

$$\triangle = \{(t,s) \in \mathbb{R}_+^2 : 0 \le s \le t \le T\}.$$

Definition 2.2. A family of linear (unbounded) operators $(A(t))_{0\le t\le T}$ on a Banach space X is a stable family if there are constants $M \ge 1$, $\omega \in \mathbb{R}$ such that $]\omega, +\infty[\subset \rho(A(t))$ for all $0 \le t \le T$ and

$$\|\prod_{i=1}^{m} R(\lambda, A(t_i))\| \le M\frac{1}{(\lambda-\omega)^m}$$

for $\lambda > \omega$ and any finite sequence $0 \le t_1 \le t_2 \le \cdots \le t_m \le T$.

Let D, X and Y be Banach spaces, D densely and continuously embedded in X. Consider families of operators $A(t) \in L(D,X)$, $L(t) \in L(D,Y)$, $\Phi(t) \in L(X,Y)$ for $0 \le t \le T$. In this section, we use the operator matrices method to prove the existence of classical solutions for the non-autonomous inhomogeneous boundary Cauchy problem (1.3). We use the following theorem due to Tanaka [9].

Theorem 2.3. *Let $(A(t))_{0\le t\le T}$ be a stable family of linear operators on a Banach space X such that*

(a) *the domain $D = (D(A(t), \|.\|_D)$ is a Banach space independent of t,*
(b) *the mapping $t \to A(t)x$ is continuously differentiable in X for every $x \in D$.*

Then there is an evolution family $(U(t,s))_{0\le s\le t\le T}$ on $\overline{D}$. Moreover, we have the following properties: (1) $U(t,s)D(s) \subset D(t)$ for all $0 \le s \le t \le T$, where

$$D(r) = \{x \in D : A(r)x \in \overline{D}\}, 0 \le r \le T;$$

(2) the mapping $t \to U(t,s)x$ is continuously differentiable in X on $[s,T]$ and

$$\frac{d}{dt}U(t,s)x = A(t)U(t,s)x$$

for all $x \in D(s)$ and $t \in [0,T]$.

We will assume that the following hypotheses:

(H1) The mapping $t \to A(t)x$ is continuously differentiable for all $x \in D$.
(H2) The family $(A_0(t))_{0\le t\le T}, A_0(t) = A(t)/\ker L(t)$ the restriction of $A(t)$ to $\ker L(t)$, is stable, with M_0 and ω_0 constants of stability.

(H3) The operator $L(t)$ is surjective for every $t \in [0,T]$ and the mapping $t \to L(t)x$ is continuously differentiable for all $x \in D$.
(H4) The mapping $t \to \Phi(t)x$ is continuously differentiable for all $x \in X$.
(H5) There exist constants $\gamma > 0$ and $\omega_1 \in \mathbb{R}$ such that

$$\|L(t)x\|_Y \geq \frac{\lambda - \omega_1}{\gamma}\|x\|_X \tag{2.1}$$

for $x \in \ker(\lambda I - A(t))$, $\omega_1 < \lambda$ and $t \in [0,T]$.

Note that under the above hypotheses, Lan [6] has showed that $A_0(t)$ generates an evolution family $(U(t,s))_{0\leq s\leq t\leq T}$ such that:

(a) $U(t,r)U(r,s) = U(t,s)$ and $U(t,t) = Id_X$ for all $0 \leq s \leq t \leq T$;
(b) $(t,s) \to U(t,s)x$ is continuously differentiable on Δ for all $x \in X$ with $\Delta = \{(t,s) \in {\mathbb{R}_+}^2 : 0 \leq s \leq t \leq T\}$;
(c) there exists constants $M_0 \geq 1$ and $\omega_0 \in \mathbb{R}$ such that $\|U(t,s)\| \leq M_0 e^{\omega_0(t-s)}$.

The following results with will be used in this article.

Lemma 2.4 ([3]). *For $t \in [0,T]$ and $\lambda \in \rho(A_0(t))$, following properties are satisfied:*

(1) $D = D(A_0(t)) \oplus \ker(\lambda I - A(t))$
(2) $L(t)/\ker(\lambda I - A(t))$ *is an isomorphism from* $\ker(\lambda I - A(t))$ *onto* Y
(3) $t \mapsto L_{\lambda,t} := (L(t)/\ker(\lambda I - A(t)))^{-1}$ *is strongly continuously differentiable.*

As a consequence of this lemma, we have $L(t)L_{\lambda,t} = Id_Y$, $L_{\lambda,t}L(t)$ and $(I - L_{\lambda,t}L(t))$ are the projections from D onto $\ker(\lambda I - A(t))$ and $D(A_0(t))$.

3. The Homogeneous Problem

In this section, we consider the Cauchy problem (1.1). A function $u : [s,T] \to X$ is called classical solution if it is continuously differentiable, $u(t) \in D$ for all $0 \leq s \leq t \leq T$ and u satisfies (1.1).

We now introduce the Banach spaces $Z = X \times Y$, $Z_0 = X \times \{0\} \subset Z$ and we consider the projection of Z onto X: $p_1(x,y) = x$. Let $M(t)$ be the matrix-valued operator defined on Z by

$$M(t) = \begin{pmatrix} A(t) & 0 \\ -L(t) + \Phi(t) & 0 \end{pmatrix} = l(t) + \phi(t),$$

where

$$l(t) = \begin{pmatrix} A(t) & 0 \\ -L(t) & 0 \end{pmatrix}, \quad \phi(t) = \begin{pmatrix} 0 & 0 \\ \Phi(t) & 0 \end{pmatrix},$$

and $D(M(t)) = D \times \{0\}$.

Now, we consider the Cauchy problem

$$\begin{gathered} \frac{d}{dt}u(t) = M(t)u(t), \quad 0 \leq s \leq t \leq T, \\ u(s) = (x,0). \end{gathered} \tag{3.1}$$

We start by proving the following lemma.

Lemma 3.1. *Assume that hypothesis (H1)–(H5) hold. Then, the family of operators $(M(t))_{0\leq t\leq T}$ is stable.*

Remark 3.2. Since $L_{\lambda,t}L(t)$ is the projection from D onto $\ker(\lambda I - A(t))$ and $x - L_{\lambda,t}L(t)x \in D(A_0(t))$, we have

$$\begin{aligned}&R(\lambda, A_0(t))((\lambda I - A(t))x) + L_{\lambda,t}L(t)x\\&= R(\lambda, A_0(t))((\lambda I - A(t))(x - L_{\lambda,t}L(t)x) + L_{\lambda,t}L(t)x\end{aligned}$$

and

$$R(\lambda, A_0(t))((\lambda I - A(t))x) + L_{\lambda,t}L(t)x = x. \tag{3.2}$$

Proof of Lemma 3.1. Since $M(t)$ is a perturbation of $l(t)$ by a linear bounded operator on E, hence, in view of the perturbation result [7, Theorem 5.2.3], it is sufficient to show the stability of $l(t)$. For $\lambda > \omega_0$ and $\lambda \neq 0$, let

$$R(\lambda) = \begin{pmatrix} R(\lambda, A_0(t)) & L_{\lambda,t} \\ 0 & 0 \end{pmatrix}.$$

We have $D(l(t)) = D \times \{0\}$ and

$$(\lambda I - l(t)) \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} (\lambda I - A(t))x \\ L(t)x \end{pmatrix}$$

for $(x$
$0) \in D \times \{0\}$. By Remark 3.2, we obtain

$$R(\lambda)(\lambda I - l(t)) \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} R(\lambda, A_0(t))((\lambda I - A(t))x) + L_{\lambda,t}L(t)x \\ 0 \end{pmatrix}.$$

So that

$$R(\lambda)(\lambda I - l(t)) \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}. \tag{3.3}$$

On the other hand, for $(x, y) \in X \times Y$, we have

$$(\lambda I - l(t))R(\lambda) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda I - A(t & 0 \\ L(t) & \lambda \end{pmatrix} \begin{pmatrix} R(\lambda, A_0(t))x + L_{\lambda,t}y \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.4}$$

from (3.3) and (3.4), we obtain that the resolvent of $l(t)$ is given by

$$R(\lambda, l(t)) = \begin{pmatrix} R(\lambda, A_0(t)) & L_{\lambda,t} \\ 0 & 0 \end{pmatrix}. \tag{3.5}$$

By a direct computation, we obtain

$$\prod_{i=1}^{m} R(\lambda, l(t_i)) = \begin{pmatrix} \prod_{i=1}^{m} R(\lambda, A_0(t_i)) & \prod_{i=1}^{m-1} R(\lambda, A_0(t_i)L_{\lambda,t_m} \\ 0 & 0 \end{pmatrix}$$

for a finite sequence $0 \leq t_1 \leq t_2 \leq \cdots \leq t_m \leq T$ and we have

$$\prod_{i=1}^{m} R(\lambda, l(t_i)) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \prod_{i=1}^{m} R(\lambda, A_0(t))x + \prod_{i=1}^{m-1} R(\lambda, A_0(t))L_{\lambda,t_m}y \\ 0 \end{pmatrix}.$$

From hypothesis (H5), we conclude that $\|L_{\lambda,t}\| \leq \frac{\gamma}{(\lambda-\omega)}$ for all $t \in [0, T]$ and $\lambda > \omega$ and by using (H2), we obtain

$$\begin{aligned}\|\prod_{i=1}^{m} R(\lambda, l(t_i)) \begin{pmatrix} x \\ y \end{pmatrix}\| &\leq \|\prod_{i=1}^{m} R(\lambda, A_0(t))x\| + \|\prod_{i=1}^{m-1} R(\lambda, A_0(t))L_{\lambda,t_m}y\| \\ &\leq \frac{M}{(\lambda-\omega_0)^m}\|x\| + \frac{\gamma M}{(\lambda-\omega_0)^{m-1}}\frac{1}{\lambda-\omega_1}\|y\|.\end{aligned} \tag{3.6}$$

For $\omega_2 = \max(\omega_0, \omega_1)$, we have

$$\| \prod_{i=1}^{m} R(\lambda, l(t_i) \begin{pmatrix} x \\ y \end{pmatrix} \| \leq \frac{M'}{(\lambda - \omega_2)^m} (\|x\| + \|y\|),$$

where $M' = \max(M, M\gamma)$. On $E = X \times Y$ equipped with the norm $\|(x,y)\|_1 = \|x\| + \|y\|$, we have:

$$\| \prod_{i=1}^{m} R(\lambda, l(t_i) \begin{pmatrix} x \\ y \end{pmatrix} \| \leq \frac{M'}{(\lambda - \omega_2)^m} (\|(x,y)\|_1).$$

□

In the following proposition we give the equivalence between the boundary problem (1.1) and the Cauchy problem (3.1).

Proposition 3.3. *Let $(x, 0) \in D \times \{0\}$.*

(1) *If the function $t \to U(t) = (u_1(t), 0)$ is a classical solution of (3.1) with an initial value $(x, 0)$ then $t \to u_1(t)$ is a classical solution of (1.1) with the initial value x.*
(2) *Let u be a classical solution of (1.1) with the initial value x. Then the function $t \to U(t) = (u(t), 0)$ is a classical solution of (3.1) with the initial value $(x, 0)$.*

Proof. (1) Since $U(t) = (u_1(t), 0)$ is a classical solution of (3.1), u_1 is continuously differentiable on $[s, T]$ and $u_1(t) \in D$. Moreover,

$$\frac{d}{dt} U(t) = \begin{pmatrix} \frac{d}{dt} u_1(t) \\ 0 \end{pmatrix} = M(t)U(t) \quad \text{and} \quad U(s) = \begin{pmatrix} x \\ 0 \end{pmatrix}. \tag{3.7}$$

Therefore,

$$\begin{gathered} \frac{d}{dt} u_1(t) = A(t)u_1(t), \quad 0 \leq s \leq t \leq T, \\ L(t)u_1(t) = \Phi(t)u_1(t), \quad 0 \leq s \leq t \leq T, \\ u_1(s) = x. \end{gathered} \tag{3.8}$$

This implies that u_1 is a classical solution of (1.1).
(2) Let u is a classical solution of (1.1), then u is continuously differentiable, $u(t) \in D$ for $t \geq s$ and

$$\begin{gathered} \frac{d}{dt} u(t) = A(t)u(t), \quad 0 \leq s \leq t \leq T, \\ L(t)u(t) = \Phi(t)u(t), \quad 0 \leq s \leq t \leq T, \\ u(s) = x. \end{gathered}$$

Hence

$$\begin{pmatrix} \frac{d}{dt} u(t) \\ 0 \end{pmatrix} = \begin{pmatrix} A(t) & 0 \\ -L(t) + \Phi(t) & 0 \end{pmatrix} \begin{pmatrix} u(t) \\ 0 \end{pmatrix},$$

with $(u(s), 0) = (x, 0)$. This implies that $U(t) = (u(t), 0)$ is a classical solution of (3.2) with the initial value $(x, 0)$. □

The above proposition allows us to get the aim of this section by showing the well-posedness of the Cauchy problem (1.1).

Theorem 3.4. *Assume that the hypotheses (H1)–(H5) hold. Then for every $x \in D$, such that $-L(s)x + \Phi(s)x = 0$, the problem (1.1) has a unique classical solution. Moreover, u is given by $t \to p_1(U(t,s)\begin{pmatrix} x \\ 0 \end{pmatrix}$, where $U(t,s)$ is the evolution family generated by $(M(t)_{0\le t\le T})$.*

Proof. For the Cauchy problem (3.1), we have the following:

(1) $D(M(t)) = D \times \{0\}$ is independent of t.
(2) $t \to M(t)\begin{pmatrix} x \\ 0 \end{pmatrix}$ is continuously differentiable for $(x,0) \in D \times \{0\}$.
(3) The family $(M(t))_{0\le t\le T}$ is stable.

Then the family $M(t)$ satisfies all conditions of Theorem 2.3. Thus, there exist an evolution family $(U(t,s))_{0\le s\le t}$ generated by the family $(M(t))_{0\le t\le T}$ such that

(a) $U(t,t) = Id_{X\times\{0\}}$,
(b) $U(t,r)U(r,s) = U(t,s)$, $0 \le s \le r \le t \le T$,
(c) $(t,s) \to U(t,s)$ is strongly continuous,
(d) the function $t \to U(t,s)\begin{pmatrix} x \\ 0 \end{pmatrix}$ is continuously differentiable in $X \times \{0\}$ on $[s,T]$, and satisfies
$$\frac{d}{dt}U(t,s)\begin{pmatrix} x \\ 0 \end{pmatrix} = M(t)U(t,s)\begin{pmatrix} x \\ 0 \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} x \\ 0 \end{pmatrix} \in D(s),$$
and
$$U(t,s)D(s) \subset D(t), \quad \text{for all } 0 \le s \le t \le T, \tag{3.9}$$
where
$$\begin{aligned} D(s) &= \{\begin{pmatrix} x \\ 0 \end{pmatrix} \in D \times \{0\} : M(s)\begin{pmatrix} x \\ 0 \end{pmatrix} \in X \times \{0\}\} \\ &= \ker\big(L(s) - \Phi(s)\big) \times \{0\}. \end{aligned} \tag{3.10}$$

Let $U(t,s)(x,0) = (u_1(t),0)$. We have
$$\begin{pmatrix} \frac{d}{dt}u_1(t) \\ 0 \end{pmatrix} = M(t)\begin{pmatrix} u_1(t) \\ 0 \end{pmatrix},$$
and for $u(t) = (u_1(t),0)$, we have $\frac{d}{dt}u(t) = M(t)u(t)$, with $u(s) = (x,0)$, thus $u(t) = (u_1(t),0)$ is a classical solution of (3.1) and from Proposition 3.3, we have u_1 is a classical solution of (1.1) and
$$u_1(t) = p_1\big(U(t,s)\begin{pmatrix} x \\ 0 \end{pmatrix}\big). \tag{3.11}$$
□

4. First Inhomogeneous Problem

In this section, we consider the inhomogeneous Cauchy problem
$$\begin{gathered} \frac{d}{dt}u(t) = A(t)u(t) + f(t), \quad 0 \le s \le t \le T, \\ L(t)u(t) = \Phi(t)u(t), \quad 0 \le s \le t \le T, \\ u(s) = x. \end{gathered} \tag{4.1}$$

A function $u : [s, T] \to X$ is called classical solution if it is continuously differentiable, $u(t) \in D$, $t \geq s$ and u satisfies (4.1).

Consider the Banach space $E = X \times Y \times C^1([0,T], X), T > 0$, where $C^1([0,T], X)$ is the space of continuously differentiable functions from $[0,T]$ into X equipped with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, for $f \in C^1([0,T], X)$. Let $B(t)$ be the operator matrices defined on E by

$$B(t) = \begin{pmatrix} A(t) & 0 & \delta_t \\ -L(t) + \Phi(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.2}$$

with $D(B(t)) = D \times \{0\} \times C^1([0,T], X)$. Where $\delta_t : C^1([0,T], X) \to X$ is the Dirac function concentrated at the point t with $\delta_t(f) = f(t)$. To the family $B(t)$ we associate the homogeneous Cauchy problem

$$\begin{gathered} \frac{d}{dt} u(t) = B(t) u(t), \quad 0 \leq s \leq t \leq T, \\ u(s) = (x, 0, f). \end{gathered} \tag{4.3}$$

with $(x, 0, f) \in D \times \{0\} \times C^1([0,T]$.

Lemma 4.1. *Assume that hypothesis (H1)–(H5) hold. Then the family operators* $(B(t))_{0 \leq t \leq T}$ *is stable.*

Proof. For $t \in [0,T]$, we write the operator $B(t)$ as $B(t) = l(t) + \phi(t)$, with

$$l(t) = \begin{pmatrix} A(t) & 0 & 0 \\ -L(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \phi(t) = \begin{pmatrix} 0 & 0 & \delta_t \\ \Phi(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We must show that $l(t)$ is stable and that

$$R(\lambda, l(t)) = \begin{pmatrix} R(\lambda, A_0(t)) & L_{\lambda,t} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix}. \tag{4.4}$$

For $\lambda > \omega_0$, $\lambda \neq 0$, and $t \in [0,T]$, let

$$R(\lambda) = \begin{pmatrix} R(\lambda, A_0(t)) & L_{\lambda,t} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix}.$$

For $(x, y, f) \in X \times Y \times C^1([0,T], X)$, we have

$$\begin{pmatrix} R(\lambda, A_0(t) & L_{\lambda,t} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ f \end{pmatrix} = \begin{pmatrix} R(\lambda, A_0(t))x + L_{\lambda,t} y \\ 0 \\ \frac{f}{\lambda} \end{pmatrix},$$

by the Remark 3.2, we obtain

$$(\lambda I - l(t)) R(\lambda) \begin{pmatrix} x \\ y \\ f \end{pmatrix} = \begin{pmatrix} (\lambda I - A(t))[R(\lambda, A_0(t))x + L_{\lambda,t} y] \\ L(t)[R(\lambda, A_0(t))x + L_{\lambda,t} y] \\ f \end{pmatrix} = \begin{pmatrix} x \\ y \\ f \end{pmatrix}. \tag{4.5}$$

On the other hand, for $(x, 0, f) \in D \times \{0\} \times C^1([0,T], X)$, we have

$$(\lambda I - l(t)) \begin{pmatrix} x \\ 0 \\ f \end{pmatrix} = \begin{pmatrix} (\lambda I - A(t))x \\ L(t)x \\ \lambda f \end{pmatrix},$$

and

$$R(\lambda)(\lambda I - l(t)) \begin{pmatrix} x \\ 0 \\ f \end{pmatrix} = \begin{pmatrix} R(\lambda, A_0(t))((\lambda I - A(t))x) + L_{\lambda,t}L(t)x \\ 0 \\ f \end{pmatrix}.$$

From Remark 3.2, we have

$$R(\lambda)(\lambda I - l(t)) \begin{pmatrix} x \\ 0 \\ f \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ f \end{pmatrix}. \tag{4.6}$$

From (4.5) and (4.6), we obtain that the resolvent of $l(t)$ is given by

$$R(\lambda, l(t)) = \begin{pmatrix} R(\lambda, A_0(t)) & L_{\lambda,t} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix}.$$

By recurrence we can obtain

$$\prod_{i=1}^{m} R(\lambda, l(t_i)) = \begin{pmatrix} \prod_{i=1}^{m} R(\lambda, A_0(t_i)) & \prod_{i=1}^{m-1} R(\lambda, A_0(t_i))L_{\lambda,t_m} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\lambda^m \end{pmatrix}.$$

For a finite sequence $0 \le t_1 \le t_2 \cdots \le t_m \le T$ and for $(x, y, f) \in E$, we have

$$\prod_{i=1}^{m} R(\lambda, l(t_i)) \begin{pmatrix} x \\ y \\ f \end{pmatrix} = \begin{pmatrix} \prod_{i=1}^{m} R(\lambda, A_0(t_i))x + \prod_{i=1}^{m-1} R(\lambda, A_0(t_i))L_{\lambda,t_m}y \\ 0 \\ f/\lambda^m \end{pmatrix}.$$

Using (H5), we obtain

$$\begin{aligned} \| \prod_{i=1}^{m} R(\lambda, l(t_i)) \begin{pmatrix} x \\ y \\ f \end{pmatrix} \| &\le \| \prod_{i=1}^{m} R(\lambda, A_0(t_i))x + \prod_{i=1}^{m-1} R(\lambda, A_0(t_i))L_{\lambda,t_m}y\| + \frac{\|f\|}{\lambda^m} \\ &\le \frac{M}{(\lambda-\omega_0)^m}\|x\| + \frac{M}{(\lambda-\omega_0)^{m-1}}\frac{\gamma}{\lambda-\omega_1}\|y\| + \frac{\|f\|}{\lambda^m}. \end{aligned}$$

Define $\omega_2 = \max(0, \omega_0, \omega_1)$. Then

$$\| \prod_{i=1}^{m} R(\lambda, l(t_i)) \begin{pmatrix} x \\ y \\ f \end{pmatrix} \| \le \frac{M'}{(\lambda-\omega_2)^m}(\|x\| + \|y\| + \|f\|),$$

where $M' = \max(M, M\gamma)$ and

$$\| \prod_{i=1}^{m} R(\lambda, l(t_i))\| \le \frac{M'}{(\lambda-\omega_2)^m}. \tag{4.7}$$

This inequality shows that the family $l(t)$ is stable and by using [7, Theorem 5.2.3], the family $B(t)$ is stable. □

Proposition 4.2. *Let* $(x, 0, f) \in D \times \{0\} \times C^1([0, T], X)$.
(1) If the function $t \to u(t) = (u_1(t), 0, u_2(t))$ *is a classical solution of* (4.3) *with an initial value* $(x, 0, f)$ *then* $t \to u_1(t)$ *is a classical solution of* (4.1) *with the initial value* x.
(2) Let u *is a classical solution of* (4.1) *with the initial value* x *.Then, the function* $t \to U(t) = (u(t), 0, f)$ *is a classical solution of* (4.3) *with the initial value* $(x, 0, f)$.

Proof. (1) If $u(t)=(u_1(t),0,u_2(t))$ is a classical solution of (4.3), then u_1 is continuously differentiable on $[s,T]$, $u_1\in D$ and we have

$$\frac{d}{dt}u(t)=\begin{pmatrix}\frac{d}{dt}u_1(t)\\ 0\\ \frac{d}{dt}u_2(t)\end{pmatrix}=B(t)u(t),$$

which implies

$$\begin{pmatrix}\frac{d}{dt}u_1(t)\\ 0\\ \frac{d}{dt}u_2(t)\end{pmatrix}=\begin{pmatrix}A(t)&0&\delta_t\\ -L(t)+\Phi(t)&0&0\\ 0&0&0\end{pmatrix}\begin{pmatrix}u_1(t)\\ 0\\ u_2(t)\end{pmatrix},$$

and

$$\begin{pmatrix}\frac{d}{dt}u_1(t)\\ 0\\ \frac{d}{dt}u_2(t)\end{pmatrix}=\begin{pmatrix}A(t)u_1(t)+\delta_t u_2(t)\\ -L(t)u_1(t)+\Phi(t)u_1(t)\\ 0\end{pmatrix},$$

with

$$u(s)=\begin{pmatrix}u_1(s)\\ 0\\ u_2(s)\end{pmatrix}=\begin{pmatrix}x\\ 0\\ f\end{pmatrix}.$$

One has $\frac{d}{dt}u_2(t)=0$. This implies $u_2(t)=u_2(s)=f$; therefore, $\delta_t u_2(t)=\delta_t f=f(t)$ and we have

$$\begin{gathered}\frac{d}{dt}u_1(t)=A(t)u_1(t)+f(t),\quad 0\le s\le t\le T,\\ L(t)u_1(t)=\Phi(t)u_1(t),\quad 0\le s\le t\le T,\\ u_1(s)=x.\end{gathered}$$

Therefore, u_1 is a classical solution of (4.1) with the initial value x.
(2) If u is a classical solution of (4.1), then u is continuously differentiable, $u(t)\in D$ and

$$\begin{gathered}\frac{d}{dt}u(t)=A(t)u(t)+f(t),\quad 0\le s\le t\le T,\\ L(t)u(t)=\Phi(t)u(t),\quad 0\le s\le t\le T,\\ u(s)=x.\end{gathered}$$

Moreover,

$$\begin{pmatrix}\frac{d}{dt}u(t)\\ 0\\ 0\end{pmatrix}=\begin{pmatrix}A(t)&0&\delta_t\\ -L(t)+\Phi(t)&0&0\\ 0&0&0\end{pmatrix}\begin{pmatrix}u(t)\\ 0\\ f\end{pmatrix}.$$

With $u(s)=x$, $U(t)=(u(t),0,f)$ is continuously differentiable, $U(t)\in D(B(t))=D\times\{0\}\times C^1([0,T],X)$ then it is a classical solution of (4.3) with the initial value $(x,0,f)$. □

Theorem 4.3. *Let $f\in C^1([0,T],X)$. Assume that the hypothesis (H1)–(H5) hold. Then for all $x\in D$, such that $-L(s)x+\Phi(s)x=0$, problem* (4.1) *has a unique classical solution solution u. Moreover, u is given by*

$$u(t)=U_\Phi(t,s)x+\int_s^t U_\Phi(t,s)f(r)dr, \tag{4.8}$$

where $U_\Phi(t,s)$ is an evolution family solution of the problem (3.1)

Proof. Consider the problem

$$\begin{gathered}\frac{d}{dt}u(t) = B(t)u(t), \quad 0 \le s \le t \le T,\\ u(s) = (x, 0, f).\end{gathered}$$

We have showed that $(B(t))_{0\le t\le T}$ is a stable family and the function $t \to B(t)y$ is continuously differentiable, for all $y \in D(B(t)) = D \times \{0\} \times C^1([0,T],X)$ and that $D(B(t))$ is independent of t. Then there exist an evolution system $U(t,s)$ on $X \times \{0\} \times C^1([0,T],X)$ such that

$$U(t,s)\begin{pmatrix} x\\ 0\\ f\end{pmatrix} = \begin{pmatrix} u_1(t)\\ 0\\ u_2(t)\end{pmatrix} = u(t)$$

is a classical solution of (4.3) and from the Proposition 4.2, u_1 is a classical solution of (4.1), for $(x, 0, f) \in \ker(L(s) - \Phi(s)) \times \{0\} \times C^1([0,T],X)$. Let $v(r) = U_\Phi(t,r)u_1(r)$. Then v is differentiable and

$$\frac{d}{dr}v(r) = -U_\Phi(t,r)A_\Phi(r)u_1(r) + U_\Phi(t,r)[A_\Phi(r)u_1(r) + f(r)],$$

where $A_\Phi(t) = A(t)/\ker(L(t) - \Phi(t))$; therefore,

$$\frac{d}{dr}v(r) = U_\Phi(t,r)f(r). \tag{4.9}$$

Integrating (4.9) from s to t, we obtain

$$u_1(t) = U_\Phi(t,s)x + \int_s^t U_\Phi(t,r)f(r)dr,$$

which completes the proof. □

5. Second Inhomogeneous Problem

In this section, we consider the Inhomogeneous Cauchy problem

$$\begin{gathered}\frac{d}{dt}u(t) = A(t)u(t) + f(t), \quad 0 \le s \le t \le T,\\ L(t)u(t) = \Phi(t)u(t) + g(t), \quad 0 \le s \le t \le T,\\ u(s) = x.\end{gathered} \tag{5.1}$$

A function $u : [s,T] \to X$ is a classical solution if it is continuously differentiable, $u(t) \in D$, for all $t \ge s$ and u satisfies (5.1).

Consider the Banach space $E = X \times Y \times C^1([0,T],X) \times C^1([0,T],Y)$, where $C^1([0,T],X)$ and $C^1([0,T],Y)$ are equipped with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ for f in $C^1([0,T],X)$ or in $C^1([0,T],Y)$. Consider the operator matrices

$$B(t) = \begin{pmatrix} A(t) & 0 & \delta_t & 0\\ -L(t)+\Phi(t) & 0 & 0 & \overline{\delta_t}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \tag{5.2}$$

with

$$D(B(t)) = D \times \{0\} \times C^1([0,T],X) \times C^1([0,T],Y)$$

where $\delta_t : C^1([0,T],X) \to X$ such that $\delta_t(f) = f(t)$ and $\overline{\delta_t} : C^1([0,T],Y) \to Y$ such that $\overline{\delta_t}(g) = g(t)$. To the family $B(t)$,we associate the homogeneous Cauchy problem

$$\begin{gathered} \frac{d}{dt}u(t) = B(t)u(t), \quad 0 \le s \le t \le T, \\ u(s) = (x, 0, f, g) \end{gathered} \tag{5.3}$$

for $(x, 0, f, g) \in D \times \{0\} \times C^1([0,T],X) \times C^1([0,T],Y) = D_1$.

Lemma 5.1. *Assume that the hypothesis (H1)–(H5) hold. Then the family operators $B(t)$ is stable.*

Proof. For $t \in [0,T]$, we write the $B(t)$ defined in (5.2) as $B(t) = l(t) + \phi(t)$, where

$$l(t) = \begin{pmatrix} A(t) & 0 & 0 & 0 \\ -L(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \phi(t) = \begin{pmatrix} 0 & 0 & \delta_t & 0 \\ \Phi(t) & 0 & 0 & \overline{\delta_t} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

we must show that the family $l(t)$ is stable. Let

$$R(\lambda) = \begin{pmatrix} R(\lambda, A_0(t)) & L_{\lambda,t} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\lambda & 0 \\ 0 & 0 & 0 & 1/\lambda \end{pmatrix}.$$

For $\lambda > \omega_0$, $\lambda \neq 0$ and $t \in [0,T]$ we show that $R(\lambda, l(t)) = R(\lambda)$. For $(x, y, f, g) \in X \times Y \times C^1([0,T],X) \times C^1([0,T],Y)$, we have

$$R(\lambda) \begin{pmatrix} x \\ y \\ f \\ g \end{pmatrix} = \begin{pmatrix} R(\lambda, A_0(t))x + L_{\lambda,t}y \\ 0 \\ f/\lambda \\ g/\lambda \end{pmatrix}, \tag{5.4}$$

by the Remark 3.2 and with the same proof as Lemma 4.1 we obtain

$$(\lambda I - l(t))R(\lambda) \begin{pmatrix} x \\ y \\ f \\ g \end{pmatrix} = \begin{pmatrix} x \\ y \\ f \\ g \end{pmatrix}. \tag{5.5}$$

On the other hand, for $(x, 0, f, g) \in D \times \{0\} \times C^1([0,T],X) \times C^1([0,T],Y)$, we have

$$(\lambda I - l(t)) \begin{pmatrix} x \\ 0 \\ f \\ g \end{pmatrix} = \begin{pmatrix} (\lambda I - A(t))x \\ L(t)x \\ \lambda f \\ \lambda g \end{pmatrix},$$

and

$$R(\lambda)(\lambda I - l(t)) \begin{pmatrix} x \\ 0 \\ f \\ g \end{pmatrix} = \begin{pmatrix} R(\lambda, A_0(t))((\lambda I - A(t))x) + L_{\lambda,t}L(t)x \\ 0 \\ f \\ g \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ f \\ g \end{pmatrix}, \tag{5.6}$$

then from (5.5), (5.6) and Remark 3.2, we have $R(\lambda) = R(\lambda I, l(t))$. By recurrence we obtain

$$\prod_{i=1}^{m} R(\lambda, l(t_i)) = \begin{pmatrix} \prod_{i=1}^{m} R(\lambda, A_0(t_i)) & \prod_{i=1}^{m-1} R(\lambda, A_0(t_i)) L_{\lambda,t_m} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\lambda^m & 0 \\ 0 & 0 & 0 & 1/\lambda^m \end{pmatrix},$$

for a finite sequence $0 \le t_1 \le t_2 \le \cdots \le t_m \le T$.

Now on the space $X \times Y \times C^1([0,T], X) \times C^1([0,T], Y)$,we consider the norm

$$\|(x, y, f, g)\| = (\|x\| + \|y\| + \|f\| + \|g\|). \tag{5.7}$$

For $(x, y, f, g) \in X \times Y \times C^1([0,T], X) \times C^1([0,T], Y)$, we have

$$\begin{aligned} \|\prod_{i=1}^{m} R(\lambda, l(t_i)) \begin{pmatrix} x \\ y \\ f \\ g \end{pmatrix} \| &\le \frac{M}{(\lambda - \omega_0)^m} \|x\| + \frac{M\gamma}{(\lambda - \omega_0)^{m-1}} \frac{1}{\lambda - \omega_1} \|y\| + \frac{\|f\|}{\lambda^m} + \frac{\|g\|}{\lambda^m} \\ &\le \frac{M'}{(\lambda - \omega_2)^m} (\|x\| + \|y\| + \|f\| + \|g\|), \end{aligned}$$

where $\omega_2 = \max(0, \omega_0, \omega_1)$ and $M' = \max(M, M\gamma)$. Since $B(t)$ is a perturbation of $l(t)$, by a linear operator $\phi(t)$ on E; hence, in view of perturbation result [7, Theorem 5.2.3], $B(t)$ is stable. □

Proposition 5.2. *Let* $(x, 0, f, g) \in D \times \{0\} \times C^1([0,T], X) \times C^1([0,T], Y)$
(1) If the function $t \to u(t) = \big(u_1(t), 0, u_2(t), u_3(t)\big)$ *is a classical solution of* (5.3) *with an initial value* $(x, 0, f, g)$ *then* $t \to u_1(t)$ *is a classical solution of* (5.1) *with the initial value* x.
(2) Let u *is a classical solution of* (5.1) *with the initial value* x. *Then, the function* $t \to U(t) = \big(u(t), 0, f, g\big)$ *is a classical solution of* (5.3) *with the initial value* $(x, 0, f, g)$.

Proof. (1) If $u(t) = \big(u_1(t), 0, u_2(t), u_3(t)\big)$ is a classical solution of (5.3), then u_1 is continuously differentiable on $[s, T]$ and we have

$$\begin{pmatrix} \frac{d}{dt} u_1(t) \\ 0 \\ \frac{d}{dt} u_2(t) \\ \frac{d}{dt} u_3(t) \end{pmatrix} = \begin{pmatrix} A(t) & 0 & \delta_t & 0 \\ -L(t) + \Phi(t) & 0 & 0 & \overline{\delta_t} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1(t) \\ 0 \\ u_2(t) \\ u_3(t) \end{pmatrix}.$$

This implies

$$\begin{gathered} \frac{d}{dt} u_1(t) = A(t) u_1(t) + \delta_t u_2(t), \quad 0 \le s \le t \le T, \\ L(t) u_1(t) = \Phi(t) u_1(t) + \overline{\delta_t} u_3(t), \quad 0 \le s \le t \le T, \\ \frac{d}{dt} u_2(t) = 0, \\ \frac{d}{dt} u_3(t) = 0. \end{gathered}$$

One has $\frac{d}{dt} u_3(t) = 0$ which implies $u_3(t) = u_3(s) = g$ and $L(t) u_1(t) = \Phi(t) u_1(t) + g(t)$. Also $\frac{d}{dt} u_2(t) = 0$ implies $u_2(t) = u_2(s) = f$ and $\frac{d}{dt} u_1(t) = A(t) u_1(t) + f(t)$.

Then

$$\begin{aligned} \frac{d}{dt}u_1(t) &= A(t)u_1(t) + f(t), \quad 0 \le s \le t \le T, \\ L(t)u_1(t) &= \Phi(t)u_1(t) + g(t), \quad 0 \le s \le t \le T, \\ u_1(s) &= x. \end{aligned}$$

Thus u_1 is a classical solution of (5.1) with the initial value x.
(2) Let u is a classical solution of (5.1). This implies that u is continuously differentiable and $u(t) \in D \times \{0\} \times C^1([0,T],X) \times C^1([0,T],Y)$. Moreover,

$$\begin{aligned} \frac{d}{dt}u(t) &= A(t)u(t) + f(t), \quad 0 \le s \le t \le T \\ L(t)u(t) &= \Phi(t)u(t) + g(t), \quad 0 \le s \le t \le T \\ u(s) &= x. \end{aligned}$$

This implies

$$\begin{pmatrix} \frac{d}{dt}u(t) \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A(t) & 0 & \delta_t & 0 \\ -L(t)+\Phi(t) & 0 & 0 & \overline{\delta_t} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u(t) \\ 0 \\ f \\ g \end{pmatrix},$$

with $u(s) = x$. Then $U(t) = (u(t), 0, f, g)$ is continuously differentiable, $U(t) \in D \times \{0\} \times C^1([0,T],X) \times C^1([0,T],Y)$, for all $t \in [s,T]$ and $U(t)$ is a classical solution of (5.1) with the initial value $(x, 0, f, g)$. □

Theorem 5.3. *Let $f \in C^1([0,T],X)$ and $g \in C^1([0,T],Y)$. Assume that the hypothesis (H1)–(H5) hold. Then for every $x \in D$ such that $-L(s)x+\Phi(s)x+g(s) = 0$, problem (5.1) has a unique classical solution.*

Proof. Consider the homogenous Cauchy problem

$$\begin{aligned} \frac{d}{dt}u(t) &= B(t)u(t), \quad 0 \le s \le t \le T, \\ u(s) &= (x, 0, f, g). \end{aligned}$$

By Lemma 5.1, $B(t)$ is a stable family and the function $t \to B(t)y$ is continuously differentiable for all $y \in D_1 = D(B(t))$ independent of t. Then there exist an evolution family $U(t,s)$ on $X \times \{0\} \times C^1([0,T],X) \times C^1([0,T],Y)$ such that

$$U(t,s)\begin{pmatrix} x \\ 0 \\ f \\ g \end{pmatrix} = \begin{pmatrix} u_1(t) \\ 0 \\ u_2(t) \\ u_3(t) \end{pmatrix} = u(t)$$

is a classical solution of (5.3) and from the Proposition 5.2, u_1 is a classical solution of (5.1). The uniqueness of u_1 comes from the uniqueness of the solution of (5.3) and Proposition 5.2. □

Theorem 5.4. *Let $f \in C^1([0,T],X)$ and $g \in C^1([0,T],Y)$. If u is a classical solution of (5.1) then u is given by the variation of constants formula*

$$u(t) = U(t,s)(I-L_{\lambda,s}L(s))x+g(t,u(t))+\int_s^t U(t,r)[\lambda g(r,u(r))-g(r,u(r))'+f(r)]dr, \tag{5.8}$$

where $U(t,s)$ is the evolution family generated by $A_0(t)$ and

$$g(t,u(t)) = L_{\lambda,t}(\Phi(t)u(t) + g(t)).$$

Proof. Let now u be a classical solution of (5.1). Take

$$u_2(t) = L_{\lambda,t}L(t)u(t) \quad \text{and} \quad u_1(t) = (I - L_{\lambda,t}L(t))u(t).$$

Then the functions

$$u_2(t) = g(t,u(t)) = L_{\lambda,t}(\Phi(t)u(t) + g(t)) \quad \text{and} \quad u_1(t)$$

are differentiable. Since $u_2(t) \in \ker(\lambda I - A(t))$, we have $A(t)u_2(t) = \lambda u_2(t)$ and

$$\begin{aligned} \frac{d}{dt}u_1(t) &= \frac{d}{dt}u(t) - \frac{d}{dt}u_2(t) \\ &= A(t)u(t) - (g(t,u(t)))' + f(t) \\ &= A(t)(u_1(t) + u_2(t)) + f(t) - (g(t,u(t)))' \\ &= A(t)u_1(t) + \lambda(g(t,u(t)) + f(t) - (g(t,u(t)))'. \end{aligned}$$

When we define $h(t) := \lambda g(t,u(t)) + f(t) - (g(t,u(t)))'$, we get

$$u_1(t) = U(t,s)u_1(s) + \int_s^t U(t,r)h(r)dr. \tag{5.9}$$

By replacing $u_1(s)$ by $(I - L_{\lambda,s}L(s))x$, we obtain

$$u_1(t) = U(t,s)(I - L_{\lambda,s}L(s))x + \int_s^t U(t,r)h(r)dr, \tag{5.10}$$

it follows that

$$\begin{aligned} u(t) &= u(t,s)(I - L_{\lambda,s}L(s))x + g(t,u(t)) \\ &\quad + \int_s^t u(t,r)[\lambda g(r,u(r)) - (g(r,u(r)))' + f(r)]dr, \end{aligned}$$

which completes the proof. □

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Mohammed Filali
Département de Mathématiques et Informatique, Faculté des Sciences, Université Mohammed 1er, Oujda, Maroc
E-mail address: filali@sciences.univ-oujda.ac.ma

Belhadj Karim
Département de Mathématiques et Informatique, Faculté des Sciences, Université Mohammed 1er, Oujda, Maroc
E-mail address, B. Karim: karim@sciences.univ-oujda.ac.ma