2005-Oujda International Conference on Nonlinear Analysis.
Electronic Journal of Differential Equations, Conference 14, 2006, pp. 35–51.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

MODELLING OF A COLLAGE PROBLEM

ABDELAZIZ AÏT MOUSSA, LOUBNA ZLAÏJI

ABSTRACT. In this paper we study the behavior of elastic adherents connected with an adhesive. We use the Γ-convergence method to approximate the problem modelling the assemblage with density energies assumed to be quasiconvex. In particular for the adhesive problem, we assume periodic density energy and some growth conditions with respect to the spherical and deviational components of the gradient. We obtain a problem depending on small parameters linked to the thickness and the stiffness of the adhesive.

1. Introduction

The problem under investigation arises in the study of adhesive bonding of elastic bodies, and the question is how to model the behavior of the adhesive material interposed between the adherents. Such problems find their applications for example in aeronautics, in the study of composites, and in other fields of engineering. In general, the computation of the solution using numerical methods is very difficult. In one hand, this is because the thickness of the adhesive requires a fine mesh, which in turn implies an increase of the degrees of freedom of the system, and in the other, the adhesive is usually more flexible than the adherents, and this produces numerical instabilities in the stiffness matrix. To overcome this difficulties, thanks to Goland and Reissner [12], it is usual to find a limit problem in which the adhesive is treated as a material surface; it disappears from a geometrical point of view, but it is represented by the energy of adhesion. In this framework, we find many works investigated on this theoretical approach; see for example Moussa [19], Suquet [20], Ganghoffer, Brillard and Schultz [10], Geymonat, Krasucki and Lenci [11], Licht and MiChaille [15], Brezis, Caffarelli and Friedman [4], Acerbi, Buttazo and Percivale [1], Klarbring [14], Caillerie [5].

This work is specially interested in approximating a minimization problem $(\mathcal{P}_r)$, where r is a small parameter linked to the thickness and the stiffness of the adhesive. In particular, we associate to each component of gradient (spherical or deviational) an independent stiffness parameter. We use the method described in [15] to find a certain limit problem denoted $(\mathcal{P})$. Precisely, by the Γ-convergence method (introduced in a paper by De Giorgi and Franzoni in 1975 [9]), we look for

2000 *Mathematics Subject Classification.* 35K22, 58D25, 73D30.
Key words and phrases. Adhesive; spherical; deviational; Γ-convergence; homogenization; quasiconvexity; subadditivity.

Published September 20, 2006.

a weak limit of a $(\mathcal{P}_r)$-minimizing sequence which is a solution of $(\mathcal{P})$. The outline of the paper is the following.

Section 2 contains some notation and a brief summary of results related to notions of Γ-convergence, quasiconvexity and subadditivity . Section 3 is devoted to Problem statement, hypothesis which we assume on his different components and existence of solutions. In section 4, we discuss topology that we shall consider for limit problem study, we compute Γ-limit of the stored strain energies represented by the functionals $(F_r)_r$ and we deduce the limit problem.

2. Notation and preliminaries

We begin by introducing some notation which is used throughout the paper. First, let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two open subsets of $\mathbb{R}^N$ with interface S. For a function v defined on $\mathcal{O}_1 \cup \mathcal{O}_2$, we call the jump of v across S the function defined on S by $[v]_S = v_{/\mathcal{O}_1} - v_{/\mathcal{O}_2}$. Let M^N be the space of $N \times N$ real matrices endowed with the Hilbert-Schmidt scalar product $A : A' = \text{trace}(A^t A')$. For a given $A \in M^N$, we call spherical part of A the matrix $A_s = \frac{\text{trace}\, A}{N} I_N$, where I_N is the unit matrix of $\mathbb{R}^N$. The deviational part will be $A_d = A - A_s$. In mechanics, the spherical part of the deformation tensor changes the volume without changing the shape whereas the deviational tensor changes the shape preserving the same volume (the trace is void, therefore there is no relative variation of volume). On the space M^N, operators $A \mapsto A_s$ and $A \mapsto A_d$ are linear continuous for matrix norm $|A| = \Sigma_{1 \le i,j \le N} |A_{i\,j}|$, where $A = (A_{i\,j})_{1 \le i,j \le N}$.

Definition. A a Carathéodory function $f : \mathbb{R}^N \times M^N \to \mathbb{R}$ satisfies condition (C_p) if there exists $\alpha_p, \beta_p, c \in \mathbb{R}^N$, such that for $x \in \mathbb{R}^N$ and all $(A, A') \in (M^N)^2$, we have

$$\begin{gathered} \alpha_p |A|^p \le f(x, A) \le \beta_p (1 + |A|^p) \\ |f(x, A) - f(x, A')| \le c|A - A'|(1 + |A|^{p-1} + |A'|^{p-1}). \end{gathered} \tag{2.1}$$

As we have already mentioned, our method will be based on the notion of Γ-convergence. Let (X, τ) be a metrisable topological space, and for every $n \in \mathbb{N}$ let $F_n, F : X \to \overline{\mathbb{R}}$ be functions defined on X. For every $x \in X$, the $\Gamma(\tau)$-liminf F_n (respectively, $\Gamma(\tau)$-limsup F_n) are defined as:

$$\Gamma(\tau) - \liminf F_n(x) = \inf\{\liminf F_n(x_n) : x_n \xrightarrow{\tau} x\}$$
$$\Gamma(\tau) - \liminf F_n(x) = \inf\{\limsup F_n(x_n) : x_n \xrightarrow{\tau} x\}$$

If the two expressions are equal to $F(x)$, then we say that the sequence (F_n) $\Gamma(\tau)$-converges to F on X and we write $F = \Gamma(\tau)\text{-lim} F_n$. An other way to define F=Γ-limF_n is the following:
$(\forall x \in X)(\exists x_{0,n} \in X)$ such that $x_{0,n} \xrightarrow{\tau} x$ and $\limsup_{n\to+\infty} F_n(x_{0,n}) \le F(x)$
$(\forall x \in X)(\forall x_n \in X)$ such that $x_n \xrightarrow{\tau} x$, $\liminf_{n\to+\infty} F_n(x_n) \ge F(x)$

The Γ-convergence method is made precise in item (1) below.

Proposition 2.1. *Suppose that $(F_n)_n$ Γ-converges to F.*
(1) [2, Theorem 2.11]. *Let $x_n \in X$ be such that $F_n(x_n) \le \inf\{F_n(x) : x \in X\} + \varepsilon_n$, where $\varepsilon_n > 0$, $\varepsilon_n \to 0$. We assume furthermore that $\{x_n, n \in \mathbb{N}\}$ is τ-relatively compact, then any cluster point $\overline{x}$ of $\{x_n, n \in \mathbb{N}\}$ is a minimizer of F and*

$$\liminf_{n\to+\infty} \{F_n(x) : x \in X\} = F(\overline{x}).$$

(2) [2, Theorem 2.15]. *If $L : X \to \mathbb{R}$ is continuous, then $(F_n + L)_n$ Γ-converges to F+L.*

For details about Γ-convergence, we refer the reader to [2, 7]. To establish existence of solutions for our initial problem, it will be useful to consider quasiconvex energy densities. So if f is a Borel measurable and locally integrable function defined on M^N, we say that f is quasiconvex if

$$f(A) \leq \frac{1}{\operatorname{meas} D} \int_D f(A + \nabla\varphi) dx$$

where D is a bounded domain of $\mathbb{R}^N$, $A \in M^N$ and $\varphi \in W_0^{1,\infty}(D, \mathbb{R}^N)$. If f is not quasiconvex, his quasiconvex envelope is given as

$$Qf = \sup\{g \leq f : \text{ g is quasiconvex }\}$$

If f is locally bounded, then the definition of Qf can be expressed as [6, Page 201]

$$Qf(A) = inf\{\frac{1}{\text{measD}} \int_D f(A + \nabla\varphi)dx : \varphi \in W_0^{1,\infty}(D, \mathbb{R}^N)\}$$

The following proposition establish sufficiency of quasiconvexity to obtain weak lower semicontinuity in $W^{1,p}$

Proposition 2.2. *Let $\mathcal{O}$ be an open bounded subset of $\mathbb{R}^N$ and $f : \mathcal{O} \times M^N \to \mathbb{R}$ a continuous quasiconvex function satisfying condition* (2.1)*, for $p \geq 1$. Then, the functional $F : u \to \int_{\mathcal{O}} f(x, \nabla u(x)))\,dx$ is weakly lower semicontinuous on $W^{1,p}(\mathcal{O}, \mathbb{R}^N)$.*

For the proof of the above proposition, see [6, Theorem 2.4 and Remark iv].

To describe a global subadditive theorem, we consider $\mathcal{B}_b(\mathbb{R}^d)$ the family of Borel bounded subsets of $\mathbb{R}^d$ and δ Euclidean distance in $\mathbb{R}^d$. for every $A \in \mathcal{B}_b(\mathbb{R}^d)$, $\rho(A) = \sup\{r \geq 0 : \exists \overline{B}_r(x) \subset A\}$, where $\overline{B}_r(x) = \{y \in \mathbb{R}^d : \delta(x, y) \leq r\}$. A sequence $(B_n)_{n\in\mathbb{N}} \subset \mathcal{B}_b(\mathbb{R}^d)$ is called regular if there exist an increasing sequence of intervals $(I_n)_n \subset \mathbb{Z}^d$ and a constant C independent of n such that $B_n \subset I_n$ and $\operatorname{meas}(I_n) \leq C \operatorname{meas}(B_n)$, $\forall n$. The global subadditive theorem is essentially based on subadditive $\mathbb{Z}^d$-periodic functions . A function $S : A \in \mathcal{B}_b(\mathbb{R}^d) \to S_A \in \mathbb{R}$ is called subadditive $\mathbb{Z}^d$-periodic if it satisfy the following conditions:

(i) For all $A, B \in \mathcal{B}_b(\mathbb{R}^d)$ such that $A \cap B = \emptyset$, $S_{A\cup B} \leq S_A + S_B$.
(ii) For all $A \in \mathcal{B}_b(\mathbb{R}^d)$, all $z \in \mathbb{Z}^d$, $S_{A+z} = S_A$.

Now, we shall see the global subadditive theorem, firstly used in the setting of the calculus of variation by Dal Maso and Modica [8], and generalized to sequences indexed by convex sets by Licht and Michaille [15]

Theorem 2.3. *Let S be a subadditive $\mathbb{Z}^d$-periodic function such that*

$$\gamma(S) = \inf\{\frac{S_I}{\operatorname{meas} I} : I = [a, b[, a, b \in \mathbb{Z}^d \text{ and } a_i < b_i \ \forall 1 \leq i \leq d\} > -\infty$$

In addition, we suppose that S satisfies the dominant property: There exists $C(S)$, for every Borel convex subset $A \subset [0, 1[^d$, $|S_A| \leq C(S)$. Let $(A_n)_n$ be a regular sequence of Borel convex subsets of $\mathcal{B}_b(\mathbb{R}^d)$ with $\lim_{n\to+\infty} \rho(A_n) = +\infty$. Then $\lim_{n\to+\infty} \frac{S_{A_n}}{\operatorname{meas} A_n}$ exists and is equal to

$$\lim_{n\to+\infty} \frac{S_{A_n}}{\operatorname{meas} A_n} = \inf_{m\in\mathbb{N}^*} \{\frac{S_{[0,m[^d}}{m^d}\} = \gamma(S)$$

For the proof of the above theorem see [16, page 24].

3. Statement of the problem

Let $\mathcal{O}$ be a domain of $\mathbb{R}^N$ with Lipschitz boundary, divided in two parts $\mathcal{O}^{\pm}$ by the plane $\{x_N = 0\}$. The common interface is noted S. The structure under study contains two adherents filling $\mathcal{O}_\varepsilon = \mathcal{O}_\varepsilon^+ \cup \mathcal{O}_\varepsilon^-$, glued perfectly with an adhesive occupying $B_\varepsilon = \{x = (\widetilde{x}, x_N) \in \mathcal{O} : \pm x_N \le \varepsilon\, \gamma^{\pm}(\frac{\widetilde{x}}{\varepsilon})\} = \mathcal{O} \setminus \overline{\mathcal{O}}_\varepsilon$, along the common surfaces $S_\varepsilon^{\pm} = \{x \in \mathcal{O} : \pm x_N = \varepsilon \gamma^{\pm}(\frac{\widetilde{x}}{\varepsilon})\}$, where ε is a small parameter intended to tend toward 0, $\gamma^{\pm} : \mathbb{R}^{N-1} \longrightarrow \mathbb{R}^+$ are two $\mathcal{C}^1$ $\widetilde{Y}$-periodic functions, $\widetilde{Y} =]0, 1[^{N-1}$. The maximum (respectively Minimum) of $\gamma^{\pm}$ on $\widetilde{Y}$ is noted $\gamma_M^{\pm}$(respectively $\gamma_m^{\pm}$). Surface forces are applied on a portion Γ_1 of $\partial\mathcal{O}$ with surface measure supposed to be positive, and the structure is clamped on his complementary Γ_0. The illustration of the domain is shown in Figure 1.

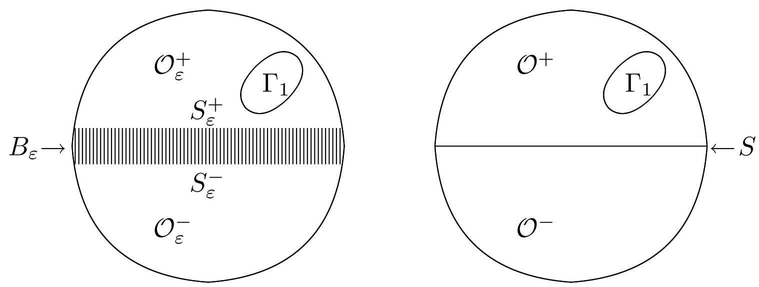

Figure 1. Initial problem (left). Limit problem (right)

Our study is focused on the minimization problem $(\mathcal{P}_r)$: Find $u \in V_\varepsilon$ such that:

$$I_r(u) = \min_{v \in V_\varepsilon} I_r(v) = \min_{v \in V_\varepsilon} F_r(v) - L(v) \tag{3.1}$$

where

- $r = (\varepsilon, \mu, \eta)$, the three parameters are positive intended to tend to 0. The first concern the thickness of adhesive and the others the stiffness connected respectively to spherical and deviational components of ∇.
- $V_\varepsilon = \{v \in W^{1,q}(\mathcal{O}_\varepsilon) \times W^{1,p}(B_\varepsilon) : \nabla_s v \in L^{p_s}(B_\varepsilon, M^N)$ and $\nabla_d v \in L^{p_d}(B_\varepsilon, M^N), [v]_{S_\varepsilon^{\pm}} = 0, v = 0$ sur $\Gamma_0\}$, ∇_s and ∇_d are respectively spherical and deviational components of ∇. p_s, p_d and q are constants with $1 < p_s, p_d \le q$ and $p = \min(p_s, p_d)$.
- For $v \in V_\varepsilon$,

$$F_r(v) = \int_{\mathcal{O}_\varepsilon} h(x, \nabla v)\, dx + \int_{B_\varepsilon} \mu\, b_s(\frac{x}{\varepsilon}, \nabla_s v) + \eta\, b_d(\frac{x}{\varepsilon}, \nabla_d v)\, dx$$
$$L(v) = \int_{\mathcal{O}} f(x)\, v(x)\, dx + \int_{\Gamma_1} g(x)\, v(x)\, d\sigma(x)$$

In the following, we denote $w = s$ or $w = d$, and we make the hypotheses:

(H1) b_w and h are Carathéodory functions defined on $\mathbb{R}^N \times M^N$. In particular, b_w is $\widetilde{Y}$-periodic with respect to first variable and satisfies the condition (C_w): There exists $\alpha_w, \beta_w, c_w \in \mathbb{R}_*^+$ such that for all $x \in \mathbb{R}^N$ and all

$(Q,Q') \in (M^N)^2$ we have

$$\begin{aligned} \alpha_w|Q_w|^{p_w} \le\ & b_w(x,Q_w) \le \beta_w(1+|Q_w|^{p_w}) \\ |b_w(x,Q_w) - b_w(x,Q'_w)| \le\ & c_w|Q_w - Q'_w|(1+|Q_w|^{p_w-1} + |Q'_w|^{p_w-1}). \end{aligned} \tag{3.2}$$

The function h satisfies the conditions C_q (2.1).

(H2) There exist a function $b^{\infty,w}$ such that $Q \mapsto b^{\infty,w}(x,Q)$ is positively p_w-homogeneous, a positive constant c'_w and $0 < m_w < p_w$ so that for all $(x,Q) \in \mathbb{R}^N \times M^N 4$,

$$|b^{\infty,w}(x,Q_w) - b_w(x,Q_w)| \le c'_w(1+|Q_w|^{p_w-m_w})$$

(H3) $(f,g) \in L^{q'}(\mathcal{O},\mathbb{R}^N) \times L^{q'}(\Gamma_1,\mathbb{R}^N)$, where q' is the conjugate exponent of q, and there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \le \varepsilon_0$: $(\operatorname{supp} f \cup \Gamma_1) \cap B_\varepsilon = \emptyset$.

To lighten notation, we shall often use const to designate different constants (independent of r) in a same proof.

Remark 3.1. *(1) If we consider the following norm on V_ε,*

$$\|v\|_{V_\varepsilon} = \|v\|_{W^{1,q}(\mathcal{O}_\varepsilon,\mathbb{R}^N)} + \|\nabla_s v\|_{L^{p_s}(B_\varepsilon,M^N)} + \|\nabla_d v\|_{L^{p_d}(B_\varepsilon,M^N)}.$$

Then V_ε will be a reflexive Banach space (because $1 < p_s, p_d$), and L is a linear continuous mapping on $(V_\varepsilon, \|\cdot\|_{V_\varepsilon})$. (2) Hypothesis (H2) implies that for all $(x,Q) \in \mathbb{R}^N \times M^N$, $\lim_{t\to+\infty} \frac{b_w(x,tQ_w)}{t^{p_w}} = b^{\infty,w}(x,Q_w)$.

Proposition 3.2. *Let b_w and h be quasiconvex, and in particular b_w is continuous on $\mathbb{R}^N \times M^N$. Then, under (H1) and (H3), problem* (3.1) *admits at least one solution.*

Proof. $(V_\varepsilon, \|\cdot\|_{V_\varepsilon})$ is a reflexive Banach space (Remark 3.1), then using the well known theorem [13, Page 135], it suffices to establish that I_r is weakly lower semicontinuous and coercive on $(V_\varepsilon, \|\cdot\|_{V_\varepsilon})$. So let $v \in V_\varepsilon$, in one hand we are $V_\varepsilon \hookrightarrow W^{1,q}(\mathcal{O}_\varepsilon)$ which implies by Proposition 2.2 that functional $v \in V_\varepsilon \mapsto \int_{\mathcal{O}_\varepsilon} h(x,\nabla v)\,dx$ is weakly lower semicontinuous on V_ε. It is the same for functional $v \mapsto \int_{B_\varepsilon} b_w(x,\nabla_w v)\,dx$. Indeed, using embedding $V_\varepsilon \hookrightarrow L^{p_w}(B_\varepsilon)$, the fact that operators $Q \in M^N \mapsto Q_w$ are linear continuous (section 2), then it suffices to adapt proof of [6, Theorem 2.4] by replacing ∇ with ∇_w. We Conclude using linearity and continuity of L on V_ε. For coercivity, it's easily seen according to (H1) and (H3) that on V_ε, $\lim_{\|v\|_{V_\varepsilon}\to+\infty} I_r(v) = +\infty$. □

Remark 3.3. In general, if a function $f : \mathbb{R}^N \times M^N \mapsto \overline{\mathbb{R}}$ satisfying condition (2.1) is not quasiconvex, then for an open bounded subset $\mathcal{O}$ of $\mathbb{R}^N$, $\inf F(u)_{W^{1,p}(\mathcal{O})} = \inf \int_\Omega f(x,\nabla u)dx$ can be not existent. In return, if we take his quasiconvex envelope $\mathcal{Q}f$ we can study existence of solutions of the problem $\inf \mathcal{Q}F = \inf \int_\Omega \mathcal{Q}f(x,\nabla .)dx$ noticing that $inf F = inf \mathcal{Q}F$ (in the sense described in [6, Corollary 2.3], and that $\mathcal{Q}F$ is weakly lower semicontinuous on $W^{1,p}(\mathcal{O})$.

4. Limit Problem

In order to determine the limit problem, we first identify the topological space that we shall consider in the following. In one hand, the space must be big enough not depending on the parameter r to include the spaces V_ε defined in section 3. In the other, topology must provide the relative compactness of a (3.1)-minimizers sequence. Let $X = W^{1,q}_{\rm loc}(\mathcal{O}\setminus\mathcal{S})$ and τ his weak topology. Let us consider the subset

$$V = \{v \in X : v \in W^{1,q}(\mathcal{O}\setminus S), v = 0 \text{ on } \Gamma_0\}$$

Proposition 4.1. *If $(v_r)_r$ is a sequence in X verifying $F_r(v_r) \le C$, then there exist $v \in V$ and a subsequence such that $v_r \xrightarrow{\tau} v$ in X. Moreover,*

1. $\mathcal{X}_{\mathcal{O}_\varepsilon}\nabla v_r \rightharpoonup \nabla v$ *in* $L^q(\mathcal{O})$.
2. $\int_{\mathbb{R}^{N-1}} |v_r(\widetilde{x}, \pm\varepsilon\gamma(\frac{\widetilde{x}}{\varepsilon})) - v^\pm(\widetilde{x})|^q d\widetilde{x} \xrightarrow{r} 0$.

For the proof of the above proposition see [15, page 9].

Remark 4.2. (1) Let $(\overline{u}_r)_r$ be a (3.1)-minimizer sequence, i.e.

$$\lim_{r\to 0} I_r(\overline{u}_r) - \inf\{I_r(v) : v \in V_\varepsilon\} = 0,$$

then $(\overline{u}_r)_r$ is relatively compact in (X,τ). It suffices to show $\liminf_{r\to 0} I_r(\overline{u}_r) < +\infty$, which implies according to conditions (3.2) and (3.2) with w replaced by q that $\liminf_{r\to 0} F_r(\overline{u}_r) < +\infty$, and we apply Proposition 4.1.

(2) Let $p = \min(p_s, p_d)$. In accordance with results of [15], we obtain (X,τ) so that:
- If $\limsup_{(\varepsilon,\mu)} \frac{\varepsilon}{\mu^{1/p_s}}$ and $\limsup_{(\varepsilon,\eta)} \frac{\varepsilon}{\eta^{1/p_d}} < +\infty$, then $X = L^\alpha(\mathcal{O})$ and τ is his strong topology for any $\alpha \in [1, p[$.
- If $\limsup_{(\varepsilon,\mu)} \frac{\varepsilon}{\mu^{1/p_s}} = \limsup_{(\varepsilon,\eta)} \frac{\varepsilon}{\eta^{1/p_d}} = 0$, then $X = L^p(\mathcal{O})$ and τ is his strong topology.

Now, we look for the Γ-limit of functionals I_r. First we have to remark that The functional L is linear continuous on (X,τ) (for the proof, is a straightforward consequence of (H3) and the compact embedding $W^{1,q}_{\rm loc}(\mathcal{O}\setminus S, \mathbb{R}^N) \hookrightarrow L^q(S)$), then according to Proposition 2.1, it suffices to study Γ-limit for functionals F_r. To this end, we extend F_r on the space (X,τ) as

$$F_r(v) = \begin{cases} \int_{\mathcal{O}_\varepsilon} h(x,\nabla v)\,dx + \int_{B_\varepsilon} \mu b_s(\frac{x}{\varepsilon}, \nabla_s v) + \eta\, b_d(\frac{x}{\varepsilon}, \nabla_d v)\,dx & \text{if } v \in V_\varepsilon \\ +\infty & \text{if } v \notin V_\varepsilon \end{cases}$$

we recall that $V_\varepsilon = \{v \in W^{1,q}(\mathcal{O}_\varepsilon) \times W^{1,p}(B_\varepsilon) : \nabla_s v \in L^{p_s}(B_\varepsilon, M^N)$ and $\nabla_d v \in L^{p_d}(B_\varepsilon, M^N), [v]_{S^\pm_\varepsilon} = 0, v = 0$ on $\Gamma_0\}$. Let

$$l_s = \lim_{(\varepsilon,\mu)} \frac{\mu}{2(2\varepsilon)^{p_s-1}} \quad \text{and} \quad l_d = \lim_{(\varepsilon,\eta)} \frac{\eta}{2(2\varepsilon)^{p_d-1}}.$$

We define functional F on X as follows:
(i) If l_s, $l_d \in [0,+\infty[$:

$$F(v) = \begin{cases} \int_{\mathcal{O}} Qh(x,\nabla v)\,dx + \int_S \{l_s\,(b^{\infty,s})^{\rm hom} + l_d\,(b^{\infty,d})^{\rm hom}\}[v] d\widetilde{x} & \text{if } v \in V \\ +\infty & \text{if } v \notin V \end{cases}$$

we recall that $V = \{v \in X : v \in W^{1,q}(\mathcal{O}\setminus S, \mathbb{R}^N)$ and $v = 0$ on $\Gamma_0\}$.
(ii) If $l_s = +\infty$ and $l_d < +\infty$:

$$F(v) = \begin{cases} \int_{\mathcal{O}} Qh(x,\nabla v)\,dx + l_d \int_S (b^{\infty,d})^{\rm hom}[v_T] d\widetilde{x} & \text{if } v \in V_{0,N} \\ +\infty & \text{if } v \notin V_{0,N} \end{cases}$$

where $V_{0,N} = \{v \in V : [v_N] = 0\}$, $v_N = (v.e_N)$ and $v_T = v - v_N e_N$.

(iii) If $l_d = +\infty$:

$$F(v) = \begin{cases} \int_{\mathcal{O}} Qh(x, \nabla v)\,dx & \text{if } v \in V_0 \\ +\infty & \text{if } v \notin V_0, \end{cases}$$

where $V_0 = \{v \in V : [v] = 0\}$. For the three functionals, Qh is the quasiconvex envelope of h, $[v]$ the jump of v across S and $(b^{\infty,w})^{\hom}$ is the function defined on $\mathbb{R}^N$ as

$$(b^{\infty,w})^{\hom}(a) = \inf_k \frac{1}{k^{N-1}} \inf\{\int_{B_k} b^{\infty,w}(y, \nabla_w \varphi)dy : \varphi \in \Psi^\gamma a + W_0^{1,p_w}(B_k, \mathbb{R}^N)\}$$

where $w = s$ or $w = d$, $B_k = \{x \in \mathbb{R}^N : \widetilde{x} \in k\widetilde{Y}, \pm x_N \leq \gamma^\pm(\widetilde{x})\}$, for $x \in \mathbb{R}^N$

$$\Psi^\gamma(x) = \operatorname{sign}(x_N)\Psi(\frac{|x_N|}{\gamma^\pm(\widetilde{x})})$$

with

$$\Psi(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \leq t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

Without loss of generality, we suppose in the following that $b_s(.,0) = 0$ on B_ε. The principal result of this section is in the following proposition

Proposition 4.3. $\Gamma(\tau) - \lim F_r = F$

To establish this result, we need some lemmas. Let $\widetilde{A} \in B_b(\mathbb{R}^{N-1})$, $a \in \mathbb{R}^N$ and we take $p = p_w$. We define

$$S_{\widetilde{A}}(a) = \inf\{\int_A b^{\infty,w}(y, \nabla_w \varphi)dy : \varphi \in \Psi^\gamma a + W_0^{1,p}(A, \mathbb{R}^N)\} \tag{4.1}$$

where

$$A = \{x \in \mathbb{R}^N : \widetilde{x} \in \widetilde{A}, \pm x_N \leq \gamma^\pm(\widetilde{x})\} \tag{4.2}$$

Lemma 4.4. *Let $\widetilde{A}$ be a convex open bounded subset of $\mathbb{R}^{N-1}$. Then for a sequence $(\varepsilon_n)_n$ of real positive, $\varepsilon_n \to 0$ we have*

$$\lim_{n\to+\infty} \frac{S_{\frac{1}{\varepsilon_n}\widetilde{A}}(a)}{\operatorname{meas}(\frac{1}{\varepsilon_n}\widetilde{A})} = (b^{\infty,w})^{\hom}(a)$$

Proof. Let $\widetilde{A} \in B_b(\mathbb{R}^{N-1})$, and the function $S : \widetilde{A} \mapsto S_{\widetilde{A}}$. Then S is a subadditive $\mathbb{Z}^{N-1}$-periodic function:
(i) Let $\widetilde{A}$, $\widetilde{B} \in B_b(\mathbb{R}^{N-1})$ such that $\widetilde{A} \cap \widetilde{B} = \emptyset$, then $S_{\widetilde{A}\cup\widetilde{B}} \leq S_{\widetilde{A}} + S_{\widetilde{B}}$. To establish this, we take $\varphi_A \in \Psi^\gamma(a) + W_0^{1,p}(A, \mathbb{R}^N)$ and $\varphi_B \in \Psi^\gamma(a) + W_0^{1,p}(B, \mathbb{R}^N)$, where A and B (in (4.2) we replace $\widetilde{A}$ by $\widetilde{B}$) are defined from $\widetilde{A}$ and $\widetilde{B}$ by (4.2).

Let us take

$$\Phi = \begin{cases} \varphi_A & \text{on } A \\ \varphi_B & \text{on } B \end{cases}$$

Since $\widetilde{A} \cap \widetilde{B} = \emptyset$, $A \cap B = \emptyset$. Thus $\Phi \in \Psi^\gamma a + W_0^{1,p}(A \cup B)$, and

$$S_{\widetilde{A}\cup\widetilde{B}} \leq \int_{A\cup B} b^{\infty,w}(y, \nabla_w \Phi)dy = \int_A b^{\infty,w}(y, \nabla_w \varphi_A)dy + \int_B b^{\infty,w}(y, \nabla_w \varphi_B)dy$$

for all φ_A and φ_B. Thus

$$S_{\widetilde{A}\cup\widetilde{B}} \leq S_{\widetilde{A}} + S_{\widetilde{B}}$$

(ii) Let $\widetilde{A} \in B_b(\mathbb{R}^{N-1})$ and $z \in \mathbb{Z}^{N-1}$. Let A and A_z subsets associated respectively to $\widetilde{A}$ and $\widetilde{A}+z$ by relation (4.2), and $\varphi \in W_0^{1,p}(A_z)$. Since b_w is $\widetilde{Y}$-periodic, it's the same for $b^{\infty,w}$. Thus

$$\begin{aligned}\int_{A_z} b^{\infty,w}(x,\nabla_w\varphi)dx &= \int_{\widetilde{A}+z}\int_{\{x_N:\pm x_N\le\gamma^\pm(\widetilde{x})\}} b^{\infty,w}(x,\nabla_w\varphi)dx_N\,d\widetilde{x}\\ &= \int_{\widetilde{A}}\int_{\{x_N:\pm x_N\le\gamma^\pm(\widetilde{x}+z)\}} b^{\infty,w}(\widetilde{x}+z,x_N,\nabla_w\varphi)dx_N\,d\widetilde{x}\\ &= \int_A b^{\infty,w}(x,\nabla_w\varphi)dx_N\,d\widetilde{x}\end{aligned}$$

Subadditivity and $\mathbb{Z}^{N-1}$-periodicity being proved for S, we have to show dominant property (Theorem 2.3). So, let $\widetilde{A} \in B_b(\mathbb{R}^{N-1})$ be a convex included in $[0,1[^{N-1}$, and A, B subsets associated respectively with $\widetilde{A}$ and $[0,1[^{N-1}$ by (4.2), $A \subset B$. Let $\Phi_0 \in W_0^{1,p}(B,\mathbb{R}^N)$ and $\Phi = \Psi^\gamma a + \Phi_0$. We take $\varphi = \Psi^\gamma a + \varphi_0$, where $\varphi_0 = \eta\Phi_0$ and $\eta \in \mathcal{D}(A)$, then $\varphi \in \Psi^\gamma a + W_0^{1,p}(A)$. If we use Remark 3.1 and condition (3.2),

$$S_{\widetilde{A}} \le \int_A b^{\infty,w}(y,\nabla_w\varphi)dy \le \int_B b^{\infty,w}(y,\nabla_w\varphi)dy \le \beta_w\int_B |\nabla_w\varphi|^p dy$$

And we have

$$|\nabla_w\varphi|^p = |(\nabla\Psi^\gamma\otimes a)_w + \nabla_w\varphi_0|^p \le \text{const}(|(\nabla\Psi^\gamma\otimes a)_w|^p + |\nabla_w\varphi_0|^p)$$

By the fact that $|\nabla\Psi^\gamma| \le 1 + \frac{\text{const}}{\gamma_m^\pm}$ and $\eta \in \mathcal{D}(A)$, we have

$$|\nabla_w\varphi|^p \le \text{const}(1 + |\nabla_w\Phi_0|^p + |\Phi_0|^p).$$

According to Poincaré inequality, we obtain

$$\begin{aligned}S_{\widetilde{A}} &\le \text{const}(\operatorname{meas} B + \int_B |\nabla_w\Phi_0|^p + \int_B |\Phi_0|^p)\\ &\le \text{const}(\operatorname{meas} B + \|\Phi_0\|^p_{W_0^{1,p}(B)}) \quad ,\forall\Phi_0 \in W_0^{1,p}(B)\\ &\le \text{const}(\operatorname{meas} B + \inf_{W_0^{1,p}(B)}\|\Phi_0\|^p_{W_0^{1,p}(B)})\end{aligned}$$

which establish the dominant property. In the other hand, $b^{\infty,w} \ge 0 \Rightarrow \gamma(S) \ge 0 > -\infty$ (see Theorem 2.3 for $\gamma(S)$ definition). Let $\widetilde{A}$ be a convex open bounded subset of $\mathbb{R}^{N-1}$, and $\widetilde{A}_n = \frac{1}{\varepsilon_n}\widetilde{A}$. $(\widetilde{A}_n)_n$ is a regular sequence. Indeed, since $\widetilde{A}$ is a bounded subset, we can find a cube $\widetilde{I} \subset \mathbb{Z}^{N-1}$ such that $\widetilde{A} \subset \widetilde{I}$ and α small enough so that $\alpha\widetilde{I} \subset \widetilde{A}$. If we take $\widetilde{I}_n = \frac{1}{\varepsilon_n}\widetilde{I}$ we obtain regularity. Now, let $\overline{B}_m(x) = \{y \in \mathbb{R}^{N-1} : \delta(x,y) \le m\}$ where $m \ge 0$, δ euclidian distance in $\mathbb{R}^{N-1}$ and $t \ge 0$. Since $t\overline{B}_m(x) = \overline{B}_{tm}(tx)$, then for $A \subset \mathbb{R}^{N-1}$ we have $\rho(tA) = t\rho(A)$ where $\rho(A) = \sup\{m \ge 0 : \exists\overline{B}_m(x) \subset A\}$. Thus

$$\rho(\widetilde{A}_n) = \rho(\frac{1}{\varepsilon_n}\widetilde{A}) = \frac{1}{\varepsilon}\rho(\widetilde{A}) \overset{n}{\to} +\infty$$

Conditions of Theorem 2.3 are then satisfied for $\widetilde{A}_n$, which prove lemma. □

Lemma 4.5. *If a sequence $(v_r)_r \subset X$ satisfies $F_r(v_r) \le c$, then*

$$\mu \int_{B_\varepsilon} |\nabla_s v_r|^{p_s} dx \le C_1 \tag{4.3}$$

$$\eta \int_{B_\varepsilon} |\nabla_d v_r|^{p_d} dx \le C_2 \tag{4.4}$$

$$\int_{\mathcal{O}_\varepsilon} |\nabla v_r|^q dx \le C_3 \tag{4.5}$$

The proof of the above lemma is a straightforward consequence of (H1). Now, we consider the following regularity condition.

(H4) l_s and $l_d \in [0, +\infty[$, $u \in V'$, $v_r \to u$ in (X, τ) and $\liminf F_r(v_r) < +\infty$. where $V' = \{v \in V : v^\pm = v_{/\mathcal{O}^\pm} \in \mathcal{C}^1(\mathcal{O}^\pm, \mathbb{R}^N)\}$. We define on V, the application

$$R_\varepsilon u(x) = \frac{u(|x_N|) - u(-|x_N|)}{2}\ \Psi_\varepsilon(x) + \frac{u(|x_N|) + u(-|x_N|)}{2}, \tag{4.6}$$

where $\Psi_\varepsilon(x) = \Psi^\gamma(\frac{x}{\varepsilon})$. We take $\theta = \nabla \Psi^\gamma \in L^\infty(\mathbb{R}^N)$. We also consider

$$\begin{gathered} t^\pm(\varepsilon) = (\int_{\mathbb{R}^{N-1}} |\ (v_r - u)(\widetilde{x}, \pm\varepsilon\gamma^\pm(\frac{\widetilde{x}}{\varepsilon}))\ |^p d\widetilde{x})^{\frac{1}{p}} \\ B'\varepsilon = \{x \in \mathcal{O} : \pm x_N \le (1 + t^\pm(\varepsilon))\varepsilon\gamma^\pm(\frac{\widetilde{x}}{\varepsilon})\} \\ \varphi_\varepsilon(\widetilde{x}, \pm x_N) = 1 - \Psi(\frac{\frac{|x_N|}{\varepsilon\gamma^\pm(\frac{\widetilde{x}}{\varepsilon})} - 1}{t^\pm(\varepsilon)}) \end{gathered} \tag{4.7}$$

Let $\alpha > 0$ such that $\alpha \to 0$. Let $(S_i)_{i \in I(\alpha)}$ be a family of open bounded disconnected cubes of $\mathbb{R}^{N-1}$ with diameter α so that $\text{meas}(\mathbb{R}^{N-1} \setminus \cup_{i \in I(\alpha)} S_i) = 0$, and $B'_{\varepsilon,i} = B'_\varepsilon \cap (S_i \times \mathbb{R})$. we denote by (λ, w) pair (μ, s) or (η, d) and $b = b_w$.

Lemma 4.6. *With condition (H4), for $\omega_r = \varphi_\varepsilon(v_r - R_\varepsilon u)$ we have*

$$\begin{aligned} &\liminf_{r\to 0} \lambda \int_{B'_{\varepsilon,i}} b(\frac{x}{\varepsilon}, \frac{1}{2\varepsilon}([u](a_i) \otimes \theta(\frac{x}{\varepsilon}))_w + \nabla_w \omega_r) dx \\ &\ge \liminf_{r\to 0} \lambda \int_{B'_{\varepsilon,i}} b(\frac{x}{\varepsilon}, \frac{1}{2\varepsilon}([u](a_i) \otimes \theta(\frac{x}{\varepsilon}))_w) dx - o(\alpha) \end{aligned}$$

Proof. We have $B'_{\varepsilon,i} = B'_\varepsilon \cap (S_i \times \mathbb{R})$, then

$$\text{meas}(B'_{\varepsilon,i}) \le \alpha^{N-1}\varepsilon(1 + t^\pm(\varepsilon))\gamma^\pm_M \quad (\pm \text{ in sense of maximum }) \tag{4.8}$$

If we take $p = \min(p_s, p_d)$

$$\begin{aligned} t^\pm(\varepsilon)^p &= \int_{\mathbb{R}^{N-1}} |\ (v_r - u)(\widetilde{x}, \pm\varepsilon\gamma^\pm(\frac{\widetilde{x}}{\varepsilon}))\ |^p d\widetilde{x} \\ &\le \text{const}(\int_{\mathbb{R}^{N-1}} |\ v_r(\widetilde{x}, \pm\varepsilon\gamma^\pm(\frac{\widetilde{x}}{\varepsilon}))\ - u(\widetilde{x}, 0)|^p d\widetilde{x} \\ &\quad + \int_{\mathbb{R}^{N-1}} |\ u(\widetilde{x}, \pm\varepsilon\gamma^\pm(\frac{\widetilde{x}}{\varepsilon}))\ - u(\widetilde{x}, 0)|^p d\widetilde{x}). \end{aligned}$$

Since $v_r \rightharpoonup u$ in $W^{1,q}_{\text{loc}}(\mathcal{O} \setminus S)$, $v_r \to u$ in $L^q_{\text{loc}}(\mathcal{O} \setminus S)$. According to Proposition 4.1, embedding $L^q \hookrightarrow L^p$ $(p \le q)$ and regularity of u, $t^\pm(\varepsilon) \xrightarrow{r} 0$. Applying this result on (4.8),

$$\mathrm{meas}(B'_{\varepsilon,i}) \le \mathrm{const}\,\alpha^{N-1}\varepsilon. \tag{4.9}$$

Since b satisfies condition (3.2) and using (4.9),

$$\begin{aligned}\lambda\int_{B'_{\varepsilon,i}} b(\frac{x}{\varepsilon},\frac{1}{2\varepsilon}([u](a_i)\otimes\theta(\frac{x}{\varepsilon}))_w)dx &\le \mathrm{const}\lambda(1+\frac{1}{(2\varepsilon)^p})\,\mathrm{meas}(B'_{\varepsilon,i})\\ &\le \mathrm{const}(\varepsilon\lambda+\frac{\lambda}{(2\varepsilon)^{p-1}})\,\alpha^{N-1}\end{aligned}$$

Thus

$$\liminf_{r\to 0}\lambda\int_{B'_{\varepsilon,i}} b(\frac{x}{\varepsilon},\frac{1}{2\varepsilon}([u](a_i)\otimes\theta(\frac{x}{\varepsilon}))_w)dx \le \mathrm{const}\,l\,\alpha^{N-1} = o(\alpha)$$

Since $b\ge 0$,

$$\begin{aligned}&\liminf_{r\to 0}\lambda\int_{B'_{\varepsilon,i}} b(\frac{x}{\varepsilon},\frac{1}{2\varepsilon}([u](a_i)\otimes\theta(\frac{x}{\varepsilon}))_w+\nabla_w\omega_r)dx\\ &\ge 0 \ge \liminf_{r\to 0}\lambda\int_{B'_{\varepsilon,i}} b(\frac{x}{\varepsilon},\frac{1}{2\varepsilon}([u](a_i)\otimes\theta(\frac{x}{\varepsilon}))_w)dx\\ &-o(\alpha)\end{aligned}$$

□

Now we are ready to establish Proposition 4.3

Proposition 4.7. *For every sequence $(v_r)_r\subset X$ and every $u\in X$ such that $v_r \xrightarrow{\tau} u$ in X, we have*

$$F(u)\le \liminf_{r\to 0} F_r(v_r).$$

Proof. Let $(v_r)_r$ be a sequence in X and $u\in X$ so that $v_r\xrightarrow{\tau} u$ in X. If $\liminf_{r\to 0}F_r(v_r)=+\infty$, then proposition is proved. If not, by Proposition 4.1 $u\in V$.

(i) **Case l_s and l_d are finite.** We begin by treating regular case; i.e., when condition (H4) is satisfied. Then, by adaptation of [15, Lemmas 4.4, 4.5, 4.6, 4.9] and by application of Lemma 4.6, we have for $\omega_r=\varphi_\varepsilon(v_r-R_\varepsilon u)$,

$$\begin{aligned}&\liminf_{r\to 0}\lambda\int_{B_\varepsilon} b(\frac{x}{\varepsilon},\nabla_w v_r)dx\\ &=\liminf_{r\to 0}\lambda\int_{B'_\varepsilon} b(\frac{x}{\varepsilon},\frac{1}{2\varepsilon}((u(|x_N|)-u(-|x_N|))\otimes\theta(\frac{x}{\varepsilon})\ )_w+\nabla_w\omega_r)dx\\ &\ge \liminf_{r\to 0}\lambda\sum_{i\in I(\alpha)}\int_{B'_{\varepsilon,i}} b(\frac{x}{\varepsilon},\frac{1}{2\varepsilon}([u](a_i)\otimes\theta(\frac{x}{\varepsilon}))_w+\nabla_w\omega_r)dx-o(\alpha)\\ &\ge \liminf_{r\to 0}\lambda\sum_{i\in I(\alpha)}\int_{B'_{\varepsilon,i}} b(\frac{x}{\varepsilon},\frac{1}{2\varepsilon}([u](a_i)\otimes\theta(\frac{x}{\varepsilon}))_w)-o(\alpha)\\ &\ge l_w\sum_{i\in I(\alpha)}\mathrm{meas}(S_i)(b^{\infty,w})^{\mathrm{hom}}([u](a_i))-o(\alpha).\end{aligned}$$

As $\alpha\to 0$, we obtain

$$\liminf_{r\to 0}\lambda\int_{B_\varepsilon} b(\frac{x}{\varepsilon},\nabla_w v_r)dx \ge l_w\int_S (b^{\infty,w})^{\mathrm{hom}}([u])d\widetilde{x}. \tag{4.10}$$

By the characterization of quasiconvex envelope (see section 2), we have

$$Qh(x,\nabla v_r(x)) = \inf\{\frac{1}{\text{meas } D}\int_D h(x,\nabla v_r(x)+\nabla\varphi(y)) : \varphi\in W_0^{1,\infty}(D)\}$$

where D is a bounded domain of $\mathbb{R}^N$. If we take $\varphi = 0$, then

$$Qh(x,\nabla v_r(x)) \le \frac{1}{\text{meas } D}\int_D h(x,\nabla v_r(x))dy = h(x,\nabla v_r(x))$$

Let δ be a fixed real less than 1. For a given ε small enough, $\mathcal{O}_\delta\subset\mathcal{O}_\varepsilon$. Thus

$$\int_{\mathcal{O}_\delta} Qh(x,\nabla v_r(x))dx \le \int_{\mathcal{O}_\varepsilon} h(x,\nabla v_r(x))dx$$

Since the sequence $(F_r(v_r))_r$ is bounded, and according to (4.5), the fact that $\mathcal{O}_\delta\subset\mathcal{O}_\varepsilon$, then $v_r \rightharpoonup u$ in $W^{1,q}(\mathcal{O}_\delta,\mathbb{R}^N)$. Qh being quasiconvex, by Proposition 2.2 the functional $I(v) = \int_{\mathcal{O}_\delta} Qh(x,\nabla v(x))dx$ is then weakly lower semicontinuous on $W^{1,q}(\mathcal{O}_\delta,\mathbb{R}^N)$. Thus

$$\liminf_{r\to 0}\int_{\mathcal{O}_\varepsilon} h(x,\nabla v_r(x))dx \ge \int_{\mathcal{O}_\delta} Qh(x,\nabla u(x))dx$$

tending δ toward 0

$$\liminf_{r\to 0}\int_{\mathcal{O}_\varepsilon} h(x,\nabla v_r(x))dx \ge \int_{\mathcal{O}} Qh(x,\nabla u(x))dx \tag{4.11}$$

According to (4.10) and (4.11)

$$\liminf_{r\to 0} F_r(v_r) \ge F(u), \text{ for } u \text{ regular} \tag{4.12}$$

If u is not regular, we consider a regular vector valued function u_δ so that $\|u - u_\delta\|_{W^{1,q}(\mathcal{O}\setminus S,\mathbb{R}^N)} \le \delta$ and we take $v_{\delta,r} = v_r - R_\varepsilon u + R_\varepsilon u_\delta$. Now, let us verify that $R_\varepsilon u \xrightarrow{\tau} u$. Since $R_\varepsilon u = u$ on $\mathcal{O}_\varepsilon$ (see (4.6)) and $\psi_\varepsilon \le 1$, it follows that for $p = \min(p_s, p_d)$,

$$\begin{aligned}\int_{\mathcal{O}} |R_\varepsilon u - u|^p dx &= \int_{B_\varepsilon} |R_\varepsilon u - u|^p dx\\ &\le \text{const}\{\int_{B_\varepsilon} |R_\varepsilon u|^p dx + \int_{B_\varepsilon} |u|^p dx\}\\ &\le \text{const}\int_{B_\varepsilon} |u|^p dx\\ &\xrightarrow{\varepsilon} 0\end{aligned} \tag{4.13}$$

So we have the result for u and u_δ, thus $v_{\delta,r} \xrightarrow{\tau} u_\delta$. Using (4.12),

$$\liminf_{r\to 0} F_r(v_{\delta,r}) \ge F(u_\delta) \tag{4.14}$$

According to conditions (3.2) and (3.2) with w replaced by q, we have

$$\begin{aligned}F_r(v_r) &= F_r(v_{\delta,r} + R_\varepsilon u - R_\varepsilon u_\delta)\\ &\ge F_r(v_{\delta,r}) - \text{const}\,\{\int_{\mathcal{O}_\varepsilon} |\nabla\,(u_\delta - u)|(1 + |\nabla v_r|^{q-1} + |\nabla v_{\delta,r}|^{q-1}\,)\\ &\quad + \sum_{(\lambda,w)} \lambda \int_{B_\varepsilon} |\nabla_w\, R_\varepsilon(u_\delta - u)|(1 + |\nabla_w v_r|^{p_w-1} + |\nabla_w v_{\delta,r}|^{p_w-1}\,)\}\end{aligned} \tag{4.15}$$

Let $v_\delta = u_\delta - u$, $v_\delta^\varepsilon = R_\varepsilon(v_\delta)$ and

$$A_1 = \int_{\mathcal{O}_\varepsilon} |\nabla\, v_\delta|(1 + |\nabla v_r|^{q-1} + |\nabla v_{\delta,r}|^{q-1})$$

$$A_2 = \sum_{(\lambda,w)} \lambda \int_{B_\varepsilon} |\nabla_w v_\delta^\varepsilon|(1 + |\nabla_w v_r|^{p_w-1} + |\nabla_w v_{\delta,r}|^{p_w-1})dx\,.$$

By Holder inequality

$$\begin{aligned} A_1 &\le (\int_{\mathcal{O}_\varepsilon} |\nabla v_\delta|^q)^{\frac{1}{q}} (\int_{\mathcal{O}_\varepsilon} (1 + |\nabla v_r|^{q-1} + |\nabla v_{\delta,r}|^{q-1})^{q'} dx)^{1/q'} \\ &\le \text{const} \|v_\delta\|_{W^{1,q}(\mathcal{O}\setminus S)} . (\int_{\mathcal{O}_\varepsilon} 1 + |\nabla v_r|^q + |\nabla u|^q + |\nabla u_\delta|^q dx)^{1/q'} \end{aligned}$$

(q' is the conjugate exponent of q). We have

$$\begin{aligned} \int_{\mathcal{O}_\varepsilon} |\nabla u_\delta|^q dx &\le \|u_\delta\|^q_{W^{1,q}(\mathcal{O}\setminus S)} \\ &\le \text{const}\, (\|v_\delta|^q_{|W^{1,q}(\mathcal{O}\setminus S)} + \|u\|^q_{W^{1,q}(\mathcal{O}\setminus S)}) \\ &\le \text{const}\, (1 + \|u\|^q_{W^{1,q}(\mathcal{O}\setminus S)}). \end{aligned}$$

Using this result and (4.5)

$$A_1 \le \text{const}\, \|v_\delta\|_{W^{1,q}(\mathcal{O}\setminus S)} (1 + \|u\|^q_{W^{1,q}(\mathcal{O}\setminus S)})^{1/q'}. \tag{4.16}$$

On the other hand, by Holder inequality,

$$A_2 \le \text{const} \sum_{(\lambda,w)} \lambda (\int_{B_\varepsilon} |\nabla v_\delta^\varepsilon|^{p_w})^{\frac{1}{p_w}} (\int_{B_\varepsilon} 1 + |\nabla_w v_r|^{p_w} + |\nabla v_\delta^\varepsilon|^{p_w} dx)^{\frac{p_w-1}{p_w}}.$$

We have

$$\begin{aligned} \int_{B_\varepsilon} |\nabla v_\delta^\varepsilon|^{p_w} dx &= \int_{B_\varepsilon} |\nabla R_\varepsilon v_\delta|^{p_w} dx \\ &= \int_{B_\varepsilon} |\frac{1}{2\varepsilon}(v_\delta |x_N| - v_\delta(-|x_N|)) \otimes \theta(\frac{x}{\varepsilon}) \\ &\quad + \frac{1}{2}(\nabla v_\delta |x_N| - \nabla v_\delta(-|x_N|))\psi_\varepsilon(x)|^{p_w} dx\,. \end{aligned}$$

Using $\theta \in L^\infty(\mathbb{R}^N)$, $\psi_\varepsilon \le 1$ and a change of variable,

$$\int_{B_\varepsilon} |\nabla u_\delta^\varepsilon|^{p_w} dx \le \text{const}(\frac{1}{(2\varepsilon)^{p_w}} \int_{B_\varepsilon} |v_\delta|^{p_w} + \int_{B_\varepsilon} |\nabla v_\delta|^{p_w} dx). \tag{4.17}$$

Since $v_\delta \in V$, by [15, Lemma 3.1],

$$\begin{aligned} \frac{\lambda}{(2\varepsilon)^{p_w}} \int_{B_\varepsilon} |v_\delta|^{p_w} dx &\le \text{const}(\lambda \int_{B_\varepsilon} |\nabla v_\delta|^{p_w} dx + \frac{\lambda}{\varepsilon^{p_w-1}} \|v_\delta\|^{p_w}_{W^{1,q}(\mathcal{O}\setminus S)}) \\ &\le \text{const}(o(r) + \frac{\lambda}{\varepsilon^{p_w-1}} \|v_\delta\|^{p_w}_{W^{1,q}(\mathcal{O}\setminus S)}). \end{aligned}$$

By (4.17), we have

$$\lambda \int_{B_\varepsilon} |\nabla u_\delta^\varepsilon|^{p_w} \le \text{const}(o(r) + \frac{\lambda}{\varepsilon^{p_w-1}} \|v_\delta\|^{p_w}_{W^{1,q}(\mathcal{O}\setminus S)}).$$

Using this result, (4.3) and (4.4),

$$A_2 \\ \leq \text{const} \sum_{(\lambda,w)} (o(r) + \frac{\lambda}{\varepsilon^{p_w-1}} \|v_\delta\|^{p_w}_{W^{1,q}(\mathcal{O}\setminus S)})^{\frac{1}{p_w}} (1 + o(r) + \frac{\lambda}{\varepsilon^{p_w-1}} \|v_\delta\|^{p_w}_{W^{1,q}(\mathcal{O}\setminus S)})^{\frac{p_w-1}{p_w}} \tag{4.18}$$

Applying (4.14), (4.15), (4.16) and (4.18), we obtain

$$\liminf_{r\to 0} F_r(v_r) \geq F(u_\delta) - C(u)\|v_\delta\|_{W^{1,q}(\mathcal{O}\setminus S)} \tag{4.19}$$

where $C(u)$ is a constant depending on u. On the other hand, since $b^{\infty,w}$ and h satisfies respectively conditions (3.2) and (3.2) with w replaced by q, $(b^{\infty,w})^{\mathrm{hom}}$ and Qh are lipshitz functions (the proof is an adaptation of the proof of [18, Proposition 2.1]). Then

$$\begin{aligned} F(u_\delta) \geq F(u) - \text{const}\{&\int_{\mathcal{O}} |\nabla v_\delta|(1 + |\nabla u|^{q-1} + |\nabla u_\delta|^{q-1}) \\ &+ \int_S |[v_\delta]|(1 + |[u]|^{p_w-1} + |[u_\delta]|^{p_w-1}) d\widetilde{x}\}. \end{aligned}$$

Using the fact that $p_w \leq q$, Holder inequality, continuity of the jump, the compact embeding $W^{1,q}(\mathcal{O}\setminus S) \hookrightarrow L^q(S)$ and that $\|u_\delta\|_{W^{1,q}(\mathcal{O}\setminus S)} \leq \|u\|_{W^{1,q}(\mathcal{O}\setminus S)} + 1$, we have

$$F(u_\delta) \geq F(u) - C(u)\|v_\delta\|_{W^{1,q}(\mathcal{O}\setminus S)}.$$

We then use this result and (4.19), and we let δ approach 0. Thus

$$\liminf_{r\to 0} F_r(v_r) \geq F(u)$$

(ii) **Case** $l_s = +\infty$ **and** $l_d < +\infty$**:** We have $[u_N] = 0$. Indeed, let $\sigma \in \mathcal{D}(\mathcal{O}, \mathcal{M}^{\mathcal{N}})$. By Green formula and Proposition 4.1

$$\begin{aligned} \int_{B_\varepsilon} \sigma : \nabla v_r \, dx &= \int_{\mathcal{O}} \sigma : \nabla v_r \, dx - \int_{\mathcal{O}} \sigma : (\mathcal{X}_{\mathcal{O}_\varepsilon} \nabla v_r) \, dx \\ &= -\int_{\mathcal{O}} div\sigma : v_r \, dx - \int_{\mathcal{O}} \sigma : (\mathcal{X}_{\mathcal{O}_\varepsilon} \nabla v_r) \, dx \\ &\xrightarrow{r} \int_S \sigma n.[u] \, d\widetilde{x}, \end{aligned}$$

where n is the unit vector normal exterior to $\mathcal{O}^+$. If we take $\sigma = \phi.I_N$, where I_N is the unit matrix of $\mathbb{R}^N$ and $\phi \in \mathcal{D}(\mathcal{O})$, we have

$$\lim_r \int_{B_\varepsilon} \phi \, div v_r \, dx = \int_S \phi.[u_N] \, d\widetilde{x} \tag{4.20}$$

According to (4.5) and that $l_s = +\infty$

$$\begin{aligned} \Big|\int_{B_\varepsilon} \phi \, div v_r \, dx\Big| &\leq \|\phi\|_{L^{p_s'}(B_\varepsilon)} \|div v_r\|_{L^{p_s}(B_\varepsilon)} \\ &\leq \text{const}(\frac{\varepsilon^{p_s-1}}{\mu})^{\frac{1}{p_s}} \xrightarrow{r} 0 \end{aligned}$$

where p_s' is the conjugate exponent of p_s. By (4.20), we obtain $[u_N] = 0$. Thus, $u \in V_{0,N}$. And we have $(b^{\infty,s})^{\mathrm{hom}}[u] = 0$. Indeed let us take $(b^{\infty,s})^{\mathrm{hom}}(a) =$

$(b^{\infty,s})^{\hom}(a,\gamma)$. According to [15, Proposition 3.8], we have

$$(b^{\infty,s})^{\hom}([u]) \leq (b^{\infty,s})^{\hom}([u],\gamma_m).$$

Let $k \in \mathbb{N}$ and $\varphi = \Psi^{\gamma_m}.[u]$, where for a given $y \in B_k$, $\Psi^{\gamma_m}(y) = sign(y_N)\Psi(\frac{|y_N|}{\gamma_m^{\pm}}) = \pm\frac{y_N}{\gamma^{\pm}{}_m}$. Definition of $(b^{\infty,s})^{\hom}$ implies

$$(b^{\infty,s})^{\hom}([u]) \leq \frac{1}{k^{N-1}}\int_{B_k} b^{\infty,s}(y,\ \nabla_s\varphi)dy.$$

Since $[u_N]=0$, $\nabla_s\varphi(y) = (\nabla\Psi^{\gamma_m}(y)\otimes[u])_s = \pm\frac{1}{\gamma^{\pm}{}_m}(e_N\otimes[u])_s = 0$. By the fact that $b_s(x,0)=0$, we deduce that $(b^{\infty,s})^{\hom}([u]) = 0$ and we conclude using result of case (i).

(iii) **case $l_d = +\infty$:** In this case, $[u]=0$. Indeed, let $\sigma \in \mathcal{D}(\mathcal{O}, M^N)$. We have

$$\lim_r \int_{B_\varepsilon} \sigma : \nabla_d v_r dx = \int_S \sigma_d n.[u]d\widetilde{x}$$

for all $\sigma \in \mathcal{D}(\mathcal{O}, M^N)$. Thus $u \in V_0$, and $\varphi = \Psi^{\gamma_m}.[u] = 0$. Consequently

$$(b^{\infty,w})^{\hom}([u]) \leq \frac{1}{k^{N-1}}\int_{B_k} b^{\infty,w}(y,\nabla_w\varphi)dy = 0$$

for $w=s$ and $w=d$. The result is then proved. □

Proposition 4.8. *If $u \in X$, then there exist a sequence $(v_r)_r \subset X$ such that $v_r \xrightarrow{\tau} u$ and*

$$\limsup_{r\to 0} F_r(v_r) \leq F(u)$$

Proof. (i) **Case l_s and l_d are finite:** Let $u \in X$. If $u \notin V$, $F(u) = +\infty$, and the result is established taking for example $v_r = u$. If not, we first take u regular. Let (S_i) be the family of open bounded disconnected cubes of $\mathbb{R}^{N-1}$ with diameter α so that $\text{meas}(\mathbb{R}^{N-1}\setminus\cup_{i\in I(\alpha)}S_i)=0$, and $v_r = R_\varepsilon u$ (4.6). By (4.13), $v_r \xrightarrow{\tau} u$. Let $a \in \mathbb{R}^N$, for $(\lambda,w) = (\mu,s)$ or (η,d) we have

$$\sum_{i\in I(\alpha)} l_w\,\text{meas}(S_i)(b^{\infty,w})^{\hom}([u](a)) \geq \lim_{r\to0}\lambda\int_{B_\varepsilon} b(\frac{x}{\varepsilon},\nabla_w v_r)dx - o(\alpha). \tag{4.21}$$

Indeed, let $u_{\varepsilon,i}$ be an ε-minimizer of $S_{\frac{1}{\varepsilon}S_i}(a)$ defined by

$$S_{\frac{1}{\varepsilon}S_i}(a) = \inf\{\int_{\frac{1}{\varepsilon}B_{\varepsilon,i}} b^{\infty,w}(y,\nabla_w\varphi)dy : \varphi \in \Psi^\gamma a + W_0^{1,p_w}(\frac{1}{\varepsilon}B_{\varepsilon,i})\},$$

where $B_{\varepsilon,i} = B_\varepsilon \cap (S_i\times\mathbb{R})$. Let $\theta = \nabla\Psi^\gamma \in L^\infty(\mathbb{R}^N)$. Using lemma 4.4 and the change of variable $x = \varepsilon y$, we have

$$\begin{aligned}
&l_w\,\text{meas}(S_i)(b^{\infty,w})^{\hom}([u](a))\\
&= \lim_{r\to0}\varepsilon^{N-1}\frac{\lambda}{2(2\varepsilon)^{p_w-1}}\int_{\frac{1}{\varepsilon}B_{\varepsilon,i}} b^{\infty,w}(y,([u](a)\otimes\theta(x))_w + \nabla_w u_{\varepsilon,i})dy\\
&= \lim_{r\to0}\frac{\lambda}{(2\varepsilon)^{p_w}}\int_{B_{\varepsilon,i}} b^{\infty,w}(\frac{x}{\varepsilon},([u](a)\otimes\theta(\frac{x}{\varepsilon}))_w + (\nabla_w u_{\varepsilon,i})(\frac{x}{\varepsilon}))dx
\end{aligned} \tag{4.22}$$

According to (H2) and the inequalities $\text{meas}(B_{\varepsilon,i}) \leq \gamma_M^{\pm}\alpha^{N-1}\varepsilon$ and

$$\int_{B_{\varepsilon,i}} |(\nabla_w u_{\varepsilon,i})(\frac{x}{\varepsilon})|^{p_w}dx \leq \text{const}\,\varepsilon(\alpha^{N-1}+\varepsilon^N)$$

(this last result is obtained using the $u_{\varepsilon,i}$ definition and condition (3.2) satisfied by $b^{\infty,w}$), (4.22) becomes

$$\begin{aligned} &l_w \operatorname{meas}(S_i)(b^{\infty,w})^{\rm hom}([u](a)) \\ &= \lim_{r\to 0} \lambda \int_{B_{\varepsilon,i}} b(\frac{x}{\varepsilon}, \frac{1}{2\varepsilon}([u](a)\otimes\theta(\frac{x}{\varepsilon}))_w + \frac{1}{2\varepsilon}(\nabla_w u_{\varepsilon,i})(\frac{x}{\varepsilon}))dx\,. \end{aligned} \tag{4.23}$$

By Holder inequality, condition (3.2) and the result $|\nabla R_\varepsilon u| \leq \text{const}(1+\frac{1}{\varepsilon})$, we deduce

$$\begin{aligned} &l_w \operatorname{meas}(S_i)(b^{\infty,w})^{\rm hom}([u](a)) \\ &\geq \lim_{r\to 0} \lambda \int_{B_{\varepsilon,i}} b(\frac{x}{\varepsilon}, \nabla_w R_\varepsilon u + \frac{1}{2\varepsilon}(\nabla_w u_{\varepsilon,i})(\frac{x}{\varepsilon}))dx - o(\alpha) \\ &\geq \lim_{r\to 0} \lambda \int_{B_{\varepsilon,i}} b(\frac{x}{\varepsilon}, \nabla_w R_\varepsilon u)dx - o(\alpha) \end{aligned}$$

Summing over $I(\alpha)$ and tending α towards 0, we deduce that

$$l_w \int_S (b^{\infty,w})^{\rm hom}([u])d\widetilde{x} \geq \lim_{r\to 0} \lambda \int_{B_\varepsilon} b(\frac{x}{\varepsilon}, \nabla_w v_r)dx\,.$$

Since $v_r = u$ on $\mathcal{O}_\varepsilon$,

$$\begin{aligned} &\lim_{r\to 0}\{\int_{\mathcal{O}_\varepsilon} h(x,\nabla v_r)dx + \sum_{\lambda,w} \lambda \int_{B_\varepsilon} b(\frac{x}{\varepsilon}, \nabla_w v_r)dx\} \\ &\leq \int_{\mathcal{O}} h(x,\nabla u)dx + \sum_w l_w \int_S (b^{\infty,w})^{\rm hom}([u])d\widetilde{x} \end{aligned}$$

Thus

$$\begin{aligned} G(u) &= \inf\{\limsup_r F_r(v_r) : v_r \xrightarrow{\tau} u\} \\ &\leq \int_{\mathcal{O}} h(x,\nabla u)dx + \sum_w l_w \int_S (b^{\infty,w})^{\rm hom}([u])d\widetilde{x} \end{aligned}$$

If we take the weak lower semicontinuous envelope on $W^{1,q}(\mathcal{O}\setminus S)$ denoted Γ_τ for the two members, we obtain

$$\Gamma_\tau G(u) \leq \int_{\mathcal{O}} Qh(x,\nabla u)dx + \sum_w l_w \int_S (b^{\infty,w})^{\rm hom}([u])d\widetilde{x}$$

(we use the integral representation of quasiconvex envelope for the first integral term and compact embeeding $W^{1,q}(\mathcal{O}\setminus S) \hookrightarrow L^{p_w}(S)$ for the second, noticing that function $(b^{\infty,w})^{\rm hom}$ is convex [17, Proposition 2.6]. Since G is the Γ-limsup of F_r, it will be τ-lower semicontinuous [2, Theorem 2.1]; thus

$$G(u) = \Gamma_\tau G(u) \leq \int_{\mathcal{O}} Qh(x,\nabla u)dx + \sum_w l_w \int_S (b^{\infty,w})^{\rm hom}([u])d\widetilde{x} \leq F(u).$$

We conclude noticing the infimum in the definition of G is attained. If u is not regular, we use a density argument like in Proposition 4.7.
(ii) **Case $l_s = +\infty$ and $l_d < +\infty$:** If $u \notin V_{0,N}$, $F(u) = +\infty$ and we take for example $v_r = u$. If not, $u \in V_{0,N} \subset V$ and it suffice to apply results of case (i) noticing that $(b^{\infty,s})^{\rm hom}([u]) = 0$.

(iii) **Case** $l_d = +\infty$**:** It is deduced from the fact that $(b^{\infty,w})^{\hom}([u]) = 0$ for $w = s$ and $w = d$. □

The proof of Proposition 4.3 is a direct consequence of Propositions 4.7 and 4.8.

Recall the functional $I_r = F_r - L$ is defined on the space (X, τ) and take $I = F - L$. Let

$$W = \begin{cases} V & \text{if } l_s \text{ and } l_d \text{ are finite} \\ V_{0,N} & \text{if } l_s = +\infty \text{ and } l_d \text{ is finite} \\ V_0 & \text{if } l_d = +\infty \end{cases}$$

Corollary 4.9. *Let $(\overline{u}_r)_r$ be a (3.1)-minimizing sequence. Thus $(\overline{u}_r)_r$ is relatively compact in (X, τ). Moreover, for every cluster point $\overline{u}$ and a subsequence, we have*

$$\lim_{r\to 0} I_r(\overline{u}_r) = I(\overline{u}) = \inf\{I(v) : v \in W\}.$$

The proof of this corollary is a straightforward application of Remark 4.2, propositions 2.1 and 4.3.

References

[1] E. Acerbi, G. Buttazzo and D. Percivale, *Thin inclusions in linear elasticity: a variational approach*, J. Reine Angew. Math., 386, P. 99-115 (1988).

[2] H. Attouch, *Variational Convergence for Functions and Operators*. Pitman Advance Publishing Program, (1984).

[3] E. Acerbi, G. Buttazzo and D. Percivale, *Thin inclusions in linear elasticity: a variational approach*, J. Reine Angew. Math., 386, P. 99-115 (1988).

[4] H. Brezis, L. A. Caffarelli and A. Friedman; *Reinforcement problems for elliptic equations and variational inequalities*. Ann. Math. Pur. Appl., 4(123), 219-246 (1980).

[5] F. Caillerie, *The effect of a thin inclusion of high rigidity in an elastic body*, Math. Meth. Appl. Sci., 2, p.251-270 (1980).

[6] B. Dacorognat, *Direct Methods in Calculus of Variations*, Applied Mathematical sciences, $n^0$78,Springer-Verlag, Berlin, (1989).

[7] G. Dal Maso, *An introduction to Γ-convergence*. Birkäuser, Boston (1993).

[8] G. Dal Maso and L. Modica,; *Non linear stochastic homogenization and ergodic theory*, J. Reine angew. Math., 363:27-43 (1986).

[9] E. De Giorgi and T.Franzoni, *Su un tipo di convergenza variazionale*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Mat. (8) 58 (1975), 842-850.

[10] J. F. Ganghoffer, A. Brillard and J. Schultz; *Modelling of the mechanical Behaviour of Joints Bonded by a nonlinear incompressible elastic adhesive*. Eur. J. Mech., A/Solids, 16, 255-276 (1997).

[11] G. Geymonat, F. krasucki, and S. Lenci, *Mathematical Analysis of a bonded joint with a soft thin adhesive*, Mathematics and Mechanics of Solids, 201-225 (1999).

[12] M. Goland and E. Reissner, *The stresses in cemented joints*, J.Appl. Mesh., ASME, 11, A17-A27 (1944).

[13] O. Kavian, *Introduction à La Théorie de Points Critiques et Applications aux Problèmes Elliptiques*, Springer-Verlag, Paris (1993).

[14] A. klarbring, *Derivation of model of adhesively bonded joints by the asymptotic expansion method*, Int. J. Engng. Sci. 29, p. 493-512 (1991).

[15] C. Licht and G. Michaille, *A Modelling of Elastic Adhesive Bonded Joints* ,Preprint 1995/12, département des sciences mathématiques, Université Montpellier II, (1995).

[16] C. Licht and G. Michaille, *Global-local subadditive ergodic theorems and application to homogenization in elasticity*, Annales Mathematiques, Blaise Pascal 9, 21-62 (2002).

[17] C. Licht, G. Michaille and Y. Abddaimi, *Stochastic homogenization for an integral of a quasiconvex function with linear growth*, asymptotic Analysis 15, IOS press 183-202 (1997).

[18] K. Messaoudi et G.Michaille, *Stochastic Homogenization of Nonconvex Integral Functionals*, Mathematical Modelling and Numerical Analysis, Vol 28, n^o3, p. 329-356 (1994).

[19] A. Ait Moussa, *Comportement Asymptotique des Solutions d'un Problème de Bandes Minces*, Thèse de Doctorat (1995).
[20] P. Suquet, *discontinuities and Plasticity, Nonsmooth Mechanics and Applications*, Moreau Panagiotopoulos Ed., Springer. p. 279-330(1988).

Abdelaziz Aït Moussa
Université Mohamed Premier, Faculté des sciences, Département de Mathématiques et Informatique, Oujda, Maroc
E-mail address: moussa@sciences.univ-oujda.ac.ma

Loubna Zlaïji
Université Mohamed Premier, Faculté des sciences, Département de Mathématiques et Informatique, Oujda, Maroc
E-mail address: l.zlaiji@yahoo.fr