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OSCILLATION AND ASYMPTOTIC STABILITY OF A DELAY DIFFERENTIAL EQUATION WITH RICHARD'S NONLINEARITY

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Abstract. We obtain sufficient conditions for oscillation of solutions, and for asymptotical stability of the positive equilibrium, of the scalar nonlinear delay differential equation

$$\frac{dN}{dt} = r(t)N(t)\Big[a - \Big(\sum_{k=1}^{m} b_k N(g_k(t))\Big)^{\gamma}\Big],$$

where $g_k(t) \le t$.

1. Introduction

Consider the following logistic differential equation which is widely used in Population Dynamics

$$\frac{dN}{dt} = rN\big(1 - \frac{N}{K}\big).$$

Here $N(t)$ is the size of a population, $r \ge 0$ is an intrinsic growth rate, K is a carrying capacity or a saturation level. A large variety of nonlinear differential equations, besides the one above, has been developed for models of Mathematical Biology; see for example [3, 9, 1].

To model processes in nature and engineering it is frequently required to know system states from the past. Depending on the phenomena under study the aftereffects represent duration of some hidden processes. In general, delay differential equations (DDE) exhibit much more complicated dynamics than ordinary differential equations (ODE) since a time lag can change a stable equilibrium into an unstable one and make populations fluctuate, they provide a richer mathematical framework (compared with ordinary differential equations) for the analysis of biosystems dynamics.

Models of Population Dynamics, based on nonlinear DDE's, have attracted much attention in recent years. The application of delay equations to biomodelling in

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many cases is associated with studies of dynamic phenomena like oscillations, bifurcations, and chaotic behavior. Time delays represent an additional level of complexity that can be incorporated in a more detailed analysis of a particular system.

The delay logistic equation

$$\frac{dN}{dt} = rN\big(1 - \frac{N_\tau}{K}\big) \tag{1.1}$$

appeared in 1948 in Hutchinson's paper [7], where $N_\tau = N(t-\tau)$, $\tau > 0$.

The autonomous equation (1.1) has been extensively investigated by numerous authors. The first paper on the oscillation of a non-autonomous logistic delay differential equation was published in [14]. Since this publication, the oscillation of the logistic DDE as well as its generalizations were studied by many mathematicians. Some of these results can be found in the monographs [6, 5, 4].

It is a well-known fact, that the traditional logistic model, in some cases, produces artificially complex dynamics. Therefore, it would be reasonable to get away from the specific logistic form in studying population dynamics and use more general classes of growth models.

For example, to drop an unnatural symmetry of the logistic curve, we consider the modified logistic form by Pella and Tomlinson [13, 12] or the Richards' growth equation with delay

$$\frac{dN}{dt} = rN\Big[1 - \Big(\frac{N_\tau}{K}\Big)^\gamma\Big]. \tag{1.2}$$

According to [13], $0 < \gamma < 1$ is used for invertebrate populations (examples of invertebrates are insects, worms, starfish, sponges, squid, plankton, crustaceans, and mollusks), and $\gamma \geq 1$ is used for the vertebrate populations (these include amphibians, birds, fish, mammals, and reptiles).

In [11] the authors considered (1.2) with several delays. They obtained conditions for existence of positive solutions and studied so-called long time average stability. In this paper we obtain oscillation and local stability results for non-autonomous (1.2) with several delays.

2. Preliminaries

Our objective is to study the scalar nonlinear delay differential equation

$$\dot N(t) = r(t)N(t)\Big[a - \Big(\sum_{k=1}^m b_k N(g_k(t))\Big)^\gamma\Big], \quad t \geq 0 \tag{2.1}$$

under the following conditions:

- (A1) $r(t)$ is Lebesgue measurable essentially bounded on $[0,\infty)$ function, $r(t) \geq 0$.
- (A2) $g_k : [0,\infty) \to \mathbb{R}$ are Lebesgue measurable functions with $g_k(t) \leq t$ and $\lim_{t\to\infty} g_k(t) = \infty$, $k = 1,\dots,m$.
- (A3) $a > 0$, $b_k > 0$, $\gamma > 0$.

Together with (2.1), we consider for $t_0 \geq 0$, the initial-value problem

$$\dot N(t) = r(t)N(t)\Big[a - \Big(\sum_{k=1}^m b_k N(g_k(t))\Big)^\gamma\Big], \quad t \geq t_0, \tag{2.2}$$

$$N(t) = \varphi(t), \quad t < t_0, \quad N(t_0) = N_0 \tag{2.3}$$

under the following conditions

(A4) $\varphi : (-\infty, t_0) \to R$ is a Borel measurable bounded function, $\varphi(t) \geq 0$, $N_0 > 0$.

Definition. A locally absolutely continuous function $x : R \to R$ is called *a solution of problem* (2.2)–(2.3), if it satisfies (2.2) for almost all $t \in [t_0, \infty)$ and (2.3) for $t \leq t_0$.

Lemma 2.1 ([11]). *Suppose Conditions (A1)–(A4) hold. Then problem (2.2)-(2.3) has a unique positive solution $N(t)$, $t \geq t_0$.*

3. Oscillation Criteria

Definition. We say that a function $y(t)$ is *non-oscillatory* about a number K if $y(t) - K$ is eventually positive or eventually negative. Otherwise $y(t)$ is *oscillatory* about K.

Note that (2.1) has a positive equilibrium,

$$N^* = a^{1/\gamma} / \sum_{k=1}^{m} b_k.$$

In this section we study oscillation of solutions of (2.1) about the value N^*.

We will present here some lemmas which will be used in this section. Consider the linear delay differential equation

$$\dot{x}(t) + \sum_{k=1}^{l} r_k(t) x(h_k(t)) = 0, \quad t \geq 0, \tag{3.1}$$

and the differential inequalities

$$\dot{x}(t) + \sum_{k=1}^{l} r_k(t) x(h_k(t)) \leq 0, \quad t \geq 0, \tag{3.2}$$

$$\dot{x}(t) + \sum_{k=1}^{l} r_k(t) x(h_k(t)) \geq 0, \quad t \geq 0. \tag{3.3}$$

Lemma 3.1 ([6]). *Let (A1)–(A2) hold for the parameters of* (3.1). *Then the following three statements are equivalent:*

1. *There exists a non-oscillatory solution of equation* (3.1).
2. *There exists an eventually positive solution of the inequality* (3.2).
3. *There exists an eventually negative solution of the inequality* (3.3).

Lemma 3.2 ([6]). *Let (A1)–(A2) hold for the parameters of* (3.1). *If*

$$\liminf_{t \to \infty} \int_{\max_k h_k(t)}^{t} \sum_{i=1}^{l} r_i(s) ds > 1/e, \tag{3.4}$$

then all solutions of (3.1) *are oscillatory.*

Theorem 3.3. *Suppose (A1)-(A4) hold and*

$$\int_0^\infty r(s) ds = \infty. \tag{3.5}$$

Then for every non-oscillatory solution $N(t)$ of (2.1) we have

$$\lim_{t \to \infty} N(t) = N^*. \tag{3.6}$$

Proof. After the substitution $N(t) = N^*(1 + x(t))$, Equation (2.1) reduced to

$$\dot{x}(t) = -ar(t)(1 + x(t))\Big[\Big(\sum_{k=1}^{m} B_k(1 + x(g_k(t)))\Big)^{\gamma} - 1\Big], \quad t \ge 0, \tag{3.7}$$

where

$$B_k = b_k / \sum_{i=1}^{m} b_i. \tag{3.8}$$

Condition (A3) implies $B_k > 0$ and $\sum_{k=1}^{m} B_k = 1$.

The zero solution is an equilibrium of (3.7), which corresponds to the equilibrium N^* of (2.1).

By Lemma 2.1 any solution of (2.1) is positive. Then for any solution of (3.7) we have $1 + x(t) > 0$. To prove the theorem we have to show that for every non-oscillatory about zero solution of (3.7) we have

$$\lim_{t\to\infty} x(t) = 0. \tag{3.9}$$

Suppose $x(t)$ is a non-oscillatory solution of (3.7). Without loss of generality we can assume $x(t) > 0$, $t \ge 0$. Hence

$$\Big(\sum_{k=1}^{m} B_k(1 + x(g_k(t)))\Big)^{\gamma} - 1 \ge \Big(\sum_{k=1}^{m} B_k\Big)^{\gamma} - 1 = 0.$$

Then $\dot{x}(t) \le 0$ and hence there exists $\lim_{t\to\infty} x(t) = l$. Suppose $l > 0$. Equality (3.7) implies

$$x(t) = x(0) - a\int_0^t r(s)(1 + x(s))\Big[\Big(\sum_{k=1}^{m} B_k(1 + x(g_k(s)))\Big)^{\gamma} - 1\Big] ds. \tag{3.10}$$

If $t \to \infty$ then the right hand side of (3.10) tends to $-\infty$, the left hand side has a finite limit. This contradiction proves the theorem. □

Theorem 3.4. *Suppose conditions (A1)–(A4) and* (3.5) *hold,* $\gamma > 1$ *and there exists* $\epsilon > 0$ *such that all solutions of the linear differential equation*

$$\dot{y}(t) = -a\gamma r(t)(1 - \epsilon)\sum_{k=1}^{m} B_k y(g_k(t)) \tag{3.11}$$

are oscillatory, were B_k *are denoted by* (3.8)*. Then all solutions of* (2.1) *are oscillatory about* N^**.*

Proof. It is sufficient to prove, that all solutions of (3.7) are oscillatory about zero. Suppose there exists a non-oscillatory solution x of (3.7). Without loss of generality we can assume, that $x(t) > 0, t \ge 0$. Theorem 3.3 implies, that for some $t_0 > 0$ and for $t \ge t_0$ we have $0 < x(t) < \epsilon$.

Consider the function

$$f(u_1, \dots, u_m) = \Big(\sum_{k=1}^{m} B_k(1 + u_k)\Big)^{\gamma} - 1 - \gamma\sum_{k=1}^{m} B_k u_k.$$

Then we have

$$\frac{\partial f}{\partial u_k} = \gamma\Big(\sum_{k=1}^m B_k(1+u_k)\Big)^{\gamma-1} B_k - \gamma B_k,$$
$$\frac{\partial^2 f}{\partial u_i \partial u_j} = \gamma(\gamma-1)\Big(\sum_{k=1}^m B_k(1+u_k)\Big)^{\gamma-2} B_i B_j.$$

Hence

$$f(0,\dots,0) = 0, \quad \frac{\partial f}{\partial u_k}(0,\dots,0) = 0, \quad \frac{\partial^2 f}{\partial u_i \partial u_j}(0,\dots,0) = \gamma(\gamma-1)B_i B_j.$$

Taylor's Formula implies

$$f(u_1,\dots,u_m) = \gamma(\gamma-1)\sum_{i=1}^m \sum_{j=1}^m B_i B_j u_i u_j + o(\Delta u),$$

where

$$\Delta u = \Big(\sum_{k=1}^m u_k^2\Big)^{1/2}, \quad \lim_{t\to 0}\frac{o(t)}{t} = 0.$$

Then for $u_k \geq 0, k = 1,\dots,m$ and Δu sufficiently small $f(u_1,\dots,u_m) \geq 0$. Hence for ϵ small enough we have

$$\dot{x}(t) \leq -a\gamma r(t)(1-\epsilon)\sum_{k=1}^m B_k x(g_k(s)), \quad t \geq 0.$$

Lemma 3.1 implies that (3.11)) has a non-oscillatory solution. We have a contradiction with our assumption. The theorem is proven. ☐

Corollary 3.5. *Suppose conditions (A1)–(A4) and* (3.5) *hold,* $\gamma > 1$,

$$\liminf_{t\to\infty} a\gamma \int_{\max_k g_k(t)}^t r(s)ds > 1/e. \tag{3.12}$$

Then all solutions of (2.1) *are oscillatory about* N^*.

Proof. Inequality (3.12) implies, that for some $\epsilon > 0$,

$$\liminf_{t\to\infty} a\gamma(1-\epsilon)\int_{\max_k g_k(t)}^t \sum_{i=1}^m B_i r(s)ds > 1/e.$$

Lemma 3.2 and Theorem 3.4 imply this corollary. ☐

4. Asymptotic Stability

Consider a general nonlinear delay differential equation

$$\dot{x}(t) = f(t, x(t), x(g_1(t)), \dots, x(g_m(t))), \quad t \geq 0, \tag{4.1}$$

with the initial function and the initial value

$$x(t) = \varphi(t), \quad t < 0, \quad x(0) = x_0, \tag{4.2}$$

under the following conditions:

(B1) $f(t, u_0, u_1, \dots, u_m)$ satisfies Caratheodory conditions: Lebesgue measurable in the first argument and continuous in other arguments, $f(t, 0, \dots, 0) = K$

(B2) $g_k(t)$ are Lebesgue measurable functions,

$$g_k(t) \le t, \quad \sup_{t\ge 0}[t-g_k(t)] < \infty;$$

(B3) $\varphi : (-\infty, 0) \to R$ is a Borel measurable bounded function.

We will assume that the initial-value problem (4.1)–(4.2) has a unique global solution $x(t)$, $t \ge 0$.

Definition. We will say that the equilibrium K of (4.1) is *(locally) stable*, if for any $\epsilon > 0$ there exists $\delta > 0$ such that for every initial conditions $|x(0)| < \delta_0$, $|\varphi(t)| < \delta_0$, $\delta_0 \le \delta$, for the solution $x(t)$ of (4.1)–(4.2) we have $|x(t)-K| < \epsilon$, $t \ge 0$.

If, in addition, $\lim_{t\to\infty}(x(t)-K) = 0$, then the equilibrium K of (4.1) is *(locally) asymptotically stable*.

Suppose there exist $M > 0$, $\gamma > 0$ such that

$$|x(t)-K| \le M \exp\{-\gamma t\}(|x(0)| + \sup_{t<0}|\varphi(t)|)$$

for all $x(0)$ and $\varphi(t)$ such that $|x(0)| + \sup_{t<0}|\varphi(t)|$ is sufficiently small. Then we will say that the equilibrium K of (4.1) is *exponentially stable*.

Lemma 4.1 ([10]). *Suppose (A1), (B2), (B3) hold for the linear equation* (3.1) *and*

$$\limsup_{t\to\infty} \sum_{k=1}^{l} r_k(t)(t-h_k(t)) < 1.$$

Then (3.1) *is exponentially stable.*

Lemma 4.2 ([2], [8]). *Suppose that (b1)-(b3) hold, and that for sufficiently small u if $|u_k| \le u$, $k = 0, \dots, m$ then*

$$|f(t, u_0, \dots, u_m) - \sum_{k=0}^{m} \frac{\partial F}{\partial u_k}(t, K, \dots, K)u_k| = o(u),$$

where $\lim_{u\to 0} o(u)/u = 0$. If the linear equation

$$\dot{y}(t) = \sum_{k=0}^{m} \frac{\partial F}{\partial u_k}(t, 0, \dots, 0)\, y(g_k(t))$$

is exponentially stable, then the equilibrium K of (4.1) *is locally asymptotically stable.*

Theorem 4.3. *Suppose that for equation (2.1) Conditions (A1), (A3), (B2), (B3) hold and*

$$\limsup_{t\to\infty} a\gamma r(t) \sum_{k=1}^{m} B_k(t-g_k(t)) < 1, \tag{4.3}$$

were B_k are denoted by (3.8). *Then the equilibrium N^* of* (2.1) *is asymptotically stable.*

Proof. The substitution $N(t) = N^*(1+x(t))$ implies that the equilibrium N^* of (2.1) is asymptotically stable if and only if the zero solution of (3.7) is asymptotically stable. Lemma 4.1 and inequality (4.3) imply that the linear equation

$$\dot{x}(t) = -a\gamma r(t) \sum_{k=1}^{m} B_k x(g_k(t))$$

is exponentially stable. Lemma 4.2 implies now that the zero solution of (3.7) is asymptotically stable. □

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