2003 Colloquium on Differential Equations and Applications, Maracaibo, Venezuela.
Electronic Journal of Differential Equations, Conference 13, 2005, pp. 21–27.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

THE RELATIVISTIC ENSKOG EQUATION NEAR THE VACUUM

RAFAEL GALEANO ANDRADES, BERNARDO OROZCO HERRERA,
MARIA OFELIA VASQUEZ AVILA

Abstract. We prove an existence and uniqueness theorem for the solution with data near the vacuum in the Hard sphere.

1. Introduction

The relativistic Boltzmann equation is written as

$$V \cdot \nabla_x F = -C(F, F),$$

where the dot represents the Lorentz inner product $(+ - --)$ of 4-vectors $v = (v_1, v_2, v_3)$, $V = (v_0, v_1, v_2, v_3)$, $X = (x_0, x_1, x_2, x_3)$, $x = (x_1, x_2, x_3)$, $x_0 = -t$ and $C(F, F)$ is the collision integral. Normalizing the speed of light $c = 1$ and the particle mass $m = 1$, we have $V \cdot V = 1$ or $v_0 = \sqrt{1 + |v|^2}$.

For convenience, we separate the time and space variables, and then divide by v_0 the relativistic Bolttzman equation to obtain,

$$\partial_t F + \hat{v} \cdot \nabla_x F = Q(F, F) \tag{1.1}$$

where

$$Q(F, F) = v_0^{-1} C(F, F) \text{ y } \hat{v} = \frac{v}{v_0} = \frac{v}{\sqrt{1 + |v|^2}}\,,$$

$$\begin{aligned} Q(F, F)(v) = &\frac{1}{2v_0} \int\int\int \delta(U^2 - 1)\delta(U'^2 - 1)\delta(V'^2 - 1) s\sigma(s, \theta)\delta^4 \\ &\times (U + V - U' - V')[F(u')F(v') - F(u)F(v)] d^4U d^4U' d^4V \end{aligned}$$

where $U^2 = U \cdot U = u_0^2 - |u|^2$, $|u|^2 = u_1^2 + u_2^2 + u_3^2$, δ is the delta function in one variable, δ^4 is the delta function in four variables, and all of the F are evaluated at the same space-time point (t, x). Furthermore $\sigma(s, \theta)$ is called the *differential cross section or the scattering kernel*; it is a function of variables s and θ which will be defined below. The delta functions express the conservation of momentum and

2000 *Mathematics Subject Classification.* 35Q75, 82-02.
Key words and phrases. Relativistic; Enskog equation; near the vacuum.

Published May 30, 2005.

energy:

$$u' + v' = u + v\,.$$
$$\sqrt{1+|u'|^2} + \sqrt{1+|v'|^2} = \sqrt{1+|u|^2} + \sqrt{1+|v|^2}$$

Let us begin by defining the remaining variables in the collision integral. We define

$$\begin{aligned} S = (U+V)^2 &= (u_0+v_0)^2 - |u+v|^2 \\ &= 2u_0v_0 - 2u\cdot v + u_0^2 - |u|^2 + v_0^2 - |v|^2 \\ &= 2(\sqrt{1+|u|^2}\sqrt{1+|v|^2} - 2u\cdot v + 1)\,. \end{aligned}$$

Now

$$\begin{aligned} 4g^2 &= -(U-V)^2 \\ &= -(u_0-v_0)^2 + |u-v|^2 \\ &= 2u_0v_0 - 2u\cdot v - u_0^2 + |u|^2 - v_0^2 + |v|^2 \\ &= 2(\sqrt{1+|u|^2}\sqrt{1+|v|^2} - u\cdot v + 1) \\ &= s - 4 \end{aligned}$$

and

$$\cos\theta = \frac{(V-U)\cdot(V'-U')}{(V-U)^2}.$$

Furthermore, we define the Moller velocity as the scalar v_M given by

$$v_M^2 = |\hat{v} - \hat{u}|^2 - |\hat{v}\times\hat{u}|^2 = \frac{s(s-4)}{4v_0^2u_0^2}$$

or

$$v_M = \frac{2g\sqrt{1+g^2}}{v_0u_0}\,.$$

The two expressions for v_M^2 are equal because

$$\begin{aligned} \frac{1}{4}s(s-4) = sg^2 &= (u_0v_0 - u\cdot v + 1)(u_0v_0 - u\cdot v - 1) \\ &= |u|^2 + |v|^2 + |u|^2|v|^2 - 2u_0v_0u\cdot v + (u\cdot v)^2 \\ &= u_0^2|v|^2 + v_0^2|u|^2 - 2u_0v_0u\cdot v - (u\times v)^2 \\ &= u_0^2v_0^2\Big[\frac{|v|^2}{v_0^2} + \frac{|u|^2}{u_0^2} - 2\frac{u}{u_0}\cdot\frac{v}{v_0} - \Big|\frac{u}{u_0}\times\frac{v}{v_0}\Big|^2\Big]\,. \end{aligned}$$

The relativistic equation resulting is

$$\partial_t F + \hat{v}\cdot\nabla_x F = \int_{\mathbb{R}^3}\int_{S^2} v_M\sigma(s,\theta)[F(u')F(v') - F(u)F(v)]d\Omega du\,,$$

where $d\Omega$ is the element of surface area on S^2 and we have to write σ as a function of g and θ. The Enskog equation has the same structure of the Boltzmann equation,

$$\frac{\partial f}{\partial t} + v\cdot\nabla_x f = E(f)\,,$$

where $E(f)$ is the Enskog's collision operator defined by $E(f) = E^+(f) - E^-(f)$. The left-hand side defines the total derivative of f that is equated by the Enskog's collision operator, which is expressed by the difference between the gain and loss terms respectively defined by

$$E^+(f)(t,x,v) = a^2 \int_{\mathbb{R}^3\times S_+^2} Y(f)\sigma(s,\theta) f(t,x+a\eta,w')d\eta dw$$

$$E^-(f)(t,x,v) = a^2 f(t,x,v) \int_{\mathbb{R}^3\times S_+^2} Y(f)\sigma(s,\theta) f(t,x-a\eta,w')d\eta dw\,,$$

where Y is a functional on M, and $S_+^2 = \{\eta \in \mathbb{R}^3 : |\eta| = 1,\ \sigma(s,\theta) \geq 0\}$, and a is the diameter of hard sphere.

A survey of mathematical results on the existence theory for the Cauchy problem for small initial data decay to zero at infinity in the phase space is proposed in [1] as well as in papers [2, 11, 12, 13]. Several other papers have been published about this type of results. Nevertheless, the main results are contained in the papers which have been cited above.

Specifically, paper [12] refers to a hard sphere gas and to initial conditions which tend exponentially to zero at infinity in the phase space. Paper [11] generalizes the result of [12]. The main result concerning the existence of solutions to the classical Boltzmann equation is a theorem by Diperna and Lions [4] that proves existence, but not uniqueness of renormalized solutions; i.e, solutions in a weak sense, which are even more general than distributional solutions. An analogous result holds in the relativistic case, as was shown by Dudynsky and Ekiel-Jezewska [5]. Regarding classical solutions, Illner and Shinbrot [12] have shown global existence of solutions to the nonrelativistic Boltzmann equation for small initial data(close to the vacuum), Galeano, Vasquez and Orozco [6] shown a result for the relativistic Boltzmann equation. When the data are close to equilibrium, global existence of classical solutions has been proved by Glassey and Strauss [8] in the relativistic case and by Ukay [14] in the nonrelativistic case. In the case of the relativistic Enskog equation we don't known results and this would be a first one. The paper is divide in two sections, we build the functional setting in the first, and we prove a lemma and the theorem of existence and uniqueness in the second one.

2. Functional Setting

For a given $\beta > 0$, let

$$M = \big\{ f \in C([0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3) : \text{there exists } c>0 \text{ such that} \\ |f(t,x,v)| \leq c e^{-\beta(\sqrt{1+|v|^2}+|x+tv|^2)} \big\}\,.$$

This space is a Banach space (See [4])

$$\|f\| = \sup_{t,x,v} e^{\beta(\sqrt{1+|v|^2}+|x+tv|^2)} |f(t,x,v)|\,.$$

We introduce the notation

$$f^{\#}(t,x,v) = f(t,x+tv,v),.$$

Then the Enskog equation can be written as

$$\frac{d}{dt} f^{\#}(t,x,v) = E^{\#}(f)\,.$$

Therefore $f^{\#}(t,x,v)=f_0(x,v)+\int_0^t E^{\#}(f)d\tau$. Now

$$E^+(f)(t,x,v)=a^2\int_{\mathbb{R}^3\times S^2_+} Y(f)\sigma(s,\theta)f(t,x,v')f(t,x+a\eta,w')d\eta dw$$

and

$$\begin{aligned}
&E^+(f^{\#})(t,x,v)\\
&=a^2\int_{\mathbb{R}^3\times S^2_+} Y(f^{\#})\sigma(s,\theta)f^{\#}(t,x,v')f^{\#}(t,x+a\eta,w')d\eta dw\\
&=a^2\int_{\mathbb{R}^3\times S^2_+} Y(f^{\#})\sigma(s,\theta)f(t,x+tv,v')f(t,x+a\eta+tv,w')d\eta dw\\
&=a^2\int_{\mathbb{R}^3\times S^2_+} Y(f^{\#})\sigma(s,\theta)f^{\#}(t,x+t(v-v'),v')f^{\#}(t,x+a\eta+t(v-w'),w')d\eta dw\,.
\end{aligned}$$

Analogously

$$\begin{aligned}
&E^-(f^{\#})(t,x,v)\\
&=a^2 f(t,x+tv,v)\int_{\mathbb{R}^3\times S^2_+} Y(f^{\#})\sigma(s,\theta)f(t,x-a\eta+tv,w)d\eta dw\\
&=a^2 f^{\#}(t,x,v)\int_{\mathbb{R}^3\times S^2_+} Y(f^{\#})\sigma(s,\theta)f^{\#}(t,x-a\eta+t(v-w),w)d\eta dw\,.
\end{aligned}$$

3. Relativistic Enskog Equation

Lemma 3.1. *Suppose that $\sigma(s,\theta)\in L^1_{\rm loc}(\Omega)$ and that there is a constant $c>0$ such that $|Y(f^{\#})|\le c\|f^{\#}\|$ for every $f^{\#}\in M$. Then for some constant $L>0$,*

$$\begin{gathered}
\int_0^t |E^+(f^{\#})|d\tau\le \frac{4acL\pi^2}{\beta^4|v|}e^{\beta\sqrt{1+|v|^2}}\|f^{\#}\|^3\\
\int_0^t |E^-(f^{\#})|d\tau\le \frac{4acL\pi^2}{\beta^4|v|}e^{\beta\sqrt{1+|v|^2}}\|f^{\#}\|^3\,.
\end{gathered}$$

Proof. Note that

$$\begin{aligned}
|E^+(f^{\#})|\le a^2\int_{\mathbb{R}^3\times S^2_+} & c\|f^{\#})\|\,|\sigma(s,\theta)|\,|f^{\#}(t,x+t(v-v'),v')|e^{\beta(\sqrt{1+|v'|^2}+|x+tv|^2)}\\
&\times|f^{\#}(t,x+a\eta+t(v-w'),w')|e^{\beta(\sqrt{1+|w'|^2}+|x+a\eta+tv|^2)}\\
&\times e^{-\beta(\sqrt{1+|v'|^2}+|x+tv|^2)}e^{-\beta(\sqrt{1+|w'|^2}+|x+a\eta+tv|^2)}d\eta dw\,.
\end{aligned}$$

Since $\sigma(s,\theta)\in L^1_{\rm loc}(\Omega)$, there is a constant $L>0$ such that

$$\begin{aligned}
&|E^+(f^{\#})|\\
&\le a^2\int_{\mathbb{R}^3\times S^2_+} cL\|f^{\#})\|^3 e^{-\beta(\sqrt{1+|v'|^2}+|x+tv|^2)}e^{-\beta(\sqrt{1+|w'|^2}+|x+a\eta+tv|^2)}d\eta dw\,.
\end{aligned}$$

Applying the conservation of energy law, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3\times S_+^2} cL\|f^{\#})\|^3 e^{-\beta(\sqrt{1+|v'|^2}+|x+tv|^2)} e^{-\beta(\sqrt{1+|w'|^2}+|x+a\eta+tv|^2)} d\eta dw \\
&= cL\|f^{\#})\|^3 \int_{\mathbb{R}^3\times S_+^2} e^{-\beta(\sqrt{1+|v|^2}+|x+tv|^2)} e^{-\beta(\sqrt{1+|w|^2}+|x+a\eta+tv|^2)} d\eta dw\,.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&cL\|f^{\#})\|^3 \int_{\mathbb{R}^3\times S_+^2} e^{-\beta(\sqrt{1+|v|^2}+|x+tv|^2)} e^{-\beta(\sqrt{1+|w|^2}+|x+a\eta+tv|^2)} d\eta dw \\
&= cL\|f^{\#})\|^3 e^{-\beta\sqrt{1+|v|^2}} e^{-\beta|x+tv|^2} \int_{\mathbb{R}^3\times S_+^2} e^{-\beta(\sqrt{1+|w|^2}+|x+a\eta+tv|^2)} d\eta dw\,.
\end{aligned}$$

By Fubbini's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^3\times S_+^2} e^{-\beta(\sqrt{1+|w|^2}+|x+a\eta+tv|^2)} d\eta dw &= \int_{\mathbb{R}^3} e^{-\beta\sqrt{1+|w|^2}} \Big(\int_{S_+^2} e^{-\beta|x+a\eta+tv|^2} d\eta\Big) dw \\
&\le \frac{4\pi}{\beta^3}\sqrt{\frac{\pi}{\beta}}\cdot\frac{1}{a}\,.
\end{aligned}$$

Therefore,

$$|E^+(f^{\#})| \le \frac{4a^2cL\pi^{\frac{3}{2}}}{\beta^{\frac{7}{2}}a}\|f^{\#})\|^3 e^{-\beta\sqrt{1+|v|^2}} e^{-\beta|x+tv|^2}\,.$$

Hence

$$\begin{aligned}
\int_0^t |E^+(f^{\#})| d\tau &\le \frac{4acL\pi^{\frac{3}{2}}}{\beta^{\frac{7}{2}}}\|f^{\#})\|^3 e^{-\beta\sqrt{1+|v|^2}} \int_0^\infty e^{-\beta|x+tv|^2} d\tau \\
&\le \frac{4acL\pi^2}{\beta^4|v|}\|f^{\#})\|^3 e^{-\beta\sqrt{1+|v|^2}}\,.
\end{aligned}$$

Then

$$\begin{aligned}
|E^-(f^{\#})| &\le a^2|f(t,x+tv,v)| e^{\beta(\sqrt{1+|v|^2}+|x+tv|^2)} e^{-\beta(\sqrt{1+|v|^2}+|x+tv|^2)} \\
&\quad\times \int_{\mathbb{R}^3\times S_+^2} |Y(f^{\#})\|\sigma(s,\theta)| f^{\#}(t,x-a\eta+tv-tw,w) \\
&\quad\times e^{\beta(\sqrt{1+|w|^2}+|x-a\eta+tv|^2)} e^{-\beta(\sqrt{1+|w|^2}+|x-a\eta+tv|^2)} d\eta dw \\
&\le a^2\|f^{\#}\|^3 cL e^{-\beta\sqrt{1+|v|^2}} e^{-\beta|x+tv|^2} \int_{\mathbb{R}^3\times S_+^2} e^{-\beta(\sqrt{1+|w|^2}+|x-a\eta+tv|^2)} d\eta dw \\
&\le a^2\|f^{\#}\|^3 cL e^{-\beta\sqrt{1+|v|^2}} e^{-\beta|x+tv|^2} \int_{\mathbb{R}^3} e^{-\beta\sqrt{1+|w|^2}} dw \int_{S_+^2} e^{-\beta|x-a\eta+tv|^2} d\eta \\
&\le \frac{4a^2cL\pi}{\beta^3} e^{-\beta\sqrt{1+|v|^2}} e^{-\beta|x+tv|^2} \sqrt{\frac{\pi}{\beta}}\frac{1}{a}\|f^{\#}\|^3 \\
&\le \frac{4acL\pi^{3/2}}{\beta^{7/2}} e^{-\beta\sqrt{1+|v|^2}} e^{-\beta|x+tv|^2}\|f^{\#}\|^3
\end{aligned}$$

and

$$\begin{aligned}\int_0^t |E^-(f^\#)|d\tau &\le \frac{4acL\pi^{3/2}}{\beta^{7/2}} e^{-\beta\sqrt{1+|v|^2}} \int_0^t e^{-\beta|x+\tau v|^2} d\tau \|f^\#\|^3 \\ &\le \frac{4acL\pi^2}{\beta^4|v|} e^{-\beta\sqrt{1+|v|^2}} \|f^\#\|^3 \,.\end{aligned}$$

So that

$$\int_0^t |E^-(f^\#)|d\tau \le \frac{4acL\pi^2}{\beta^4|v|} \|f^\#)\|^3 e^{-\beta\sqrt{1+|v|^2}}$$

which completes the proof □

Theorem 3.2. *Suppose that $\sigma(s,\theta) \in L^1_{\rm loc}(\Omega)$ and there exists $c > 0$ such that $|Y(f^\#| \le c\|f^\#)\|$ for every $f^\# \in M_R = \{f \in M : \|f\| \le R\}$ with $R^2 < \frac{\beta^4|v|}{16\pi^2 cLa}$ and $\|f_0\| < \frac{R}{2e^{-\beta|x|^2}}$. Then the Enskog relativistic equation has solution in M_R.*

Proof. We define the operator $\boldsymbol{\mathcal{F}}$ on M by

$$\boldsymbol{\mathcal{F}} f^\# = f_0(x,v) + \int_0^t |E^\#(f)|d\tau \,.$$

Then

$$\begin{aligned}|\boldsymbol{\mathcal{F}} f^\#| &\le |f_0(x,v)| + \Big|\int_0^t E^\#(f)d\tau\Big| \\ &\le |f_0(x,v)| e^{\beta(\sqrt{1+|v|^2}+|x|^2)} e^{-\beta(\sqrt{1+|v|^2}+|x|^2)} + \Big|\int_0^t E^+(f^\#) - E^-(f^\#)d\tau\Big| \\ &\le \|f_0\| e^{-\beta\sqrt{1+|v|^2}} e^{-\beta|x|^2} + \int_0^t |E^+(f^\#) - E^-(f^\#)|d\tau \\ &\le \|f_0\| e^{-\beta\sqrt{1+|v|^2}} e^{-\beta|x|^2} + \frac{8acL\pi^2}{\beta^4|v|}\|f^\#)\|^3 e^{-\beta\sqrt{1+|v|^2}} \\ &\le \big[\frac{R}{2} + \frac{8acL\pi^2}{\beta^4|v|} R^3\big] e^{-\beta\sqrt{1+|v|^2}} \\ &= Re^{-\beta\sqrt{1+|v|^2}}\big[\frac{1}{2} + \frac{8acL\pi^2}{\beta^4|v|}R^2\big] \\ &\le Re^{-\beta\sqrt{1+|v|^2}}\big[\frac{1}{2} + \frac{8acL\pi^2}{\beta^4|v|}\frac{\beta^4|v|}{16\pi^2 cLa}\big] \\ &\le Re^{-\beta\sqrt{1+|v|^2}}\big[\frac{1}{2} + \frac{1}{2}\big] \\ &= Re^{-\beta\sqrt{1+|v|^2}} < R \,.\end{aligned}$$

Therefore, $\boldsymbol{\mathcal{F}}$ maps M_R into itself. Similarly, we show that $\boldsymbol{\mathcal{F}}$ is a contraction on M_R. Since elements of M_R are continuous, the continuity of $\boldsymbol{\mathcal{F}} f^\#$ is evident. □

References

[1] N. Bellomo, A. Palczewski, G. Toscani; *Mathematical Topics in nonlinear kinetic Theory*World Scientific, (1998).

[2] N. Bellomo and G. Toscani; *On the Cauchy Problem for the nonlinear Boltzmann equation: Global existence, uniqueness and asymptotic behaviour*,J. math. phys., **26**, 334-338, (1985).

[3] C. Cercignani and Kremer G. Medeiros; *The relativistic Boltzmann equation:theory and applications*, Birkhauser Verlag, (2002).

[4] Diperna and Lions; *On the Cauchy problem for Boltzmann equation: Global existence and weak stability*,Ann. Math., **130**, 321–366, (1989).
[5] M. Dudynski and Jezewska M. Ekiel; *On the linearized relativistic Boltzmann equation*,Comm. Math.Phys., **115**, 607–629, (1988).
[6] R. Galeano, O. Vasquez, B. Orozco; *Ecuacion de Boltzmann relativistica cerca al vacio*, Revista Colombiana de Matematicas.2005.
[7] Glassey Robert; *The Cauchy Problem in kinetic theory* SIAM, (1996).
[8] R. Glassey and W. Strauss; *On the derivatives of the collision map of relativistic particles*, T.T.S.P., **20**, 55–68, (1991).
[9] R. Glassey and W. Strauss; *Asymptotic stability of the relativisticMaxwellian*, Publi. RIMS Kyoto Univ, **29**, 301–347, (1993).
[10] R. Glassey and W. Strauss; *Asymptotic stability of the relativistic Maxwellian via fourteen moments*, T.T.S.P. , **24**, 657–678, (1995).
[11] R. Hamdache; *Existence in the Large and Asymptotic behaviour for the Boltzmann equation*,Japan J. Appl. Math., **2**, 1-15, (1985).
[12] R. Illner, M. Shimbrot; *The Boltzman equation: Global existence for a rare gas in an Infinite Vacuum*, Commun. math. phys. **95**, 217-226, (1984).
[13] G. Toscani; *On the Boltzmann equation in unbounded domains*,Arch. Rat. Mech. Anal. **95**, 37-49, (1996).
[14] S. Ukay; *Solutions of the Boltzmann equation*, studies in Math Appl., **18**, 37–96, (1986).
[15] S. Ukay; *On the existence of global solutions of a mixed problem for the nonlinear Boltzmann equation*, Proc., Japan Acad., **50**, 179–184, (1974)

Rafael Galeano Andrades
Program de Matematicas, Universidad de Cartagena, Cartagena, Colombia
E-mail address: `ecuacionesdif@yahoo.com`

Bernardo Orozco Herrera
Program de Matematicas, Universidad de Cartagena, Cartagena, Colombia
E-mail address: `orozcoberna@hotmail.com`

Maria Ofelia Vasquez Avila
Program de Matematicas, Universidad de Cartagena, Cartagena, Colombia
E-mail address: `movasquez@epm.net.co`