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A GENERALIZATION OF EKELAND'S VARIATIONAL PRINCIPLE WITH APPLICATIONS

ABDEL R. EL AMROUSS, NAJIB TSOULI

Abstract. In this paper, we establish a variant of Ekeland's variational principle. This result suggest to introduce a generalization of the famous Palais-Smale condition. An example is provided showing how it is used to give the existence of minimizer for functions for which the Palais-Smale condition and the one introduced by Cerami are not satisfied.

1. Introduction

Let E be a complete metric space with metric d and $\Phi : E \to \mathbb{R} \cup \{\infty\}$ a lower semicontinuous function which is bounded from below and not identically to $+\infty$. The Ekeland's variational principle, see [1], allows for each $\varepsilon > 0$, each $\delta > 0$ and each $x \in E$ such as

$$\Phi(x) \leq \inf_E \Phi + \varepsilon,$$

to build an element $v \in E$ minimizing the functional Φ_v given by

$$\Phi_v(x) = \Phi(x) + \frac{\varepsilon}{\delta} d(x, v).$$

This principle has wide applications in optimization and nonlinear analysis [1, 2, 4].

If E is a Banach space and $\Phi : E \to \mathbb{R}$ is Gâteaux differentiable, lower semicontinuous and bounded from below, then the Ekeland's variational principle provides the existence of a minimizing sequence (u_n) such as $\Phi'(u_n) \to 0$, when $n \to \infty$. It is well known that if Φ satisfies the Palais-Smale condition then Φ reaches its minimum. But, it is possible to find a minimizing sequence (u_n) such as $\Phi'(u_n) \to 0$, when $n \to \infty$, not having any convergent subsequence. Let us take the example of the function $\Phi(s) = \arctan(s)$.

Ekeland [2] prove that if Φ is bounded below and satisfies the Cerami condition for every $c \in \mathbb{R}$, introduced by [3], then Φ has a minimal point.

In this note, we prove a variant of Ekeland's variational principle. This result suggest to introduce a generalization of the classical Palais-Smale condition. An example is provided showing how it is used to give the existence of minimizer for functions for which the Palais-Smale condition or the Cerami condition are not

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satisfied. We also generalize some results cited in [1], [5], which the Palais-Smale condition or Cerami condition has failed.

2. Variants of Ekeland's variational principle

In this section we will prove the following variant of Ekeland's variational principle. We start with a definition.

Definition 2.1. We say that $\alpha : [0, \infty[\to]0, \infty[$ is a comparison function of order k if for every $q \geq k$ there exist $c, d \geq 0$ such that

$$\frac{\alpha((t+1)s)}{\alpha(t)} \leq cs^q + d, \forall t, s \in \mathbb{R}^+.$$

Examples:

(1) $\alpha(s) = (1+s)^k$
(2) $\alpha(s) = (1+s)^k Log(2+s)$

Let (E, d) be a complet space metric and $u \in E$. Denote by $\bar{B}(u, r) = \{x \in E \mid d(u, x) \leq r\}$ the closed boule and $B(u, r) = \{x \in E \mid d(u, x) < r\}$ the open boule.

Theorem 2.2. *Let (E, d) be a complete space metric, $x_0 \in E$ fixed, $\Phi : E \to \mathbb{R}$ a lower semi-continuous and bounded below. Let $\alpha : [0, \infty[\to]0, \infty[$ be a comparison function of order k continuous nondecreasing. Thus for each $\varepsilon > 0$, each $\delta > 0$ and each $u \in E$ such that*

$$\Phi(u) \leq \inf_E \Phi + \varepsilon$$

there exists a convergent sequence $(z_n)_{n \geq 1}$ of E satisfies:

(i) *$z_1 = u, z_n \in \bar{B}(u, \gamma(u))$ with $\gamma(u)$ be a positive constant such that $u \mapsto \frac{\gamma(u)}{1+d(x_0,u)}$ is bounded in E*
(ii) *The sequence $(d(x_0, z_n))_{n \geq 1}$ is nondecreasing*
(iii) *$\sum_{n=1}^{j} \frac{d(z_n, z_{n+1})}{\alpha(d(x_0, z_{n+1}))} < 2\delta$, for all $j \geq 1$*
(iv) *for $v = \lim_{n\to\infty} z_n, \Phi(v) \leq \Phi(u)$*
(v) *$d(u, v) \leq min\{\delta\alpha(d(x_0, v)), \gamma(u)\}$*
(vi) *for every $w \in \bar{B}(u, \gamma(u)) \setminus B(u, d(x_0, u))$,*

$$\Phi(w) \geq \Phi(v) - \frac{\varepsilon}{\delta\alpha(d(x_0, w))} d(v, w).$$

Proof. Let us define a partial order in E by letting

$$\Phi(r) \leq \Phi(s) - \frac{\varepsilon}{\delta\alpha(d(x_0, r))} d(r, s) \tag{2.1}$$

and

$$d(x_0, r) \geq d(x_0, s). \tag{2.2}$$

This relation is easily seen to be reflexive, antisymmetry and transitive. Indeed, it is clear that $r \prec r$, for every $r \in E$. The partial order $\prec$ is antisymmetry. Indeed, if $r \prec s$ and $s \prec r$ then $d(x_0, r) = d(x_0, s)$,

$$\Phi(r) \leq \Phi(s) - \frac{\varepsilon}{\delta\alpha(d(x_0, r))} d(r, s) \leq \Phi(r) - \frac{2\varepsilon}{\delta\alpha(d(x_0, r))} d(r, s).$$

However $d(r, s) = 0$ and so $r = s$. $\prec$ is transitive, because if $r \prec s$ and $s \prec t$ then $d(x_0, r) \geq d(x_0, s) \geq d(x_0, t)$ and

$$\Phi(r) \leq \Phi(s) - \frac{\varepsilon}{\delta\alpha(d(x_0, r))} d(r, s), \quad \Phi(s) \leq \Phi(t) - \frac{\varepsilon}{\delta\alpha(d(x_0, s))} d(t, s). \tag{2.3}$$

From $d(t,s) \le d(t,r) + d(r,s)$, (2.3) becomes

$$\Phi(r) \le \Phi(s) - \frac{\varepsilon}{\delta\alpha(d(x_0,r))}[d(t,r) - d(t,s)], \quad \Phi(s) \le \Phi(t) - \frac{\varepsilon}{\delta\alpha(d(x_0,s))} d(t,s).$$

This implies

$$\Phi(r) \le \Phi(t) + \Big[\frac{\varepsilon}{\delta\alpha(d(x_0,r))} - \frac{\varepsilon}{\delta\alpha(d(x_0,s))}\Big] d(t,s) - \frac{\varepsilon}{\delta\alpha(d(x_0,r))} d(r,t).$$

Since $\alpha(.)$ is nondecreasing and $d(x_0,r) \ge d(x_0,s)$, we obtain

$$\begin{aligned} r \prec s \text{ and } s \prec t &\Rightarrow \begin{cases} \Phi(r) \le \Phi(t) - \frac{\varepsilon}{\delta\alpha(d(x_0,r))} d(r,t) \\ d(x_0,r) \ge d(x_0,t) \end{cases} \\ &\Rightarrow r \prec t. \end{aligned}$$

Moreover, if we denote $S = \{r \in E \mid r \prec s\}$, by lower semi-continuity of E, S is closed.

Let ε, δ, u and $\gamma(u)$ given by theorem. Now we define a sequence S_n of subsets as follows. Start with $z_1 = u$ and define

$$S_1 = \{w \in E \mid w \prec z_1\} \cap \bar{B}(u, \gamma(u)),$$

and inductively

$$S_n = \{w \in E \mid w \prec z_n\} \cap \bar{B}(u, \gamma(u)), z_{n+1} \in S_n$$

such that

$$\Phi(z_{n+1}) \le \inf_{S_n} \Phi + \frac{1}{(n+1)\alpha(d(x_0,z_n))}. \tag{2.4}$$

Clearly by transitivity of $\prec$ the sequence $(S_n)_n$ is a decreasing sequence of non empty closed sets. Hence also $(d(x_0,z_n))_n$ is a bounded nondecreasing sequence and converges in $[d(x_0,u), d(x_0,u) + \gamma(u)]$.

Now we prove that the diameters of these sets go to zero: $diam S_n \to 0$. Indeed, on one hand $w \in S_{n+1}$ implies

$$\Phi(w) \le \Phi(z_{n+1}) - \frac{\varepsilon}{\delta\alpha(d(x_0,w))} d(w, z_{n+1}) \quad \text{and} \quad d(x_0,w) \ge d(x_0,z_{n+1}).$$

From (2.4), it results

$$\Phi(w) \le \inf_{S_n} \Phi + \frac{1}{(n+1)\alpha(d(x_0,z_n))} - \frac{\varepsilon}{\delta\alpha(d(x_0,w))} d(w, z_{n+1}).$$

This implies

$$d(w, z_{n+1}) \le \frac{\delta}{\varepsilon(n+1)} \frac{\alpha(d(x_0,w))}{\alpha(d(x_0,z_n))}.$$

On the other hand, we have that w belongs to $\bar{B}(u,\gamma(u))$, we obtain

$$d(w, z_{n+1}) \le \frac{\delta}{\varepsilon(n+1)} \frac{\alpha(\gamma(u) + d(x_0,u))}{\alpha(d(x_0,z_n))}. \tag{2.5}$$

Since the function $u \mapsto \frac{\gamma(u)}{1+d(x_0,u)}$ is bounded, then there exists $M > 0$ such that

$$\gamma(u) \le M(1 + d(x_0,u)). \tag{2.6}$$

From (2.5), (2.6) and $\alpha(.)$ is a nondecreasing function, it results

$$d(w, z_{n+1}) \le \frac{\delta}{\varepsilon(n+1)} \frac{\alpha((M+1)(1 + d(x_0,z_n)))}{\alpha(d(x_0,z_n))}. \tag{2.7}$$

By (2.7) and $\alpha(.)$ is a comparison function of order k, there exist $c, d > 0$ such that

$$d(w, z_{n+1}) \le \frac{\delta}{\varepsilon(n+1)}(c(M+1)^k + d), n \in \mathbb{N}$$

which gives diam S_{n+1} go to 0, when $n \to \infty$.

Now we claim that the unique point $v \in E$ in the intersection of the S_n's satisfies conditions (iii)–(vi) of Theorem 2.2. Let then $\cap_n S_n = \{v\}$ and z_n converges to v. Since $z_j \prec z_{j-1} \prec \dots \prec z_1$; and by (2.1), we have

$$\begin{aligned}\Phi(z_{j+1}) &\le \Phi(z_j) - \frac{\varepsilon}{\delta\alpha(d(x_0, z_{j+1}))} d(z_j, z_{j+1}) \\ &\le \Phi(z_1) - \sum_{n=1}^{j} \frac{\varepsilon d(z_j, z_{j+1})}{\delta\alpha(d(x_0, z_{j+1}))}\end{aligned}$$

or

$$\begin{aligned}\sum_{n=1}^{j} \frac{\varepsilon d(z_j, z_{j+1})}{\delta\alpha(d(x_0, z_{j+1}))} &\le \Phi(u) - \Phi(z_{j+1}) \\ &\le \inf_E \Phi + \varepsilon - \Phi(z_{j+1}) \le \varepsilon.\end{aligned}$$

Thus assertion (iii) is shown. Since $v \in S_1$, (iv) is clear. It also results from it that

$$\frac{\varepsilon}{\delta\alpha(d(x_0, v))} d(v, u) \le \Phi(u) - \Phi(v) \le \inf_E \Phi + \varepsilon - \Phi(v) \le \varepsilon.$$

The assertion (v) is shown.

Finally, we prove (vi), let $w \in E$ such that $w \prec v$ and $w \in \bar{B}(u, \gamma(u))$, then we have $w \prec z_n$ for every n. This gives $w \in \cap_n S_n$ and $w = v$, which means that v be an minimal element in $\bar{B}(u, \gamma(u))$, i.e.

$$w \in \bar{B}(u, \gamma(u)) \quad \text{and} \quad w \prec v \Rightarrow w = v.$$

Consequently,

$$\Phi(w) > \Phi(v) - \frac{\varepsilon}{\delta\alpha(d(x_0, w))} d(v, w)$$

for every $w \in \bar{B}(u, \gamma(u)) \setminus B(x_0, d(x_0, v))$. The proof is complete. □

3. Applications

In this section H denotes a Hilbert space, recall that a function $\Phi : H \to \mathbb{R}$ is called Gâteaux differerentiable if at every point x_0, there exists a continuous linear functional $f'(x_0)$ such that, for every $e \in X$,

$$\lim_{t \to 0} \frac{f(x_0 + te) - f(x_0)}{t} = \langle f'(x_0), e \rangle.$$

We always assume that $\alpha : [0, \infty[\to]0, \infty[$ is a continuous nondecreasing comparison function of order k. For the rest of the text we will write

$$\Phi^c = \{u \in H : \Phi(u) \le c\},$$

for the sublevel sets as usual.

Definition 3.1. We say that Φ satisfies (C_c^α) if: Every sequence $(u_n)_n \subset H$ such that $\Phi(u_n) \to c$ and $\Phi'(u_n)\alpha(\|u_n\|) \to 0$ possesses a convergent subsequence.

Remark 3.2. Note that if $\alpha(s) = cte$, the (C_c^α) condition is just the famous Palais-Smale condition and if $\alpha(s) = s + 1$, (C_c^α) is (C) condition introduced by Cerami in [3].

We can now state the following result.

Theorem 3.3. *Let H be a Hilbert space, $\Phi : H \to \mathbb{R}$ lower semi-continuous, bounded below and Gâteaux differentiable. Let $\alpha : [0, \infty[\to]0, \infty[$ a continuous non-decreasing comparison function of order k. Assume that for every $\varepsilon > 0$,*

$$\Phi^{a+\varepsilon} \cap K \neq \emptyset,$$

with K is bounded in H and Φ satisfies (C_a^α), with $a = \inf_H \Phi$, then Φ has a minimal point.

For the proof of this theorem we will use the following lemmas.

Lemma 3.4. *Under the conditions of Theorem 3.3, for every $\varepsilon > 0$, every $u \in H$ such that $\Phi(u) \leq \inf_H \Phi + \varepsilon$ and every $\delta > 0$ such that*

$$\delta \leq \frac{\|u\| + 1}{2\alpha(3(1 + \|u\|))}$$

there exists $v \in H$ that satisfies

(1) $\Phi(v) \leq \Phi(u)$
(2) $\frac{\|v-u\|}{\alpha(\|v\|)} \leq \delta$
(3) *for every $h \in H, t \in \mathbb{R}$ such that $\|h\| = 1, |t| \leq 1$ and $t < v, h > \geq 0$ we have*
$$\Phi(v + th) \geq \Phi(v) - \frac{\varepsilon}{\delta\alpha(\|v + th\|)}|t|.$$

Proof. Let in Theorem 2.2, $x_0 = 0, \gamma(u) = 2(\|u\| + 1)$ and $d(x, y) = \|x - y\|$ for every $x, y \in H$. Then, by $iv)$ and $v)$ of Theorem 2.2, there exists $v \in H$ ($v = \lim_{n\to\infty} z_n, (z_n)$ the sequence built in theorem 2.2) such that

$$\Phi(v) \leq \Phi(u) \text{ and } \|v - u\| \leq \delta\alpha(\|v\|). \tag{3.1}$$

Thus assertions 1. and 2. follow.

Now we prove the assertion 3. Let $h \in H$ such that $\|h\| = 1$ and $|t| \leq 1$ we have $v \in \bar{B}(u, \|u\| + 1)$. Indeed, if not $\|u\| + 1 < \|v - u\|$. Since $\alpha(.)$ is nondecreasing, $\delta \leq \frac{\|u\|+1}{2\alpha(3(1+\|u\|))}$ and by (3.1), it results

$$\|u\| + 1 < \|v - u\| \leq \delta\alpha(\|v\|) \leq \delta\alpha(3(1 + \|u\|))) \leq \frac{\|u\| + 1}{2}.$$

This is a contradiction. Furthermore, we have

$$\begin{aligned}\|v + th\| &\leq \|v\| + |t|\|h\| = \|v\| + |t| \\ &\leq 2\|u\| + 1 + 1 = \gamma(u).\end{aligned}$$

On the other hand, it is clear , since $t\langle v, h\rangle \geq 0$, that

$$\|v + th\| = [\|v\|^2 + \|th\|^2 + 2t < v, h >]^{1/2} \geq \|v\|.$$

Thus, by (iv), (v), (vi) of Theorem 2.2, assertions 1, 2, 3 of the lemma follow. □

Lemma 3.5. *Under the conditions of Theorem 3.3, we have*

$$|\langle\Phi'(v), h\rangle| \leq \frac{\varepsilon}{\delta\alpha(\|v\|)}, \quad \forall h \in H, \|h\| = 1. \tag{3.2}$$

Proof. Let $h \in H$ such that $\|h\| = 1$ and consider two cases:
Case 1. If $\langle v, h\rangle \geq 0$ and $t > 0$, from 3. of Lemma 3.4 and Φ being Gâteaux differentiable, letting t approach 0, we obtain

$$\langle \Phi'(v), h\rangle \geq -\frac{\varepsilon}{\delta\alpha(\|v\|)}.$$

Case 2. In the similar way, if $\langle v, h\rangle \leq 0$ and $t < 0$ goes to 0, we have

$$\langle \Phi'(v), h\rangle \leq \frac{\varepsilon}{\delta\alpha(\|v\|)}, \quad \forall h, \|h\| = 1.$$

Thus the Lemma 3.5 follows. □

Proof of Theorem 3.3. For $\varepsilon = \frac{1}{n}$, with $n \geq 1$, there exists a sequence $(u_n) \subset K$ such that

$$\Phi(u_n) \leq a + \frac{1}{n}$$

and, since (u_n) is bounded, there exists $\delta > 0$ such that

$$\delta \leq \frac{\|u_n\| + 1}{\alpha(3(1 + \|u_n\|))}, \forall n \geq 1.$$

Consequently, by Lemma 3.4 and Lemma 3.5, there exists a sequence (v_n) satisfying

(i) $\Phi(v_n) \leq \Phi(u_n)$
(ii) $\|\Phi'(v_n)\|\alpha(\|v_n\|) \to 0$, as $n \to \infty$.

The (C_a^α) condition implies that (v_n) has a subsequence (v_{n_k}) convergent to some point u. Since Φ is lower semi-continuous, we get

$$\inf_H \Phi \leq \Phi(u) \leq \liminf_{k\to\infty} \Phi(v_{n_k}) \leq \inf_H \Phi.$$

Therefore, $\Phi(u) = \inf_H \Phi$. □

Now, we illustrate Theorem 3.3 by an example where the function Φ checks the conditions of Theorem 3.3, but the Palais-Smale condition and Cerami condition do not hold.

Example. Consider

$$f(s) = \begin{cases} \arctan(s) & \text{if } s \leq 0 \\ \sin(s) & \text{if } 0 \leq s \leq 2\pi \\ \arctan(s - 2\pi) & \text{if } s \geq 2\pi. \end{cases}$$

and $\Phi(u) = f(2\pi + Log(\|u\|^2 + 1) - (\|u\|^2 + 1)^{\frac{1}{2}})$ for $u \in H$. It is clear that Φ is C^1 functional and $a = \inf_H \Phi = -1$. Take

$$K = \{u \in H : \log(\|u\|^2 + 1) - (\|u\|^2 + 1)^{\frac{1}{2}} \in [-2\pi, 0]\},$$

it is easy to see that $\Phi^{-1+\varepsilon} \cap K \neq \emptyset$ for every $\varepsilon > 0$. On the other hand, Φ satisfies (C_c^α), with $\alpha(s) = s^2 + 1$, and by Theorem 3.3, Φ has a minimal point u_0 which $\Phi'(u_0) = 0$.

Theorem 3.6. *Let $\Phi : H \to \mathbb{R}$ be Gâteaux differentiable and bounded below, says $a \inf_H \Phi$. Assume that $\alpha : [0, \infty[\to]0, \infty[$ be a continuous nondecreasing function such that $\int_1^\infty \frac{1}{\alpha(s)}\, ds = +\infty$. If Φ satisfies (C_a^α) then the set $\Phi^{a+\beta}$ is bounded, for some $\beta > 0$.*

The main point to prove Theorem 3.6 is the following.

Lemma 3.7. *Under the conditions of Theorem 3.6, for every $\varepsilon > 0$, every $u \in H$ such that $\Phi(u) \le \inf_H \Phi + \varepsilon$ and every $\delta > 0$ there exists $v \in H$ satisfies*

(1) $\Phi(v) \le \Phi(u)$
(2) $\frac{\|v-u\|}{\alpha(\|v\|)} \le \delta$
(3) *for every $h \in H$ such that $\|h\| = 1$, we have*
$$|\langle \Phi'(v), h\rangle| \le \frac{\varepsilon}{\delta\alpha(\|v\|)}.$$

Proof. Let in Theorem 2.2, $x_0 = 0$ and $d(x,y) = \|x-y\|$ for every $x, y \in H$. From theorem 2.2 there exists a sequence $(z_n)_{n\ge1}$ satisfying $(\|z_n\|)$ is nondecreasing and
$$\sum_{n=1}^{j} \frac{\|z_n - z_{n+1}\|}{\alpha(\|z_{n+1}\|)} < 2\delta, \quad \forall j \ge 1. \tag{3.3}$$
However, since $\int_1^\infty \frac{1}{\alpha(s)}\, ds = +\infty$ there exists $\gamma > 0$ such that
$$\delta \le \frac{1}{2}\int_{\|u\|}^{\|u\|+\gamma} \frac{1}{\alpha(s)}\, ds. \tag{3.4}$$
Put $v = \lim_{n\to\infty} z_n$ and $\gamma(u) = 2\|u\| + \gamma + 1$ in Theorem 2.2. Thus, by (iv)-(v) of Theorem 2.2, we obtain
$$\Phi(v) \le \Phi(u) \quad \text{and} \quad \|v-u\| \le \delta\alpha(\|v\|).$$
For the proof of assertion 3, it is enough to verify that $h \in H$ such that $\|h\| = 1$ we have $v + th \in \bar{B}(u, \gamma(u))$ for every t sufficiently small. Now we prove that
$$\|z_n\| \le \|u\| + \gamma, \quad \forall n \ge 1. \tag{3.5}$$
If not, there exists $j \ge 1$ such that $\|z_{j+1}\| > \|u\| + \gamma$. However, by (3.4) and α is nondecreasing, we obtain
$$\begin{aligned} 2\delta &\le \int_{\|z_1\|}^{\|z_{j+1}\|} \frac{1}{\alpha(s)}\, ds \\ &\le \sum_{n=1}^{j} \int_{\|z_n\|}^{\|z_{n+1}\|} \frac{1}{\alpha(s)}\, ds \\ &\le \sum_{n=1}^{j} \frac{\|z_{n+1}\| - \|z_n\|}{\alpha(\|z_{n+1}\|)} \\ &\le \sum_{n=1}^{j} \frac{\|z_n - z_{n+1}\|}{\alpha(\|z_{n+1}\|)}. \end{aligned}$$
This contradicts (3.3). Using (3.5), we have
$$\|v-u\| \le 2\|u\| + \gamma. \tag{3.6}$$
Thus, for $|t| \le 1$ and $h \in H$ such that $\|h\| = 1$ and by (3.6), it results
$$\|v + th - u\| \le 2\|u\| + \gamma + 1 = \gamma(u).$$
Finally, the Lemma 3.5 allows to conclude. The proof is complete. $\square$

Proof of theorem 3.6. Suppose, by contradiction, that $\Phi^{a+\beta}$ is unbounded for all $\beta > 0$. Then, there exists (u_n) such that $\|u_n\| \geq n$ and

$$\Phi(u_n) \leq a + \frac{1}{n}.$$

and Lemma 3.7 with $\varepsilon = (\frac{1}{n})^2, \delta = \frac{1}{n}$ implies the existence of (v_n) satisfying

(i) $\Phi(v_n) \leq \Phi(u_n)$
(ii) $\|v_n - u_n\| \leq \frac{1}{n}\alpha(\|v_n\|)$
(iii) $\|\Phi'(v_n)\|\alpha(\|v_n\|) \to 0$, as $n \to \infty$.

We reach a contradiction with (C_a^α), since (i)-(iii) give respectively

(1) $\Phi(v_n) \to a$, as $n \to \infty$,
(2) $\|v_n\| \to \infty$, as $n \to \infty$,
(3) $\|\Phi'(v_n)\|\alpha(\|v_n\|) \to 0$, as $n \to \infty$.

□

As an immediate consequence of the above results we have the following result.

Corollary 3.8. *Let H be a Hilbert space, $\Phi : H \to \mathbb{R}$ lower semi-continuous, bounded below and Gâteaux differentiable. Assume that $\alpha : [0, \infty[\to]0, \infty[$ be a continuous nondecreasing function such that $\int_1^\infty \frac{1}{\alpha(s)}\, ds = +\infty$. If Φ satisfies (C_a^α), with $a = \inf_H \Phi$, then Φ has a minimal point.*

References

[1] Ekeland I., *On the variational principle*, J. Math. Anal. Applic., 47 (1974) 324-357.
[2] Ekeland I., *Convexity methods in Hamiltonian mechanics*, Springer, Berlin, (1990).
[3] Cerami G., *Un criterio de esistenza per i punti critici su varietá ilimitate*, Rc. Ist. Lomb. Sci. Lett. 121,(1978) 332-336.
[4] Clark, F. H. *Optimisation and Nonsmooth Analysis*, Wiley, NewYork1983.
[5] Costa, D. G., Elves De B. eSilva; *The Palais-Smale condition versus coercivity*, Nonlinear Analysis, T. M. A. 16 (1991) 371-381.

Abdel R. El Amrouss
University Mohamed 1er, Faculty of sciences, Department of Mathematics, Oujda, Morocco
E-mail address: amrouss@sciences.univ-oujda.ac.ma

Najib Tsouli
University Mohamed 1er, Faculty of sciences, Department of Mathematics, Oujda, Morocco
E-mail address: tsouli@sciences.univ-oujda.ac.ma