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ON A PROBLEM OF SHALLOW WATER TYPE

MOHAMED ELALAOUI TALIBI, MOULAY HICHAM TBER

Abstract. In this paper we present an existence theorem for a problem of shallow water kind. We take into account a general friction term depending on water depth and the norm of velocity, which is the main difficulty. We present also a numerical study in the case which we consider the above problem as a perturbation of shallow water equations in the non conservative dept-mean velocity form.

1. Introduction and setting of the problem

The two-dimensional shallow water equations (briefly SWE) are deduced by integrating, with respect to depth, the continuity and the momentum equations of the three-dimensional incompressible Navier-Stokes system, neglecting the influence of the vertical component of acceleration, the pressure is then supposed hydrostatic [1]. They provide a model allowing to describe the flows of water in domains characterized by small ratio between vertical and horizontal length scales, therefore typical physical situations modelled are: tidal waves, currents in portual basins, lagoon, ..etc. But their use is surprisingly extended to very different phenomena even with discontinuous behavior, like the "dam break" problem [11].

The shallow-water system we are studying in this work reads

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu_1 \boldsymbol{\Delta}\mathbf{u} + C(h)\,|\mathbf{u}|\,\mathbf{u} + \mathbf{l}\times\mathbf{u} + g\nabla h = \mathbf{f} \quad \text{on } \Omega, \tag{1.1}$$

$$\frac{\partial h}{\partial t} - \nu_2 \triangle \mathbf{h} + \nabla\cdot(h\mathbf{u}) = f \quad \text{on } \Omega, \tag{1.2}$$

where $\mathbf{u} = (u_1, u_2)^{\perp}$ is the velocity vector and h is the depth of studied layer, it can be considered as sum of the bottom topography which is given and the topography of the free surface. $\Omega \in \mathbb{R}^2$ is the projection of the domain of the study on the horizontal plane. Γ denotes its boundary. $\mathbf{l}$ is the Coriolis force defined by $(0, 0, 2\omega sin(\phi))$, where ω is the rotation rate of the earth and ϕ the latitude. g denotes the acceleration of the gravity. The bottom friction effect is presented by the term $C(h)\,|\mathbf{u}|\,\mathbf{u}$ where $C(.)$ is a continuous function satisfying the condition $0 \le C(.) < \overline{\varepsilon}$ which physically justified by the Manning-Strickler's formula and by

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the Chezy's one if the free surface elevation remain larger than minimal level. ν_1, ν_2 are respectively the eddy viscosity and diffusivity coefficients which we consider as an artificial viscosity taken, numerically, equal to zero to have the shallow-water equations in the nonconservative depth-mean velocity form. The right-hand side terms $\mathbf{f}$ and f represent, respectively, the outside stress and the fluid exchanges (rain, evaporation, etc.).

To solve these equations we take homogeneous boundary conditions and we set the initial data as

$$\begin{gathered}(\mathbf{u}, h) = (0,0) \quad \text{on } \Gamma,\\ (\mathbf{u}, h)(t=0) = (\mathbf{u}_0, h_0) \quad \text{in } \Omega\end{gathered}$$

Remark 1.1. To be compatible with the physical situation for which the friction formulation is justified, we assume that $h = h_B \geq h_{min} > 0$ on Γ and $h(0) \geq h_{min} > 0$ in Ω. However by setting $h := h + h_L$ where h_L is the solution of the problem

$$\begin{gathered}\frac{\partial h_L}{\partial t} - \nu_2 \triangle h_L = 0 \quad \text{in } \Omega\\ h_L(0) = 0 \quad \text{in } \Omega\\ h_L = h_B \quad \text{on } \Gamma\,.\end{gathered}$$

(As shown in [8], this problem has a solution in $L^2(0,T,H^1(\Omega)) \cap L^\infty(0,T,L^\infty(\Omega))$ for $h_B \in L^2(0,T,H^{\frac{1}{2}}(\Gamma)) \cap L^\infty(0,T,L^\infty(\Gamma)))$ we find again the homogeneous boundary conditions modulo a constant in the momentum equation and a linear term in the continuity one changing quit the reasoning done below. Therefore we will consider, for convenience, only the homogeneous case.

2. Notation and variational formulation

We introduce the following functional spaces: $V_1 = (H_0^1(\Omega))^2$,$H_1 = (L^2(\Omega))^2$, $V_2 = H_0^1(\Omega)$, $H_2 = L^2(\Omega)$, $V = V_1 \times V_2$, $H = H_1 \times H_2$. The norm and semi-norm defined on $H^1(\Omega)$ are equivalent in V_1, V_2, and V.
Then we set $\|\mathbf{u}\| = \|\mathbf{u}\|_{V_1}$, $\|h\| = \|h\|_{V_2}$ and $\|X\| = \|X\|_V$ for $\mathbf{u} \in V_1$, $h \in V_2$, and $X \in V$. $|\cdot|$ denotes the norm in $L^2(\Omega)$, $|\cdot|_2$ denotes the Euclidean norm in $\mathbb{R}^2$, $(\cdot,\cdot)$ is the scalar product in H_1, H_2 or H and $(\cdot,\cdot)_2$ the scalar product in $\mathbb{R}^2$. We define

$$\begin{gathered}a_1(\mathbf{u}, \mathbf{v}) = \nu_1(\nabla \mathbf{u}, \nabla \mathbf{v}),\\ a_2(h, \beta) = \nu_2(\nabla h, \nabla \beta),\\ a(X, Y) = a_1(\mathbf{u}, \mathbf{v}) + a_2(h, \beta)\end{gathered}$$

with $X = (\mathbf{u}, h)$ and $Y = (\mathbf{v}, \beta)$. Note that a_1, a_2 and a are bilinear continuous coercive forms, respectively, on V_1, V_2 and V.

We denote by $\bar{\varepsilon}$, ν, A, B, C, λ and θ constants such that:

$$\begin{gathered}0 \leq C(.) \leq \bar{\varepsilon}\\ a(X, Y) \geq \nu \|X\| \|Y\| \text{ for } (X, Y) \in V \times V\\ \lambda > 0, \quad C_g = \text{constant} \cdot g, \quad B = 2\nu - C_g - \lambda,\end{gathered}$$

and $C = \text{constant} \cdot C_G$ where C_G is the best constant of the Gagliardo-Nirenberg inequality [2]:

$$\|\mathbf{u}\|^2_{L^4(\Omega)^2} \le C_G \|\mathbf{u}\| |\mathbf{u}|. \tag{2.1}$$

In what follows we take homogeneous boundary conditions and we write

$$(\mathbf{u}.\nabla)\mathbf{u} = \frac{1}{2}\operatorname{grad}(|\mathbf{u}|_2^2) + \operatorname{curl}(\mathbf{u})\alpha(u)$$

where $\operatorname{curl}(\mathbf{u}) = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$ and $\alpha(\mathbf{u}) = (-u_2, u_1)$. Now we can set the weak formulation of the problem:

($\mathcal{V}$) Find $(\mathbf{u}, h) \in L^2(0, T, V) \cap L^\infty(0, T, H)$ such that

$$\begin{aligned}\big(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\big) + a_1(\mathbf{u}, \mathbf{v}) + (curl(\mathbf{u})\alpha(\mathbf{u}), \mathbf{v}) + \frac{1}{2}(\operatorname{grad}|u|_2^2, \mathbf{v})\\ +(C(h)|\mathbf{u}|_2\mathbf{u}, \mathbf{v}) + (\mathbf{l} \wedge \mathbf{u}, \mathbf{v}) - g(\operatorname{div}(\mathbf{v}), h) = (\mathbf{f}, \mathbf{v})\end{aligned} \tag{2.2}$$

$$\big(\frac{\partial h}{\partial t}, \beta\big) + a_2(h, \beta) + (\operatorname{div}(h\mathbf{u}), \beta) = (f, \beta) \quad \forall(\mathbf{v}, \beta) \in V\,, \tag{2.3}$$

$$(\mathbf{u}, h)(t = 0) = (\mathbf{u_0}, h_0). \tag{2.4}$$

3. Existence theorem

Theorem 3.1. *Assume that* $\mathbf{F} = (\mathbf{f}, f) \in L^2(0, T, H), X_0 = (\mathbf{u_0}, h_0) \in V \cap L^\infty(\Omega)^3$. *Also assume the following conditions are satisfied,*

(1) $B = 2\nu - C_g - \lambda > 0$
(2) $|X_0| < B/C$
(3) $(B/C)^2 > |X_0|^2 + \frac{1}{\lambda}|\mathbf{F}|$

where constants are defined above. Then the variational problem ($\mathcal{V}$) admits at last one solution $(\mathbf{u}, h)$ *in* $L^2(0, T, V) \cap L^\infty(0, T, H)$.

The proof of the theorem is based on the three next lemmas.

Lemma 3.2. *Let* $X = (\mathbf{u}, h)$ *be a classic solution of the problem (V), on* $[0, T]$. *Under the same hypothesis in the theorem, we have*

$$\|X\|_{L^\infty(0,T,H)} + (B - C\|X\|_{L^\infty(0,T,H)})\|X\|_{L^2(0,T,V)} \le \frac{1}{\lambda}\|\mathbf{F}\|_{L^2(0,T,H)} + |X_0|$$
$$(B - C\|X\|_{L^\infty(0,T,H)}) > 0.$$
$$\|X\|_{L^\infty(0,T,H)} + \|X\|_{L^2(0,T,V)} \le \textit{constant}$$

Proof. By writing the energy inequality and using the hypothesis above, we find the result via Green's formula and Gagliardo-Neirenberg inequality. □

Lemma 3.3. *Let* (X_n) *be a sequence of classic solution of (V) on* $[0, T]$ *satisfying*

$$\|X_n\|_{L^\infty(0,T,H)} + \|X_n\|_{L^2(0,T,V)} \le C'. \tag{3.1}$$

where C' *is a constant independent of* n. *Then there exist a subsequence also denoted by* X_n *and* $X = (\mathbf{u}, h) \in L^\infty(0, T, H) \cap L^2(0, T, V)$ *such that*

$$X_n \longrightarrow X \quad \textit{weakly in } L^\infty(0, T, H), \tag{3.2}$$
$$X_n \longrightarrow X \quad \textit{weakly in } L^2(0, T, V), \tag{3.3}$$
$$X_n \longrightarrow X \quad \textit{strongly in } L^2(0, T, H). \tag{3.4}$$

Proof. Statements (3.2) and (3.3) are immediate consequences of (3.1). On the other hand we can show that the sequence X_n is uniformly bounded in the set

$$Y = \{\mathbf{v} \in L^2(0,t,V), \frac{\partial \mathbf{v}}{\partial t} \in L^1(0,T,V')\}.$$

According to [15], the injection of Y into $L^2(0,T,H)$ is compact. Then we can extract from X_n a subsequence also denoted by X_n such that we have (3.4) □

Lemma 3.4. *let $(\mathbf{u_n}, h_n)$ be a sequence converging toward (u,h) in $L^2(0,T,H)$ strongly and $L^2(0,T,V)$ weakly. Then for any $\varphi(t) \in \mathcal{C}^1(0,T)$ and $(\mathbf{v},\beta) \in V \cap L^\infty(\Omega)^3$ we have*

$$\int_0^T (C(h_n)|\mathbf{u_n}|\mathbf{u_n}, \varphi(t)\mathbf{v})dt \longrightarrow \int_0^T (C(h)|u|u, \varphi(t)\mathbf{v})dt\,,$$
$$\int_0^T (\operatorname{div}(h_m\mathbf{u_n}), \varphi(t)\beta)dt \longrightarrow \int_0^T (\operatorname{div}(h\mathbf{u}), \varphi(t)\beta)dt\,,$$
$$\int_0^T (curl(\mathbf{u_n})\alpha(\mathbf{u_n}), \varphi(t)\mathbf{v})dt \longrightarrow \int_0^T (curl(\mathbf{u})\alpha(u), \varphi(t)\mathbf{v})dt\,,$$
$$\frac{1}{2}\int_0^T (\operatorname{grad}|u|_2^2, \varphi(t)\mathbf{v})dt \longrightarrow \frac{1}{2}\int_0^T (\operatorname{grad}|u|_2^2, \varphi(t)\mathbf{v})dt\,.$$

We can proof this lemma using Shwartz inequality and appropriated Sobolev injections.

Proof of the Theorem 3.1. The proof is based on the construction of sequence of finite dimensional Problems $(\mathcal{V}_n)$ of which the solutions (X_n) (by using lemmas 3.2 and 3.3) converge strongly in H and weakly in V to $X \in (\mathbf{u},h) \in L^2(0,T,V) \cap L^\infty(0,T,H)$. Then by third Lemma we can show that X is a solution of the problem. □

4. Numerical studies

The goal of this numerical studies is to know how the solution of the problem varies when the included artificial diffusivity coefficients ν_2 tend to zero. The approach we are using here is based on the finite elements for the space discretization and on the discretization of the Lagrangian derivative along the characteristics. This method provides a centred scheme which have the advantage of stabilizing the convection and allow large time steps to be taken when compared to standard time-stepping methods [4].

Similar numerical schemes were considered in [13] for the incompressible Navier-Stokes problem. Within the framework of the shallow water problems, this approach combined with the method of the fractional steps, is adopted in [10] to simulate transcritical flows, and applied later in TELEMAC project [9].

Temporal discretization. The characteristic methods consists in approaching the lagrangian derivative of a function S in time step t^{n+1} by:

$$\frac{dS}{dt}(\mathbf{x}, t^{n+1}) \simeq \frac{S(\mathbf{x}, t^{n+1}) - S(\mathbf{X}(\mathbf{x}, t^{n+1}; t^n), t^n)}{\Delta t} \tag{4.1}$$

where $\mathbf{X}^n = \mathbf{X}(\mathbf{x}, t^{n+1}; t^n)$ is the position in the time step t^n of the particle positioning at the geometrical point $\mathbf{x}$ in time step t^{n+1} and $\mathbf{X}^n(\mathbf{x}, t^{n+1}; \tau)$ is the solution of

$$\frac{d\mathbf{X}^n}{d\tau}(\mathbf{x},t^{n+1};\tau)=\mathbf{u}^n(\mathbf{X}^n(\mathbf{x},t^{n+1};\tau)),\quad \text{for } t^n\le\tau\le t^{n+1},$$
$$\mathbf{X}^n(\mathbf{x},t^{n+1};t^{n+1})=\mathbf{x}.$$

Using (4.1), the semi-implicit time discretization of (1.1), (1.2) is

$$\begin{gathered}\frac{\mathbf{u}^{n+1}-\mathbf{u}^n\circ\mathbf{X}^n}{\Delta t}-\nu_1\boldsymbol{\Delta}\mathbf{u}^{n+1},\\ C(h^n)\left|\mathbf{u}^n\right|\mathbf{u}^{n+1}+\mathbf{l}\times\mathbf{u}^n+g\nabla h^{n+1},=\mathbf{f}^n\end{gathered}\tag{4.2}$$

$$\frac{h^{n+1}-h^n\circ\mathbf{X}^n}{\Delta t}-\nu_2\boldsymbol{\Delta}\mathbf{u}^{n+1}+h^n\nabla\cdot\mathbf{u}^{n+1}=f^n,\tag{4.3}$$

where h^n and $\mathbf{u}^n$ are the approximations of h and $\mathbf{u}$ respectively in time step t^n.

Variational formulation. let us introduce the spaces

$$\begin{gathered}V_\phi^1=\left\{\mathbf{v}\in H^1(\Omega)\times H^1(\Omega);\mathbf{v}=\phi\quad\text{on }\Gamma\right\}\\ V_\eta^2=\left\{h\in H^1(\Omega);h=\eta\quad\text{on }\Gamma\right\}\,.\end{gathered}$$

Multiplying (4.2) and (4.3) by $\mathbf{v}\in V_1$ and $q\in V_2$ respectively, and integrating by part on Ω we obtain

$$\begin{aligned}&\big(\frac{\mathbf{u}^{n+1}}{\Delta t},\mathbf{v}\big)+\nu_1\left(\nabla\mathbf{u}^{n+1},\nabla\mathbf{v}\right)+\left(C(h^n)\left|\mathbf{u}^n\right|\mathbf{u}^{n+1},\mathbf{v}\right)-g\left(h^{n+1},\nabla\cdot\mathbf{v}\right)\\ &=\big(\frac{\mathbf{u}^n\circ\mathbf{X}^n}{\Delta t}+\mathbf{f}^n-\mathbf{l}\times\mathbf{u}^n,\mathbf{v}\big),\end{aligned}\tag{4.4}$$

$$\big(\frac{h^{n+1}}{\Delta t},q\big)+\nu_1\left(\nabla\mathbf{u}^{n+1},\nabla\mathbf{v}\right)+\left(h^n\nabla\cdot\mathbf{u}^{n+1},q\right)=\left(f^n+\frac{h^n\circ\mathbf{X}^n}{\Delta t},q\right).\tag{4.5}$$

Then we write the time-discretized variational formulation as follows:

$(\mathcal{V})^n$ Find $(\mathbf{u}^{n+1},h^{n+1})$ in $V_\phi^1\times V_\eta^2$ such that

$$\begin{gathered}e(\mathbf{u}^{n+1},\mathbf{v})+b(\mathbf{v},h^{n+1})=(\mathbf{f^n},\mathbf{v})\,,\quad\forall\mathbf{v}\in\mathbf{V_1}\,,\\ -b(\mathbf{u^{n+1}},h^nq)+e'(h^{n+1},q)+=(f^n,q)\,,\forall q\in V_2\,.\end{gathered}$$

where

$$\begin{gathered}e(\mathbf{u},\mathbf{v})=\frac{1}{g\triangle t}(\mathbf{u},\mathbf{v})+\frac{\nu_1}{g}(\nabla\mathbf{u},\nabla\mathbf{v})+\frac{1}{g}\left(C(h^n)\left|\mathbf{u}^n\right|\mathbf{u},\mathbf{v}\right)\,,\\ e'(h,q)=\frac{1}{\triangle t}(h,q)+\nu_2(\nabla h,\nabla q)\,,\\ b(\mathbf{v},q)=-\left(q,\nabla\cdot\mathbf{v}\right)\,,\mathbf{f}^n:=\frac{1}{g}\left(\mathbf{f}^n+\frac{\mathbf{u}^n\circ\mathbf{X}^n}{\Delta t}-\mathbf{l}\times\mathbf{u}^n\right)\,,\\ f^n:=f^n+\frac{h^n\circ\mathbf{X}^n}{\Delta t}\,.\end{gathered}$$

Finite element discretization. Let $V_{\phi h}^1$ and $V_{\eta h}^2$ (resp V_{1h} and V_{2h}) two finite elements spaces approaching V_ϕ^1 and V_η^2 (resp V_1 and V_2) such that the LBB condition is satisfied [3]. Then the discrete problem is written as

$(\mathcal{V})_h^n$ Find $(\mathbf{u}_h^{n+1}, h_h^{n+1})$ in $V^1_{\phi h} \times V^2_{\eta h}$ such that

$$e(\mathbf{u}_h^{n+1}, \mathbf{v}_h) + b(\mathbf{v}, h_h^{n+1}) = (\mathbf{f}_h^n, \mathbf{v}_h), \quad \forall \mathbf{v_h} \in V_{1h},$$
$$-b(\mathbf{u}_h^{n+1}, h_h^n q_h) + e'(h_h^{n+1}, q_h) + = (f_h^n, q_h), \quad \forall q_h \in V_{2h}.$$

The value $X_h^m(x)$ is approximated by $X((n+1)\triangle t, x)$, the solution of the problem

$$\frac{dX}{d\tau} = \mathbf{u}_h^n(X(\tau), \tau), \quad X((n+1)\triangle \tau) = x,$$

therefore, at each time step we have to solve the linear system

$$\begin{pmatrix} A & B \\ -\overline{B}^\top & -D \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ H \end{pmatrix} = \begin{pmatrix} \mathbf{F}_U \\ F_H \end{pmatrix}$$

where A and D are two definite positive matrices, and B, $-\overline{B}^\top$ are two matrices approaching operator of divergence type.

We can show easily (see for example [14] [7]) that the problem $(\mathcal{V})^n$ (resp $(\mathcal{V})_h^n$) is well posed if h^n (resp h_h^n) remain larger than one level $\xi > 0$. Moreover in [6] and [10], a preconditionner of Cahouet-Chabard kind [5] are proposed for the linear system.

Numerical results. The studied domain is a square with $1km$ in length with mean water elevation of $1m$. We suppose there is no exchange with the external medium and the surface stress is reduced to the wind stress tensor defined by

$$\mathbf{f}_{wind} = \frac{1}{h} \frac{\rho_{water}}{\rho_{air}} a_{wind} |\mathbf{u}_{\text{wind}}|_2 \mathbf{u}_{wind},$$

where ρ_{water}, ρ_{air} are the density of the water and the air respectively, and a_{wind} is an adimensional empiric coefficient. On the other hand, if we choose the Manning-Strickler's formula for the bottom friction we obtain

$$C(h) = \frac{gn^2}{h^{\frac{4}{3}}} \tag{4.6}$$

where n is the Manning coefficient.

Table 1. Physical parameters

$g(m/s^{-2})$	$\rho_{water}(kg/m^3)$	$\rho_{air}(kg/m^3)$	n	$\nu_2(m^2/s)$
1	999.00	1.225	0.03	0.1

w (rad/s)	ϕ ()	$\nu_1(m^2/s)$	$\Delta t(s)$	a_{wind}
0-100	0.1	0.56510^{-3}	7.29210^{-10}	45

Conclusions. Although the continuous problem (2.2) requires a condition on it, we can take the diffusion coefficient of continuity equation ν_2 numerically as small as we want, without any explosion of the solution (see figure 1). Then for $\nu_2 = 0$ and $f = 0$ we find the shallow water equations established in [1]. Moreover for this case we can prove formally by characteristics that the free surface elevation remain larger than minimal level if the initial one it is. Therefore the choice of Manning-Strickler's formula (4.6) is justified and the numerical results are satisfactory.

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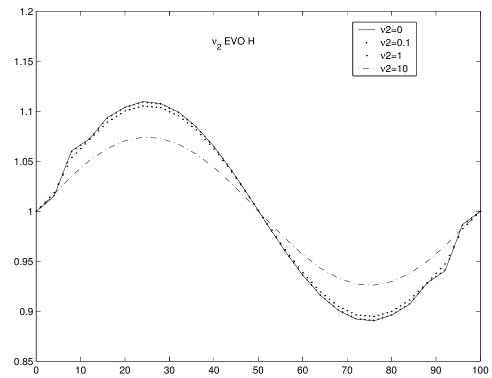

FIGURE 1. The section h(x,250m) for different values of $\nu 2$

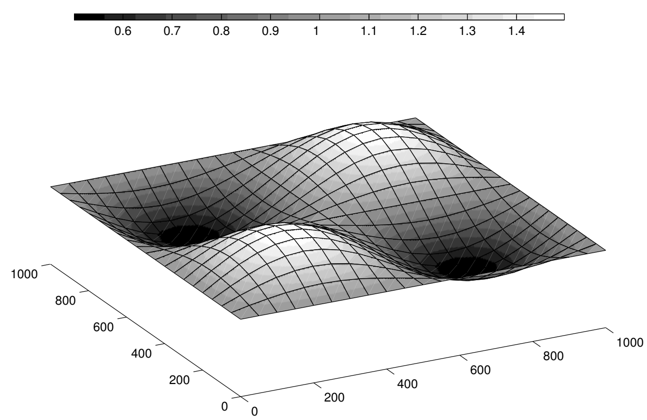

FIGURE 2. Water elevation for $\nu 2 = 0$ in $t = 10s$

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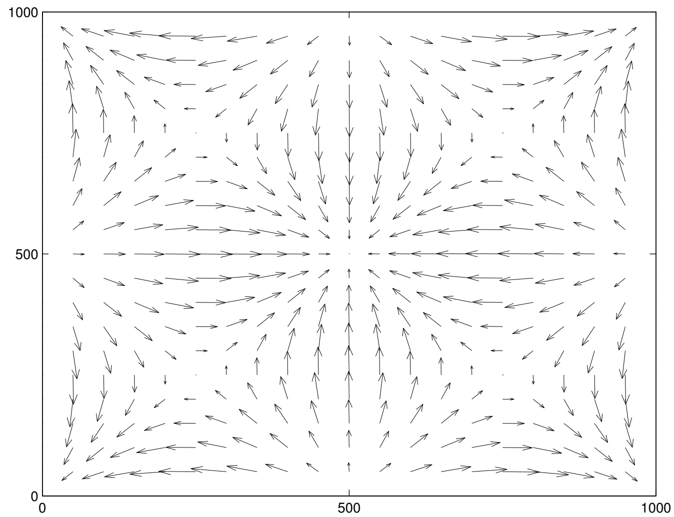

FIGURE 3. Velocity field for $\nu 2 = 0$ in $t = 10s$

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DEPARTEMENT DE MATHÉMATIQUES, FACULTÉ DES SCIENCES SEMLALIA, MAROC

E-mail address, M. ElAlaoui: elalaoui@ucam.ac.ma

E-mail address, M. H. Tber: tber.hicham@eudoramail.com