2003 Colloquium on Differential Equations and Applications, Maracaibo, Venezuela.
Electronic Journal of Differential Equations, Conference 13, 2005, pp. 35–47.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

A SYSTEM OF SEMILINEAR EVOLUTION EQUATIONS WITH HOMOGENEOUS BOUNDARY CONDITIONS FOR THIN PLATES COUPLED WITH MEMBRANES

JAIRO HERNÁNDEZ

Abstract. In this work we consider a semilinear initial boundary-value problem modelling an elastic thin plate (in the context of the so-called Kirchhoff-Love theory) coupled with an elastic membrane, regarding homogeneous boundary conditions. By means of the theory of strongly continuous semigroups of linear operators applied to abstract semilinear initial valued problems [16], we obtain existence and uniqueness of a weak solution which is defined in a suitable way.

1. Introduction

In this work we consider a semilinear evolution problem which we pose as follows: Let Ω and Ω_m be two open bounded connected subsets of $\mathbb{R}^2$ with sufficiently smooth boundary $\partial\Omega$ and $\partial\Omega_m$ so that $\Omega_m \subset\subset \Omega$. Let $\Omega_p := \Omega\backslash\overline{\Omega}_m$ and $\Gamma_1 := \partial\Omega_m$. We decompose $\partial\Omega$ in two connected parts Γ_2 and Γ_3 with $\Gamma_2 \cap \Gamma_3 = \emptyset$, $\sigma_1(\Gamma_2) \neq 0$ and $\sigma_1(\Gamma_3) \neq 0$, where σ_1 is the surface measure on $\partial\Omega$, induced by the Lebesgue measure on $\mathbb{R}$ (see figure 1). Then we consider the system of partial differential equations

$$\begin{aligned} &\rho_p h \frac{\partial^2 u_p}{\partial t^2}(t,x) + \frac{h^3}{12} \sum_{\alpha,\beta\gamma,\theta=1}^{2} \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\big(A_{\alpha\beta\gamma\theta}(x)\frac{\partial^2 u_p}{\partial x_\gamma \partial x_\theta}(t,x)\big) \\ &= f_p(t,x,u_p(t,x)) \quad \text{in }]0,T] \times \Omega_p \end{aligned} \tag{1.1}$$

$$\rho_m \frac{\partial^2 u_m}{\partial t^2}(t,x) - C\Delta u_m(t,x) = f_m(t,x,u_m(t,x)) \quad \text{in }]0,T] \times \Omega_m\,, \tag{1.2}$$

$$\begin{aligned} &\frac{h^3}{12} \sum_{\alpha,\beta,\gamma,\theta=1}^{2} \nu_\alpha \frac{\partial}{\partial x_\beta}\big(A_{\alpha\beta\gamma\theta}\frac{\partial^2 u_p}{\partial x_\gamma \partial x_\theta}\big) + \frac{h^3}{12}\frac{\partial}{\partial \vec{\tau}}\Big(\sum_{\alpha,\beta,\gamma,\theta=1}^{2} \nu_\alpha \tau_\beta A_{\alpha\beta\gamma\theta}\frac{\partial^2 u_p}{\partial x_\gamma \partial x_\theta}\Big) \\ &= 0 \quad \text{on }]0,T] \times \Gamma_2\,, \end{aligned} \tag{1.3}$$

2000 *Mathematics Subject Classification.* 74H20, 74H25, 74K15.
Key words and phrases. Plates, membranes, coupled structures, transmission problems, semilinear evolution equations.

Published May 30, 2005.

$$\begin{aligned}&\frac{h^3}{12}\sum_{\alpha,\beta,\gamma,\theta=1}^{2}\nu_\alpha\frac{\partial}{\partial x_\beta}\big(A_{\alpha\beta\gamma\theta}\frac{\partial^2 u_p}{\partial x_\gamma\partial x_\theta}\big)+\frac{h^3}{12}\frac{\partial}{\partial\vec\tau}\Big(\sum_{\alpha,\beta,\gamma,\theta=1}^{2}\nu_\alpha\tau_\beta A_{\alpha\beta\gamma\theta}\frac{\partial^2 u_p}{\partial x_\gamma\partial x_\theta}\Big)\\&+C\frac{\partial u_m}{\partial\vec\nu}=0\quad\text{on }]0,T]\times\Gamma_1\,,\end{aligned}\tag{1.4}$$

$$\sum_{\alpha,\beta,\gamma,\theta=1}^{2}\nu_\alpha\nu_\beta A_{\alpha\beta\gamma\theta}\frac{\partial^2 u_p}{\partial x_\gamma\partial x_\theta}=0\quad\text{on }]0,T]\times(\partial\Omega_p\setminus\Gamma_3),\tag{1.5}$$

$$u_p=\frac{\partial u_p}{\partial\vec\nu}=0\quad\text{on }]0,T]\times\Gamma_3,\tag{1.6}$$

$$u_p=u_m\quad\text{on }]0,T]\times\Gamma_1\,,\tag{1.7}$$

with the initial conditions

$$u_p(0,\cdot)=g_p^0\quad\text{in }\Omega_p,\tag{1.8}$$

$$u_m(0,\cdot)=g_m^0\quad\text{in }\Omega_m,\tag{1.9}$$

$$\frac{\partial u_p}{\partial t}(0,\cdot)=g_p^1\quad\text{in }\Omega_p,\tag{1.10}$$

$$\frac{\partial u_m}{\partial t}(0,\cdot)=g_m^1\quad\text{in }\Omega_m.\tag{1.11}$$

Equations (1.1)-(1.11) describe the vibrations of a structure which consists of a thin elastic anisotropic plate (in the context of the so called Kirchhoff-Love theory) with its middle surface occupying the domain Ω_p, coupled with a membrane occupying the domain Ω_m (see figure 1).

It is supposed that ρ_p and ρ_m are positive constants, where ρ_p (resp. ρ_m) is the density of the middle surface of the plate (resp. the membrane) and h is the thickness of the plate. The coefficients $A_{\alpha\beta\gamma\theta}$ depend on the elastic modulus of the plate and are assumed as C^∞ functions on $\overline{\Omega}_p$; they satisfy the symmetry assumption

$$A_{\alpha\beta\gamma\theta}=A_{\beta\alpha\gamma\theta},\quad A_{\alpha\beta\gamma\theta}=A_{\alpha\beta\theta\gamma},\quad A_{\alpha\beta\gamma\theta}=A_{\gamma\theta\alpha\beta}\tag{1.12}$$

and the coercivity hypothesis

$$\sum_{\alpha,\beta,\gamma,\theta=1}^{2}A_{\alpha\beta\gamma\theta}(x)\xi_{\gamma\theta}\xi_{\alpha\beta}\ge\rho\sum_{\alpha,\beta=1}^{2}\xi_{\alpha\beta}^2\tag{1.13}$$

for all $x\in\Omega_p$ and for all real matrices $(\xi_{\alpha\beta})_{2\times2}$ with $\xi_{\alpha\beta}=\xi_{\beta\alpha}$ for $\alpha,\beta\in\{1,2\}$, where $\rho>0$ is a constant. Moreover it is supposed that the plate is clamped on Γ_3 (equation (1.6)) and is free on Γ_2 (see figure 1).

The vector $\vec\nu=(\nu_1,\nu_2)$ is the unitary outward normal to $\partial\Omega_p$ and $\tau=(\tau_1,\tau_2)=(-\nu_2,\nu_1)$ is the positive oriented unitary tangent vector. C is a positive constant depending on the material forming the membrane. f_p (resp. f_m) is the pressure supported by the plate (resp. the membrane) and depend on the transverse displacement u_p (resp. u_m) of the plate (resp. the membrane). The initial conditions g_p^0 and g_p^1 (resp. g_m^0 and g_m^1) are real functions defined on Ω_p (resp. Ω_m). The equations (1.4) and (1.7) are the boundary conditions expressing the coupling between the plate and the membrane.

We give the definition of weak solution for our semilinear problem (1.1)-(1.11) and with help of the theory of C^0-semigroups of linear operators we obtain a result of existence and uniqueness for this type of solution. For other works in the area of

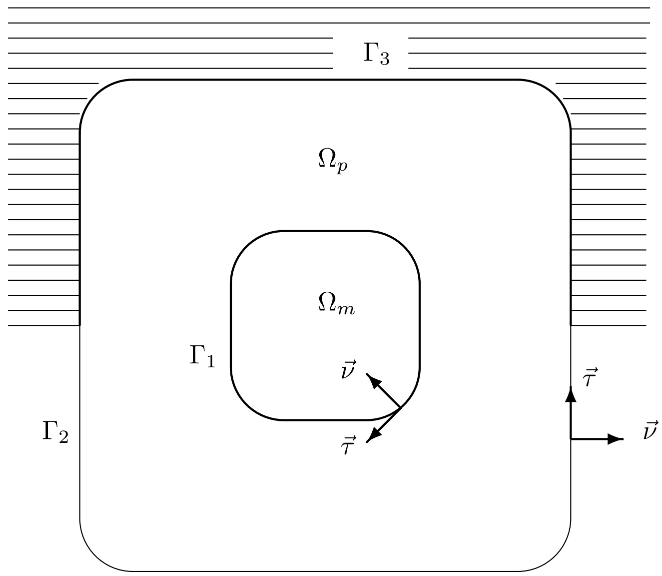

FIGURE 1. $\overline{\Omega}_m$ (resp. $\overline{\Omega}_p$) is occupied by the membrane (resp. the middle surface of the Plate). The Plate is clamped on Γ_3.

transmission problems and networks we refer the reader to [2, 3, 4, 6, 7, 10, 11, 12, 13, 14, 15].

2. Notation and mathematical preliminaries

In this section we shall present the concepts and abstract framework that we need for the treatment of our problem (1.1)-(1.11). We shall consider only real valued functions. Let n a positive integer. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_2)$ (i.e. $\alpha \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of all nonnegative integers), we write

$$\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \quad \text{where} \quad |\alpha| := \alpha_1 + \dots + \alpha_n.$$

Sometimes we write ∂_i for $\frac{\partial}{\partial x_i}$, $i = 1, \dots, n$. For the rest of this section, let Ω be an open bounded connected set in $\mathbb{R}^n$ with sufficiently smooth boundary.

For any nonnegative integer k let $C^k(\Omega)$ be the vector space consisting of all functions ϕ which, together with all their partial derivatives $\partial^\alpha \phi$ of orders $|\alpha| \le k$, are continuous in Ω. $C^\infty(\Omega)$ is the vector space consisting of all functions ϕ, such that $\phi \in C^k(\Omega)$ for all nonnegative integer k.

We write $C^k(\overline{\Omega})$ for the Banach space consisting of all functions $\phi \in C^k(\Omega)$ for which $\partial^\alpha \phi$ is bounded and uniformly continuous on Ω for $|\alpha| \le k$, with norm given by

$$\|\phi\|_{C^k(\overline{\Omega})} := \max_{|\alpha| \le k} \sup_{x \in \Omega} |\partial^\alpha \phi(x)|.$$

For a nonnegative integer k and $1 \leq p \leq \infty$ let $W^{k,p}(\Omega)$ be the usual Sobolev space defined as

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega); \partial^\alpha u \in L^p(\Omega) \text{for all} \alpha \in \mathbb{N}_0^n, |\alpha| \leq k\}, \tag{2.1}$$

where $\partial^\alpha u$ is understood in distributional (or weak) sense, with the usual norm

$$\|u\|_{k,p,\Omega} := \Big\{ \sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha u(x)|^p dx \Big\}^{1/p} \quad \text{if} 1 \leq p < \infty, \tag{2.2}$$

$$\|u\|_{k,\infty,\Omega} := \max_{|\alpha| \leq k} \operatorname*{ess\,sup}_{x \in \Omega} |\partial^\alpha u(x)|. \tag{2.3}$$

As usual we shall write $H^k(\Omega) := W^{k,2}(\Omega)$.

Lemma 2.1. *The set $\mathcal{D}(\overline{\Omega})$ of restrictions to Ω of functions in $C_c^\infty(\mathbb{R}^n)$ (i.e. the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support) is dense in $W^{k,p}(\Omega)$ for $1 \leq p < \infty$.*

For the proof of the above lemma, see Adams [1, theorem 3.18,].

Lemma 2.2. *If $kp = n$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $p \leq q < \infty$.*

For the proof of the above lemma, see Adams [1, lemma 5.14].

Lemma 2.3. *If $kp > n$, then $W^{k,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$.*

The proof of the above lemma can be found in Evans [9, sec. 5.6, Theorem 6] and in Adams [1, lemma 5.17].

Lemma 2.4. *Let $1 \leq p < \infty$. Then there exists a linear operator*

$$\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega) \tag{2.4}$$

such that

(i) *$\gamma_0 u = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$.*
(ii) *$\|\gamma_0 u\|_{L^p(\partial\Omega)} \leq c(p,\Omega)\|u\|_{1,p,\Omega}$ for each $u \in W^{1,p}(\Omega)$, where $c(p,\Omega)$ is a constant depending only on p and Ω.*

For the proof of the above lemma, see Evans [9, theorem 5.5.1].

Remark 2.5. We call $\gamma_0 u$ the trace of order zero of u on $\partial\Omega$.

Definition 2.6. Let $j, k \in \mathbb{N}$, $k > 1$, $1 \leq j \leq k-1$ and $u \in W^{k,p}(\Omega)$. We define the trace of order j of u on $\partial\Omega$ by

$$\gamma_j u := \sum_{|\alpha|=j} \frac{j!}{\alpha_1! \cdots \alpha_n!} \gamma_0(\partial^\alpha u) \nu_1^{\alpha_1} \cdots \nu_n^{\alpha_n}, \tag{2.5}$$

where $\vec{\nu} = (\nu_1, \dots, \nu_n)$ is the unit outward normal along $\partial\Omega$.

Remark 2.7. $\gamma_j : W^{k,p}(\Omega) \to L^p(\partial\Omega)$ is a linear operator with

(i) $\gamma_j u = \frac{\partial^j u}{\partial \vec{\nu}^j}\Big|_{\partial\Omega} := \sum_{|\alpha|=j} \frac{j!}{\alpha_1! \cdots \alpha_n!} \partial^\alpha u|_{\partial\Omega} \nu_1^{\alpha_1} \cdots \nu_n^{\alpha_n}$ for $j = 1, \dots, k-1$ if $u \in W^{k,p}(\Omega) \cap C^{k-1}(\overline{\Omega})$.
(ii) $\|\gamma_j u\|_{L^p(\partial\Omega)} \leq c(k,p,\Omega)\|u\|_{k,p,\Omega}$ for each $u \in W^{k,p}(\Omega)$ and for all $j = 1, \dots, k-1$.

Now for $j, k \in \mathbb{N}_0$, $0 \le j \le k$, and $1 \le p < \infty$ we define the the functional given by

$$|u|_{j,p,\Omega} := \Big\{ \sum_{|\alpha|=j} \int_\Omega |\partial^\alpha u(x)|^p dx \Big\}^{1/p}, \quad u \in W^{k,p}(\Omega). \tag{2.6}$$

Clearly, $|u|_{0,p,\Omega} = \|u\|_{0,p,\Omega} = \|u\|_{L^p(\Omega)}$. We have the following statement.

Lemma 2.8. *The functional*

$$((u))_{k,p,\Omega} = \Big\{ |u|^p_{k,p,\Omega} + |u|^p_{0,p,\Omega} \Big\}^{1/p}$$

is a norm on $W^{k,p}(\Omega)$, equivalent to the usual norm $\|\cdot\|_{k,p,\Omega}$.

The proof of the above lemma can be found in Adams [1, corollary 4.16].

We need some crucial results of the theory of semigroups of linear operators in Banach spaces. We refer to Pazy [16] or Dautray-Lions [8], chapter XVII, with respect to this theory.

Let V (resp. H) be a real separable Hilbert space with scalar product $(\cdot|\cdot)_V$ (resp. $(\cdot|\cdot)_H$) and norm $\|\cdot\|_V$ (resp. $\|\cdot\|_H$). We assume $V \hookrightarrow H$ and V dense in H.
Let $a(\cdot|\cdot) : V \times V \to \mathbb{R}$ be a continuous bilinear form, V-coercive with respect to H i.e., there exists $\lambda_0 \in \mathbb{R}$ and $c_0 > 0$ such that

$$a(v|v) + \lambda_0 \|v\|_H^2 \ge c_0 \|v\|_V^2, \quad \forall v \in V. \tag{2.7}$$

We put

$$D(\mathcal{A}) := \{u \in V; V \ni v \mapsto a(u|v) \text{is continuous for the topology of} H\}. \tag{2.8}$$

Theorem 2.9. *Let $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ be the operator given by $(\mathcal{A}u|v)_H = a(u|v)$ $\forall u \in D(\mathcal{A})$ and $\forall v \in V$. Then $-\mathcal{A}$ is the infinitesimal generator of a C^0- semigroup $\{T(t)\}_{t\ge0}$ in H which satisfies*

$$\|T(t)\|_{\mathcal{L}(H)} \le e^{\lambda_0 t} \quad \forall t \ge 0\,.$$

For a proof of the above theorem, see Dautray-Lions [8, theorem XVII.3.3].

Now we assume furthermore that $a(\cdot|\cdot)$ is symmetrical ($a(u|v) = a(v|u)$ $\forall u, v \in V$). Let $\mathcal{H} := V \times H$. $\mathcal{H}$ equipped with the scalar product defined by $(u|v)_{\mathcal{H}} := a(u_1|v_1) + (u_2|v_2)_H$ for $u = (u_1, u_2)^t, v = (v_1, v_2)^t \in \mathcal{H}$ (we write the elements of $\mathcal{H}$ as columns) is a Hilbert space (cf. Dautray-Lions [8], Section VII.3.4., p. 331).

Let $D(\mathbb{A}) := D(\mathcal{A}) \times V$. We define the operator $\mathbb{A}$ over $D(\mathbb{A})$ by

$$\mathbb{A}u := \begin{pmatrix} 0 & -id \\ \mathcal{A} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -u_2 \\ \mathcal{A}u_1 \end{pmatrix}, \quad \forall u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in D(\mathbb{A}). \tag{2.9}$$

It follows that $D(\mathbb{A})$ is dense in $\mathcal{H}$ and $\mathbb{A}$ is a closed operator.

Theorem 2.10. *$-\mathbb{A}$ is the infinitesimal generator of a C^0-semigroup in $\mathcal{H}$.*

For the proof of the above theroem, see Dautray-Lions [8, theorem XVII.3.4].

Theorem 2.11. *Let $-A$ be the infinitesimal generator of a C^0-semigroup of linear operators on a Banach space X and $u_0 \in D(A)$. If $f : [t_0, T] \times X \to X$*

is continuously differentiable with bounded partial derivatives then there exists a unique classical solution $u \in C^1([t_0, T]; X)$ *of the initial value problem*

$$\begin{gathered} \frac{du(t)}{dt} + Au(t) = f(t, u(t)) \quad \text{in } X, \text{ on }]t_0, T] \\ u(t_0) = u_0 \,. \end{gathered} \tag{2.10}$$

The proof of this lemma ca be found in Pazy [16, theorem 6.1.5].

3. Function spaces and bilinear forms for the semilinear problem plate-membrane

We define the vector space (with the usual vectorial sum and multiplication by scalars)

$$V := \big\{ (u_p, u_m) \in H^2(\Omega_p) \times H^1(\Omega_m); u_p|_{\Gamma_3} = \gamma_1 u_p|_{\Gamma_3} = 0, u_p|_{\Gamma_1} = \gamma_0 u_m|_{\Gamma_1} \big\} \tag{3.1}$$

(In this work we only consider real vector spaces). The vector space V, endowed with the inner product

$$((u_p, u_m)|(v_p, v_m))_V := (u_p|v_p)_{H^2(\Omega_p)} + (u_m|v_m)_{H^1(\Omega_m)}, \tag{3.2}$$

is a separable Hilbert space. The norm in V is given by

$$\|(u_p, u_m)\|_V := \big(\|u_p\|^2_{2,2,\Omega_p} + \|u_m\|^2_{1,2,\Omega_m} \big)^{1/2}. \tag{3.3}$$

We consider also

$$H := L^2(\Omega_p) \times L^2(\Omega_m) \tag{3.4}$$

with inner product and norm given by

$$((u_p, u_m)|(v_p, v_m))_H := (u_p|v_p)_{L^2(\Omega_p)} + (u_m|v_m)_{L^2(\Omega_m)} \tag{3.5}$$

and

$$\|(u_p, u_m)\|_H := \Big(\|u_p\|^2_{0,2,\Omega_p} + \|u_m\|^2_{0,2,\Omega_m} \Big)^{1/2}. \tag{3.6}$$

Also we consider

$$\tilde V := \big\{ (\tilde u_p, \tilde u_m) \in H^2(\Omega_p) \times H^1(\Omega_m); \big(\frac{1}{\sqrt{\rho_p h}} \tilde u_p, \frac{1}{\sqrt{\rho_m}} \tilde u_m \big) \in V \big\}, \tag{3.7}$$

endowed with the norm

$$\|(\tilde u_p, \tilde u_m)\|_{\tilde V} := \big(\frac{1}{\rho_p h} \|\tilde u_p\|^2_{2,2,\Omega_p} + \frac{1}{\rho_m} \|\tilde u_m\|^2_{1,2,\Omega_m} \big)^{1/2}. \tag{3.8}$$

We have the imbedding $\tilde V \hookrightarrow H$ with $\tilde V$ dense in H. Identifying H with its dual H' we obtain $\tilde V \overset{i}{\hookrightarrow} H = H' \overset{i'}{\hookrightarrow} \tilde V'$, where $i : \tilde V \to H$ is the identity operator and $i' : H \to \tilde V'$ is the dual operator of $i : V \to H$. Since $i : \tilde V \to H$ is injective and its range is dense in H, the same holds for $i' : H \to \tilde V'$. Furthermore we identify $i'f$ with f for $f \in H$. Therefore we regard H as subspace of $\tilde V'$.

We consider the symmetric bilinear form

$$\begin{aligned} &a((u_p, u_m)|(v_p, v_m)) \\ &:= \frac{h^3}{12} \sum_{\alpha,\beta,\gamma,\theta=1}^{2} \int_{\Omega_p} A_{\alpha\beta\gamma\theta} \frac{\partial^2 u_p}{\partial x_\gamma \partial x_\theta} \frac{\partial^2 v_p}{\partial x_\alpha \partial x_\beta} dx + C \int_{\Omega_m} \nabla u_m \cdot \nabla v_m dx \end{aligned} \tag{3.9}$$

for $(u_p, u_m), (v_p, v_m) \in V$ (The symmetry is a consequence of the assumption (1.12)). For technical reasons it is convenient to consider also

$$\tilde{a}\left((\tilde{u}_p, \tilde{u}_m)|(\tilde{v}_p, \tilde{v}_m)\right) := a\Big(\big(\frac{1}{\sqrt{\rho_p h}}\tilde{u}_p, \frac{1}{\sqrt{\rho_m}}\tilde{u}_m\big)\big|\big(\frac{1}{\sqrt{\rho_p h}}\tilde{v}_p, \frac{1}{\sqrt{\rho_m}}\tilde{v}_m\big)\Big) \tag{3.10}$$

for $(\tilde{u}_p, \tilde{u}_m), (\tilde{v}_p, \tilde{v}_m) \in \tilde{V}$.

Lemma 3.1. *Under the assumptions introduced for the coefficients $A_{\alpha\beta\gamma\theta}$, the bilinear form (3.9) (resp. (3.10)) is continuous and V-coercive (resp. $\tilde{V}-coercive$) with respect to H.*

Proof. From the Schwarz inequality we have the continuity of the bilinear forms (3.9) and (3.10). Now let $u = (u_p, u_m) \in V$. From Lemma 2.8 we have that there exists $c_p > 0$ such that

$$((u_p))_{2,2,\Omega_p} \geq c_p \|u_p\|_{2,2,\Omega_p}.$$

Then

$$\begin{aligned} a(u|u) &= \frac{h^3}{12} \sum_{\alpha,\beta,\gamma,\theta=1}^{2} \int_{\Omega_p} A_{\alpha,\beta,\gamma,\theta} \frac{\partial^2 u_p}{\partial x_\gamma \partial x_\theta} \frac{\partial^2 u_p}{\partial x_\alpha \partial x_\beta} dx + C \int_{\Omega_m} |\nabla u_m|^2 dx \\ &\geq \frac{h^3}{12}\rho \sum_{\alpha,\beta=1}^{2} \int_{\Omega_p} \Big|\frac{\partial^2 u_p}{\partial x_\alpha \partial x_\beta}\Big|^2 dx + C|u_m|^2_{1,2,\Omega_m} \\ &= \frac{h^3}{12}\rho |u_p|^2_{2,2,\Omega_p} + C|u_m|^2_{1,2,\Omega_m} \\ &\geq \frac{h^3}{12}\rho c_p \|u_p\|^2_{2,2,\Omega_p} - \frac{h^3}{12}\rho |u_p|^2_{0,2,\Omega_p} + C\|u_m\|^2_{1,2,\Omega_m} - C|u_m|^2_{0,2,\Omega_m}. \end{aligned}$$

With $\lambda_0 := \max\left\{\frac{h^3}{12}\rho, C\right\}$ and $c_0 := \min\left\{\frac{h^3}{12}\rho c_p, C\right\}$ we obtain the V-coerciveness of $a(\cdot|\cdot)$ with respect to H. From this follows immediately the $\tilde{V}$-coerciveness of $\tilde{a}(\cdot|\cdot)$ with respect to H. ☐

Let$D(\tilde{\mathcal{A}}) := \tilde{A}^{-1}(H)$ and $\tilde{\mathcal{A}} := \tilde{A}|_{D(\tilde{\mathcal{A}})}$, where $\tilde{A} : \tilde{V} \to \tilde{V}'$ is given by $\langle \tilde{A}\tilde{u}|\tilde{v}\rangle = \tilde{a}(\tilde{u}|\tilde{v})$, for all $\tilde{u}, \tilde{v} \in \tilde{V}$. We have that $-\tilde{\mathcal{A}}$ is the infinitesimal generator of a C^0-semigroup in H (see [11, p. 54].

4. Weak solution

For the function

$$(t, x, u) \mapsto f_p(t, x, u) : [0, T] \times \Omega_p \times \mathbb{R} \to \mathbb{R} \tag{4.1}$$

we assume the following:

- (i) For all $t \in [0, T]$, $x \mapsto f_p(t, x, u(x)) : \Omega_p \to \mathbb{R}$ is measurable, if $u : \Omega_p \to \mathbb{R}$ is measurable.
- (ii) $|f_p(t, x, u)| \leq q_p(t, x) + k_p|u|$ for all $(t, x, u) \in [0, T] \times \Omega_p \times \mathbb{R}$, where $q_p(t, \cdot) \in L^2(\Omega_p)$ for all $t \in [0, T]$ and $k_p > 0$ is a constant.
- (iii) $\frac{\partial f_p}{\partial t}(t, x, u)$ exists for all $(t, x, u) \in [0, T] \times \Omega_p \times \mathbb{R}$. It is bounded and Lipschitz continuous on $[0, T] \times \Omega_p \times \mathbb{R}$.
- (iv) $\frac{\partial f_p}{\partial u}(t, x, u)$ exists for all $(t, x, u) \in [0, T] \times \Omega_p \times \mathbb{R}$. It is bounded and Lipschitz continuous on $[0, T] \times \Omega_p \times \mathbb{R}$.

For the function

$$(t,x,u)\mapsto f_m(t,x,u):[0,T]\times\Omega_m\times\mathbb{R}\to\mathbb{R} \tag{4.2}$$

we assume the following:

(i) For all $t\in[0,T]$, $x\mapsto f_m(t,x,u(x)):\Omega_m\to\mathbb{R}$ is measurable, if $u:\Omega_m\to\mathbb{R}$ is measurable.

(ii) $|f_m(t,x,u)|\le q_m(t,x)+k_m|u|$, for all $(t,x,u)\in[0,T]\times\Omega_m\times\mathbb{R}$, where $q_m(t,\cdot)\in L^2(\Omega_m)$ for all $t\in[0,T]$ and $k_m>0$ a constant.

(iii) $\frac{\partial f_m}{\partial t}(t,x,u)$ exists for all $(t,x,u)\in[0,T]\times\Omega_m\times\mathbb{R}$. It is bounded and Lipschitz continuous on $[0,T]\times\Omega_m\times\mathbb{R}$.

(iv) $\frac{\partial f_m}{\partial u}(t,x,u)$ exists for all $(t,x,u)\in[0,T]\times\Omega_m\times\mathbb{R}$. It is bounded and Lipschitz continuous on $[0,T]\times\Omega_m\times\mathbb{R}$.

Let $\mathbf{f}_p:[0,T]\times L^2(\Omega_p)\to L^2(\Omega_p)$ and $\mathbf{f}_m:[0,T]\times L^2(\Omega_m)\to L^2(\Omega_m)$ be defined by

$$[\mathbf{f}_p(t,u_p)](x):=f_p(t,x,u_p(x))\quad\text{for } (t,x)\in[0,T]\times\Omega_p\ u_p\in L^2(\Omega_p)\,, \tag{4.3}$$

$$[\mathbf{f}_m(t,u_m)](x):=f_m(t,x,u_m(x))\quad\text{for } (t,x)\in[0,T]\times\Omega_m\ u_m\in L^2(\Omega_m). \tag{4.4}$$

From assumptions on (4.1) and (4.2), we see that $\mathbf{f}_p(t,u_p)\in L^2(\Omega_p)$ and $\mathbf{f}_m(t,u_m)\in L^2(\Omega_m)$, for $u_p\in L^2(\Omega_p)$ and $u_m\in L^2(\Omega_m)$.

For technical reasons we introduce the following functions:

$$\tilde{\mathbf{f}}_p(t,u_p):=\frac{1}{\sqrt{\rho_p h}}\mathbf{f}_p\big(t,\frac{1}{\sqrt{\rho_p h}}u_p\big)\quad\text{for } t\in[0,T]\ u_p\in L^2(\Omega_p)\,, \tag{4.5}$$

$$\tilde{\mathbf{f}}_m(t,u_m):=\frac{1}{\sqrt{\rho_m}}\mathbf{f}_m\big(t,\frac{1}{\sqrt{\rho_m}}u_m\big)\quad\text{for } t\in[0,T]\ u_m\in L^2(\Omega_m)\,. \tag{4.6}$$

Let us suppose that $u_p:[0,T]\times\overline{\Omega}_p\to\mathbb{R}$ and $u_m:[0,T]\times\overline{\Omega}_m\to\mathbb{R}$ are smooth enough in such a way that the system (1.1) - (1.11) for (u_p,u_m) holds; i.e., we suppose that (u_p,u_m) is a classical solution of the semilinear problem (1.1)-(1.11). Furthermore we assume that $(\tilde u_p(t,.),\tilde u_m(t,.))\in D(\tilde{\mathcal{A}})$ for $t\in]0,T]$, where $(\tilde u_p,\tilde u_m):=\big(\sqrt{\rho_p h}u_p,\sqrt{\rho_m}u_m\big)$. If we multiply (1.1) (resp. (1.2)) with $\frac{1}{\sqrt{\rho_p h}}\tilde v_p$ (resp. $\frac{1}{\sqrt{\rho_m}}\tilde v_m$), where $(\tilde v_p,\tilde v_m)\in\tilde V$, by use of integration by parts, (1.3)-(1.7) and the fact that $\tilde V$ is dense in H we obtain

$$\big(\frac{\partial^2\tilde u_p}{\partial t^2}(t,\cdot),\frac{\partial^2\tilde u_m}{\partial t^2}(t,\cdot)\big)+\tilde{\mathcal{A}}(\tilde u_p(t,\cdot),\tilde u_m(t,\cdot))=\big(\tilde{\mathbf{f}}_p(t,\tilde u_p(t,\cdot)),\tilde{\mathbf{f}}_m(t,\tilde u_m(t,\cdot))\big) \tag{4.7}$$

in H, for $t\in]0,T]$. On the other hand we have

$$\tilde u_p(0,\cdot)=\tilde g_p^0,\quad \tilde u_m(0,\cdot)=\tilde g_m^0,\quad \frac{\partial\tilde u_p}{\partial t}(0,\cdot)=\tilde g_p^1,\quad \frac{\partial\tilde u_p}{\partial t}(0,\cdot)=\tilde g_m^1, \tag{4.8}$$

where $\tilde g_p^0:=\sqrt{\rho_p h}g_p^0$, $\tilde g_m^0:=\sqrt{\rho_m}g_m^0$, $\tilde g_p^1:=\sqrt{\rho_p h}g_p^1$ and $\tilde g_m^1:=\sqrt{\rho_m}g_m^1$.
We suppose

$$(i)\ (g_p^0,g_m^0)\in A^{-1}(H),\quad (ii)\ (g_p^1,g_m^1)\in V \tag{4.9}$$

where $A:V\to V'$ is given by $\langle Au|v\rangle=a(u|v)$, for all $u,v\in V$.

Equations (4.7) and (4.8) motivate the following definition: Consider the Hilbert space $\mathcal{H} := \tilde{V} \times H$ endowed with the inner product

$$\Big(\begin{pmatrix} (\tilde{u}_p^1, \tilde{u}_m^1) \\ (\tilde{u}_p^2, \tilde{u}_m^2) \end{pmatrix} \Big| \begin{pmatrix} (\tilde{v}_p^1, \tilde{v}_m^1) \\ (\tilde{v}_p^2, \tilde{v}_m^2) \end{pmatrix} \Big)_{\mathcal{H}} := a((\tilde{u}_p^1, \tilde{u}_m^1)|(\tilde{v}_p^1, \tilde{v}_m^1)) + ((\tilde{u}_p^2, \tilde{u}_m^2)|(\tilde{v}_p^2, \tilde{v}_m^2))_H. \tag{4.10}$$

Moreover let $D(\tilde{\mathbb{A}}) := D(\tilde{\mathcal{A}}) \times \tilde{V}$ and $\tilde{\mathbb{A}} := \begin{pmatrix} 0 & -id \\ \tilde{\mathcal{A}} & 0 \end{pmatrix}$. It follows from theorem 2.10 that $-\tilde{\mathbb{A}}$ is the infinitesimal generator of a C^0-semigroup of contractions in $\mathcal{H}$. We put

$$\tilde{\mathbb{F}}(t, \tilde{\mathbb{U}}) := \begin{pmatrix} 0 \\ \big(\tilde{\mathbf{f}}_p(t, \tilde{\mathbf{u}}_p^1), \tilde{\mathbf{f}}_m(t, \tilde{\mathbf{u}}_m^1)\big) \end{pmatrix} \quad \text{for } \tilde{\mathbb{U}} := \begin{pmatrix} (\tilde{\mathbf{u}}_p^1, \tilde{\mathbf{u}}_m^1) \\ (\tilde{\mathbf{u}}_p^2, \tilde{\mathbf{u}}_m^2) \end{pmatrix} \in \mathcal{H}, \tag{4.11}$$

$$\tilde{\mathbb{G}} := \begin{pmatrix} (\tilde{g}_p^0, \tilde{g}_m^0) \\ (\tilde{g}_p^1, \tilde{g}_m^1) \end{pmatrix}. \tag{4.12}$$

Next we define weak solution for our semilinear problem.

Definition 4.1. Assume that (1.12), (1.13), (4.1), (4.2) and (4.9) are satisfied. We say that a function $(\mathbf{u}_p, \mathbf{u}_m) \in C^1([0,T]; V) \cap C^2([0,T]; H)$ is a weak solution of the semilinear problem (1.1)-(1.11) if the function

$$(\tilde{\mathbf{u}}_p, \tilde{\mathbf{u}}_m) := \big(\sqrt{\rho_p h}\mathbf{u}_p, \sqrt{\rho_m}\mathbf{u}_m\big) \in C^1([0,T]; \tilde{V}) \cap C^2([0,T]; H)$$

has the following properties:

$$\begin{aligned} &(i)\big(\frac{d^2\tilde{\mathbf{u}}_p(t)}{dt^2}, \frac{d^2\tilde{\mathbf{u}}_m(t)}{dt^2}\big) + \tilde{\mathcal{A}}(\tilde{\mathbf{u}}_p(t), \tilde{\mathbf{u}}_m(t)) = \big(\tilde{\mathbf{f}}_p(t, \tilde{\mathbf{u}}_p(t)), \tilde{\mathbf{f}}_m(t, \tilde{\mathbf{u}}_m(t))\big) \\ &\quad \text{in } H, \text{ on }]0,T] \\ &(ii)(\tilde{\mathbf{u}}_p(0), \tilde{\mathbf{u}}_m(0)) = (\tilde{g}_p^0, \tilde{g}_m^0). \\ &(iii)\big(\frac{d\tilde{\mathbf{u}}_\mathbf{p}}{dt}(0), \frac{d\tilde{\mathbf{u}}_\mathbf{m}}{dt}(0)\big) = (\tilde{g}_p^1, \tilde{g}_m^1). \end{aligned} \tag{4.13}$$

Lemma 4.2. *Assume* (1.12)*,* (1.13)*,* (4.1) *and* (4.2)*. Then the function* $(t, \mathbb{U}) \mapsto \mathbb{F}(t, \mathbb{U}) : [0,T] \times \mathcal{H} \to \mathcal{H}$ *which is defined by* (4.11)*, is continuously differentiable with bounded partial derivatives.*

Proof. **1.** The assumptions (4.1)(i),(ii) and (4.2)(i),(ii) lead to

$$\tilde{\mathbf{f}}_p(t, \tilde{\mathbf{u}}_p^1) \in L^2(\Omega_p) \quad \text{and} \quad \tilde{\mathbf{f}}_m(t, \tilde{\mathbf{u}}_m^1) \in L^2(\Omega_m)$$

for $\tilde{\mathbf{u}}_p^1 \in L^2(\Omega_p)$ and $\tilde{\mathbf{u}}_m^1 \in L^2(\Omega_m)$ and for all $t \in [0,T]$ (cf. [5, theorem 2.1]). Then we have $\tilde{\mathbb{F}}(t, \tilde{\mathbb{U}}) \in \mathcal{H}$ for $(t, \tilde{\mathbb{U}}) \in [0,T] \times \mathcal{H}$.

2. It follows from (4.1)(iii) that

$$\frac{\partial f_p}{\partial t}\big(t, \cdot, \frac{1}{\sqrt{\rho_p h}}\tilde{\mathbf{u}}_p^1(\cdot)\big) \in L^2(\Omega_p) \quad \forall t \in [0,T]\ \forall \tilde{\mathbf{u}}_p^1 \in L^2(\Omega_p).$$

Let $t \in [0,T]$. For $\tau \in \mathbb{R}$ with $-t \le \tau \le T-t$ we have

$$
\begin{aligned}
&\Big\| \frac{\tilde{\mathbf{f}}_p(t+\tau, \tilde{\mathbf{u}}_p^1) - \tilde{\mathbf{f}}_p(t, \tilde{\mathbf{u}}_p^1)}{\tau} - \frac{1}{\sqrt{\rho_p h}} \frac{\partial f_p}{\partial t}\big(t, \cdot, \frac{1}{\sqrt{\rho_p h}} \tilde{\mathbf{u}}_p^1(\cdot)\big) \Big\|^2_{L^2(\Omega_p)} \\
&= \int_{\Omega_p} \frac{1}{\rho_p h} \Big| \int_0^1 \Big[\frac{\partial f_p}{\partial t}\big(t+\xi\tau, x, \frac{1}{\sqrt{\rho_p h}} \tilde{\mathbf{u}}_p^1(x)\big) - \frac{\partial f_p}{\partial t}\big(t, x, \frac{1}{\sqrt{\rho_p h}} \tilde{\mathbf{u}}_p^1(x)\big)\Big] d\xi \Big|^2 dx \\
&\le \int_{\Omega_p} \frac{1}{\rho_p h} \Big[\int_0^1 \Big| \frac{\partial f_p}{\partial t}\big(t+\xi\tau, x, \frac{1}{\sqrt{\rho_p h}} \tilde{\mathbf{u}}_p^1(x)\big) - \frac{\partial f_p}{\partial t}\big(t, x, \frac{1}{\sqrt{\rho_p h}} \tilde{\mathbf{u}}_p^1(x)\big)\Big| d\xi \Big]^2 dx \\
&\le \frac{1}{\rho_p h} \text{const.} \mu_p(\Omega_p) \tau^2 \xrightarrow[\tau \to 0]{} 0
\end{aligned} \tag{4.14}
$$

The above inequality because the Lipschitz continuity of $\frac{\partial f_p}{\partial t}$.

3. It follows from (4.2)(iii) that

$$
\frac{\partial f_m}{\partial t}\big(t, \cdot, \frac{1}{\sqrt{\rho_m}} \tilde{\mathbf{u}}_m^1(\cdot)\big) \in L^2(\Omega_m) \quad \forall t \in [0,T]\ \forall \tilde{\mathbf{u}}_m^1 \in L^2(\Omega_m).
$$

Let $t \in [0,T]$. For $\tau \in \mathbb{R}$ with $-t \le \tau \le T-t$ we have as above

$$
\Big\| \frac{\tilde{\mathbf{f}}_m(t+\tau, \tilde{\mathbf{u}}_m^1) - \tilde{\mathbf{f}}_m(t, \tilde{\mathbf{u}}_m^1)}{\tau} - \frac{1}{\sqrt{\rho_m}} \frac{\partial f_m}{\partial t}\big(t, \cdot, \frac{1}{\sqrt{\rho_m}} \tilde{\mathbf{u}}_m^1(\cdot)\big) \Big\|^2_{L^2(\Omega_m)} \tag{4.15}
$$

approaches zero as $\tau \to 0$.

4. Let $(t, \tilde{\mathbb{U}}) \in [0,T] \times \mathcal{H}$ with $\tilde{\mathbb{U}} := \begin{pmatrix} (\tilde{\mathbf{u}}_p^1, \tilde{\mathbf{u}}_m^1) \\ (\tilde{\mathbf{u}}_p^2, \tilde{\mathbf{u}}_m^2) \end{pmatrix}$. We consider the operator $D_1\tilde{\mathbb{F}}(t, \tilde{\mathbb{U}}) \in \mathcal{L}(\mathbb{R}; \mathcal{H})$ which is defined by

$$
D_1\tilde{\mathbb{F}}(t, \tilde{\mathbb{U}})\tau := \begin{pmatrix} 0 \\ \big(\frac{1}{\sqrt{\rho_p h}} \frac{\partial f_p}{\partial t}\big(t, \cdot, \frac{1}{\sqrt{\rho_p h}} \tilde{\mathbf{u}}_p^1(\cdot)\big)\tau, \frac{1}{\sqrt{\rho_m}} \frac{\partial f_m}{\partial t}\big(t, \cdot, \frac{1}{\sqrt{\rho_m}} \tilde{\mathbf{u}}_m^1(\cdot)\big)\tau \big) \end{pmatrix} \tag{4.16}
$$

For $(t, \tilde{\mathbb{U}}) \in [0,T] \times \mathcal{H}$ and from (4.14) and (4.15) we have that

$$
\frac{\|\tilde{\mathbb{F}}(t+\tau, \tilde{\mathbb{U}}) - \tilde{\mathbb{F}}(t, \tilde{\mathbb{U}}) - D_1\tilde{\mathbb{F}}(t, \tilde{\mathbb{U}})\tau\|_{\mathcal{H}}}{|\tau|} \xrightarrow[-t \le \tau \le T-t,\ \tau \ne 0,\ \tau \to 0]{0} . \tag{4.17}
$$

Then there exists the partial derivative of $\tilde{\mathbb{F}}$ with respect to t for all $(t, \tilde{\mathbb{U}}) \in [0,T] \times \mathcal{H}$ and it is equal to $D_1\tilde{\mathbb{F}}(t, \tilde{\mathbb{U}})$. By the Lipschitz continuity of $\frac{\partial f_p}{\partial t}$ and $\frac{\partial f_m}{\partial t}$ it can be showed that

$$
\|D_1\tilde{\mathbb{F}}(t_1, \tilde{\mathbb{U}}_1) - D_1\tilde{\mathbb{F}}(t_2, \tilde{\mathbb{U}}_2)\|_{\mathcal{L}(\mathbb{R};\mathcal{H})} \le \text{const.}\big(|t_1 - t_2| + \|\tilde{\mathbb{U}}_1 - \tilde{\mathbb{U}}_2\|_{\mathcal{H}}\big). \tag{4.18}
$$

Then the maping

$$
(t, \tilde{\mathbb{U}}) \mapsto D_1\tilde{\mathbb{F}}(t, \tilde{\mathbb{U}}) : [0,T] \times \mathcal{H} \to \mathcal{L}(\mathbb{R}; \mathcal{H})
$$

is continuous. The boundedness of $\frac{\partial f_p}{\partial t}$ and $\frac{\partial f_m}{\partial t}$ implied by the boundedness of $D_1\tilde{\mathbb{F}}$.

5. From (4.1)(iv) and (4.2)(iv) we have

$$
\frac{\partial f_p}{\partial u}\big(t, \cdot, \frac{1}{\sqrt{\rho_p h}} \tilde{\mathbf{u}}_p^1(\cdot)\big)\tilde{\mathbf{v}}_p^1 \in L^2(\Omega_p)
$$

and

$$
\frac{\partial f_m}{\partial u}\big(t, \cdot, \frac{1}{\sqrt{\rho_m}} \tilde{\mathbf{u}}_m^1(\cdot)\big)\tilde{\mathbf{v}}_m^1 \in L^2(\Omega_m)
$$

for all $t \in [0,T]$ and all $(\tilde{\mathbf{u}}_p^1, \tilde{\mathbf{u}}_m^1), (\tilde{\mathbf{v}}_p^1, \tilde{\mathbf{v}}_m^1) \in H$. For $t \in [0,T]$, $\tilde{\mathbb{U}} := \begin{pmatrix} (\tilde{\mathbf{u}}_p^1, \tilde{\mathbf{u}}_m^1) \\ (\tilde{\mathbf{u}}_p^2, \tilde{\mathbf{u}}_m^2) \end{pmatrix} \in \mathcal{H}$ and $\tilde{\mathbb{V}} := \begin{pmatrix} (\tilde{\mathbf{v}}_p^1, \tilde{\mathbf{v}}_m^1) \\ (\tilde{\mathbf{v}}_p^2, \tilde{\mathbf{v}}_m^2) \end{pmatrix} \in \mathcal{H}$ we put

$$D_2\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}})\tilde{\mathbb{V}} := \begin{pmatrix} 0 \\ \big(\frac{1}{\rho_p h}\frac{\partial f_p}{\partial u}\big(t,\cdot,\frac{1}{\sqrt{\rho_p h}}\tilde{\mathbf{u}}_p^1(\cdot)\big)\tilde{\mathbf{v}}_p^1, \frac{1}{\rho_m}\frac{\partial f_m}{\partial u}\big(t,\cdot,\frac{1}{\sqrt{\rho_m}}\tilde{\mathbf{u}}_m^1(\cdot)\big)\tilde{\mathbf{v}}_m^1\big) \end{pmatrix} \tag{4.19}$$

Since $\frac{\partial f_p}{\partial u}$ (resp. $\frac{\partial f_m}{\partial u}$) is bounded on $[0,T]\times\Omega_p\times\mathbb{R}$ (resp. $[0,T]\times\Omega_m\times\mathbb{R}$), we see that $D_2\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}}) \in \mathcal{L}(\mathcal{H})$ for all $(t,\tilde{\mathbb{U}}) \in [0,T]\times\mathcal{H}$.

For $(t,\tilde{\mathbb{U}}) \in [0,T]\times\mathcal{H}$ and $\tilde{\mathbb{V}} \in \mathcal{H}$ with $\|\tilde{\mathbb{V}}\|_{\mathcal{H}} \neq 0$ we have (with "const" denoting different constants)

$$\begin{aligned}
&\frac{\|\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}}+\tilde{\mathbb{V}}) - \tilde{\mathbb{F}}(t,\tilde{\mathbb{U}}) - D_2\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}})\tilde{\mathbb{V}}\|_{\mathcal{H}}^2}{\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^2} \\
&\le \frac{\text{const}}{\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^2}\Big\{\int_{\Omega_p}\Big[\int_0^1 \big|\frac{\partial f_p}{\partial u}\big(t,x,\frac{1}{\sqrt{\rho_p h}}(\tilde{\mathbf{u}}_p^1(x)+\xi\tilde{\mathbf{v}}_p^1(x))\big) \\
&\quad - \frac{\partial f_p}{\partial u}\big(t,x,\frac{\tilde{\mathbf{u}}_p^1(x)}{\sqrt{\rho_p h}}\big)\big|d\xi\Big]^2 \frac{|\tilde{\mathbf{v}}_p^1(x)|^2}{\rho_p h}dx \\
&\quad + \int_{\Omega_m}\Big[\int_0^1 \big|\frac{\partial f_m}{\partial u}\big(t,x,\frac{1}{\sqrt{\rho_m}}(\tilde{\mathbf{u}}_m^1(x)+\xi\tilde{\mathbf{v}}_m^1(x))\big) \\
&\quad - \frac{\partial f_m}{\partial u}\big(t,x,\frac{\tilde{\mathbf{u}}_m^1(x)}{\sqrt{\rho_m}}\big)\big|d\xi\Big]^2 \frac{|\tilde{\mathbf{v}}_m^1(x)|^2}{\rho_m}dx\Big\} \\
&\le \frac{\text{const}}{\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^2}\Big\{\frac{1}{\rho_p^2 h^2}\int_{\Omega_p}|\tilde{\mathbf{v}}_p^1(x)|^4 dx + \frac{1}{\rho_m^2}\int_{\Omega_m}|\tilde{\mathbf{v}}_m^1(x)|^4 dx\Big\}.
\end{aligned} \tag{4.20}$$

The above holds because of the Lipschitz continuity of $\frac{\partial f_p}{\partial u}$ and $\frac{\partial f_m}{\partial u}$. Since

$$\tilde{\mathbf{v}}_p^1 \in H^2(\Omega_p) \hookrightarrow C^0(\overline{\Omega}_p) \hookrightarrow L^4(\Omega_p) \quad \text{and} \quad \tilde{\mathbf{v}}_m^1 \in H^1(\Omega_m) \hookrightarrow L^4(\Omega_m)$$

(see lemmas 2.2 and 2.3), from (4.20), we have

$$\begin{aligned}
&\frac{\|\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}}+\tilde{\mathbb{V}}) - \tilde{\mathbb{F}}(t,\tilde{\mathbb{U}}) - D_2\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}})\tilde{\mathbb{V}}\|_{\mathcal{H}}^2}{\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^2} \\
&\le \frac{\text{const.}}{\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^2}\big(\frac{1}{\rho_p^2 h^2}\|\tilde{\mathbf{v}}_p^1\|_{H^2(\Omega_p)}^4 + \frac{1}{\rho_m^2}\|\tilde{\mathbf{v}}_m^1\|_{H^1(\Omega_m)}^4\big) \\
&\le \frac{\text{const.}}{\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^2}\|(\tilde{\mathbf{v}}_p^1, \tilde{\mathbf{v}}_m^1)\|_{\tilde{V}}^4 \\
&\le \frac{\text{const.}}{\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^2}\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^4 = \text{const.}\|\tilde{\mathbb{V}}\|_{\mathcal{H}}^2.
\end{aligned} \tag{4.21}$$

It follows that the partial derivative of $\tilde{\mathbb{F}}$ with respect to the second variable $\tilde{\mathbb{U}}$ exists and it is equal to $D_2\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}})$ for all $(t,\tilde{\mathbb{U}}) \in [0,T]\times\mathcal{H}$. We can show similarly that the Lipschitz continuity (resp. the boundedness) of $\frac{\partial f_p}{\partial u}$ and $\frac{\partial f_m}{\partial u}$ leads to the continuity (resp. the boundedness) of

$$(t,\tilde{\mathbb{U}}) \mapsto D_2\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}}) : [0,T]\times\mathcal{H} \to \mathcal{L}(\mathcal{H}).$$

So the proof is complete. □

Lemma 4.3. *Let* $\tilde{\mathbb{F}} : [0,T]\times\mathcal{H}\to\mathcal{H}$ *(resp.* $\tilde{\mathbb{G}}$*) be defined by* (4.11) *(resp. (4.12)). Under assumptions* (1.12)*,* (1.13)*,* (4.9)*,* (4.1) *and* (4.2)*, there exists a unique function* $\tilde{\mathbb{U}} : [0,T]\to\mathcal{H}$ *with the following properties:*

$$\begin{gathered}(i)\tilde{\mathbb{U}}\in C^1([0,T];\mathcal{H}).\\ (ii)\frac{d\tilde{\mathbb{U}}(t)}{dt}+\tilde{\mathbb{A}}\tilde{\mathbb{U}}(t)=\tilde{\mathbb{F}}(t,\tilde{\mathbb{U}}(t))\quad in\ \mathcal{H}\quad \text{on}]0,T].\\ (iii)\tilde{\mathbb{U}}(0)=\tilde{\mathbb{G}}.\end{gathered}\tag{4.22}$$

Proof. **1.** It follows from theorem 2.10 that $-\tilde{\mathbb{A}}$ is the infinitesimal generator of a C^0-semigroup of linear operators in $\mathcal{H}$.
2. From lemma 4.2 we have that $\tilde{\mathbb{F}} : [0,T]\times\mathcal{H}\to\mathcal{H}$ is continuously differentiable with bounded partial derivatives.
3. It can be seen that $\tilde{\mathbb{G}}$ belongs to $D(\tilde{\mathbb{A}})$.
4. From theorem 2.11 we have the desired result. □

Theorem 4.4. *Under assumptions* (1.12)*,* (1.13)*,* (4.9)*,* (4.1) *and* (4.2)*, there exists a unique weak solution of the semilinear problem* (1.1)*-*(1.11)*.*

Proof. Let

$$\tilde{\mathbb{U}} := \begin{pmatrix}(\tilde{\mathbf{u}}_p^1,\tilde{\mathbf{u}}_m^1)\\(\tilde{\mathbf{u}}_p^2,\tilde{\mathbf{u}}_m^2)\end{pmatrix} : [0,T]\to\mathcal{H}$$

be the unique function satisfying (4.22) (Lemma 4.3). It can be showed that $(\tilde{\mathbf{u}}_p^1,\tilde{\mathbf{u}}_m^1)$ belongs to $C^1([0,T];\tilde{V})\cap C^2([0,T];H)$ and that it satisfies (4.13). Then $(\frac{1}{\sqrt{\rho_p h}}\tilde{\mathbf{u}}_p^1,\frac{1}{\sqrt{\rho_m}}\tilde{\mathbf{u}}_m^1)$ is the desired weak solution. The uniqueness follows from the uniqueness of $\tilde{\mathbb{U}}$. □

Remark 4.5. For sufficiently smooth solutions in the sense of definition 4.1 we can obtain as usual a classical pointwise solution of system (1.1)-(1.11). See [12].

References

[1] Adams, R. A.; *Sobolev Spaces*, Academic Press, Inc., Boston. 1978.
[2] Ali Mehmeti, F.[*Lokale und globale Lösungen linearer und nichtlinearer hyperbolischer Evolutionsgleichungen mit Transmission*, Dissertation, Johannes Gutenberg-Universität Mainz. 1987.
[3] Ali Mehmeti, F. *Regular Solutions of Transmission and Interaction Problems for Wave Equations*, Mathematical Methods in the Applied Sciences, Vol. 11 (1989), 665-685.
[4] Ali Mehmeti, F.; *Nonlinear Waves in Networks*, Mathematical Research, volume 80, Akademie-Verlag, Berlin. 1994.
[5] Appell??
[6] Arango, J. A., Lebedev, L. P. and Vorovich, I. I.; *Some boundary value problems and models for coupled elastic bodies*, Quarterly of Applied Mathematics, Vol LVI, Number 1 (March 1998), 157-172.
[7] Ciarlet, P. G., Le Dret, H. and Nzengwa, R.; *Junctions between three-dimensional and two-dimensional linearly elastic structures*, J. Math. pures et appl. 68 (1989), 261-295.
[8] Dautray, R., Lions, J. L.; *Mathematical Analysis and Numerical Methods for Science and Technology. Vol. 5. Evolution Problems I*, Springer-Verlag, Berlin. 1992.
[9] Evans, L. C.; *Partial Differential Equations*, Graduate Studies in Mathematics, Volume 19, American Mathematical Society, Providence, Rhode Island. 1998.

[10] Hernández, J.; *Modelos Matemáticos para la deformación de placas y membranas acopladas*, Tesis de Maestría, Universidad del Norte-Universidad del Valle, Barranquilla. 1997.

[11] Hernández, J.; *Evolutionsgleichungen für gekoppelte elastische dünne Platten mit Membranen*, Johannes Gutenberg - Universität Mainz, Preprint-Reihe des Fachbereichs Mathematik, Preprint Nr. 12. 2002.

[12] Hernández, J.; *Evolutionsgleichungen für gekoppelte elastische dünne Platten mit Membranen*, Dissertation, Johannes Gutenberg-Universität Mainz. 2002.

[13] Mercier, D. *Some systems of PDE on polygonal networks*, In: Ali Mehmeti, F., von Below, J. and Nicaise, S. eds, Partial differential equations on multistructures, Lecture notes in pure and applied mathematics, Vol. 219, Marcel Dekker, Inc., New York (2001), 163-182.

[14] Nicaise, S., Sändig, A-M.; *General Interface Problems-I*, Mathematical Methods in the Applied Sciences, Vol. 17 (1994), 395-429.

[15] Nicaise, S., Sändig, A-M.; *General Interface Problems-II*, Mathematical Methods in the Applied Sciences, Vol. 17 (1994), 431-450.

[16] Pazy, A.; *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York. 1983 .

Universidad del Norte, Km 5 via a Puerto Colombia, Barranquilla, Colombia
E-mail address: jahernan@uninorte.edu.co