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DOUBLY NONLINEAR PARABOLIC EQUATIONS RELATED TO THE P-LAPLACIAN OPERATOR

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Abstract. This paper concerns the doubly nonlinear parabolic P.D.E.

$$\frac{\partial\beta(u)}{\partial t} - \Delta_p u + f(x,t,u) = 0 \quad \text{in } \Omega\times\mathbb{R}^+,$$

with Dirichlet boundary conditions and initial data. We investigate here a time-discretization of the continuous problem by the Euler forward scheme. In addition to existence, uniqueness and stability questions, we study the long-time behavior of the solution to the discrete problem. We prove the existence of a global attractor, and obtain regularity results under certain restrictions.

1. Introduction

We consider problems of the form

$$\begin{gathered}
\frac{\partial\beta(u)}{\partial t} - \Delta_p u + f(x,t,u) = 0 \quad \text{in } \Omega\times]0,\infty[,\\
u = 0 \quad \text{on } \partial\Omega\times]0,\infty[,\\
\beta\big(u(.,0)\big) = \beta(u_0) \quad \text{in } \Omega,
\end{gathered} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$, $1 < p < +\infty$, β is a nonlinearity of porous medium type and f is a nonlinearity of reaction-diffusion type.

The continuous problem (1.1) has already been treated quite completely in [7] for $p > 1$, and in the case p = 2 in [3]. Here, we shall discretize (1.1) and replace it by

$$\begin{gathered}
\beta(U^n) - \tau\Delta_p U^n + \tau f(x,n\tau,U^n) = \beta(U^{n-1}) \quad \text{in } \Omega,\\
U^n = 0 \quad \text{on } \partial\Omega,\\
\beta(U^0) = \beta(u_0) \quad \text{in } \Omega.
\end{gathered} \tag{1.2}$$

The case $p = 2$ is completely studied in [4]. Here we shall treat the case $p > 1$, and obtain existence, uniqueness and stability results for the solutions of (1.2). Then, existence of absorbing sets is given and the global attractor is shown to exist as well. Under restrictive conditions on f and p, a supplementary regularity result for

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the global attractor and, as a consequence, a stabilization result for the solutions of (1.2) are obtained in the case when $\beta(u) = u$.

The paper is organized as follows: in section 2, we give some preliminaries and in section 3, we deal with existence and uniqueness of solutions of the problem (1.2). The question of stability is studied in section 4, while the semi-discrete dynamical system study is done in section 5. finally, section 6 is dedicated to obtain more regularity for the attractor.

2. Preliminaries

Notation. Let β a continuous function with $\beta(0) = 0$. we define, for $t \in \mathbb{R}$,

$$\psi(t) = \int_0^t \beta(s)ds.$$

The Legendre transform ψ^* of ψ is defined by

$$\psi^*(\tau) = \sup_{s\in\mathbb{R}}\{\tau s - \psi(s)\}.$$

Here Ω stand for a regular open bounded set of $\mathbb{R}^d$, $d \geq 1$ and $\partial\Omega$ is it's boundary. The norm in a space X will be denoted as follows:
$\|\cdot\|_r$ if $X = L^r(\Omega)$, $1 \leq r \leq +\infty$;
$\|\cdot\|_{1,q}$ if $X = W^{1,q}(\Omega)$, $1 \leq q \leq +\infty$;
$\|\cdot\|_X$ otherwise;
and $\langle .,. \rangle$ denotes the duality between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$. For any $p \geq 1$ we define it's conjugate p' by $\frac{1}{p} + \frac{1}{p'} = 1$. On this paper, C_i and C will denote various positive constants.

Assumptions and definition of solution. We consider the following Euler forward scheme for (1.1):

$$\begin{gathered}\beta(U^n) - \tau\Delta_p U^n + \tau f(x, n\tau, U^n) = \beta(U^{n-1}) \quad \text{in } \Omega,\\ U^n = 0 \quad \text{on } \partial\Omega,\\ \beta(U^0) = \beta(u_0) \quad \text{in } \Omega,\end{gathered}$$

where $N\tau = T$ is a fixed positive real, and $1 \leq n \leq N$. We shall consider the case $u_0 \in L^2(\Omega)$, and we assume the following hypotheses.

(H1) β is an increasing continuous function from $\mathbb{R}$ to $\mathbb{R}$, $\beta(0) = 0$, and for some $C_1 > 0$, $C_2 > 0$, $\beta(\xi) \leq C_1|\xi| + C_2$ for any $\xi \in \mathbb{R}$

(H2) For any ξ in $\mathbb{R}$, the map $(x, t) \mapsto f(x, t, \xi)$ is measurable, and $\xi \mapsto f(x, t, \xi)$ is continuous a.e. in $\Omega \times \mathbb{R}^+$. Furthermore we assume that there exist $q > \sup(2, p)$ and positives constants C_3, C_4 and C_5 such that

$$\begin{gathered}\operatorname{sign}(\xi)f(x, t, \xi) \geq C_3|\xi|^{q-1} - C_4,\\ |f(x, t, \xi)| \leq a(|\xi|)\end{gathered}$$

where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing, and

$$\limsup_{t\to 0^+} |f(x, t, \xi)| \leq C_5(|\xi|^{q-1} + 1).$$

(H3) There is $C_6 > 0$ such that for almost $(x, t) \in \Omega\times\mathbb{R}^+$, $\xi \mapsto f(x, t, \xi) + C_6\beta(\xi)$ is increasing.

Definition 2.1. By a weak solution of the discretized problem, we mean a sequence $(U^n)_{0\le n\le N}$ such that $\beta(U^0)=\beta(u_0)$, and U^n is defined by induction as a weak solution of the problem

$$\begin{gathered}\beta(U)-\tau\Delta_p U+\tau f(x,n\tau,U)=\beta(U^{n-1}) \quad \text{in } \Omega,\\ U\in W_0^{1,p}(\Omega).\end{gathered}$$

3. Existence and uniqueness result

Theorem 3.1. *If $p\ge \frac{2d}{d+2}$, then for each $n=1,\dots,N$ there exists a unique solution U^n of (1.2) in $W^{-1,p}(\Omega)$ provided that $0<\tau<\frac{1}{C_6}$.*

Proof. We can write (1.2) as

$$\begin{gathered}-\tau\Delta_p U+F(x,U)=h,\\ U\in W_0^{1,p}(\Omega),\end{gathered}$$

where $U=U^n$, $h=\beta(U^{n-1})$ and $F(x,\xi)=\tau f(x,n\tau,\xi)+\beta(\xi)$. From (H1) and (H2) we obtain

$$rmsign(\xi)F(x,\xi)\ge -\tau C_4 \quad \text{and} \quad h\in W^{-1,p'}(\Omega) \text{ for } p\ge \frac{2d}{d+2}.$$

Hence the existence follows from a slight modification of a result in [1] (there, $C_4=0$). To obtain uniqueness, we set for simplicity

$$w=U^n, \quad \overline{f}(x,w)=f(x,n\tau,U^n), \quad g(x)=\beta(U^{n-1})\,.$$

Then problem (1.2) reads

$$\begin{gathered}-\tau\Delta_p w+\tau\overline{f}(x,w)+\beta(w)=g(x),\\ w\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).\end{gathered} \tag{3.1}$$

If w_1 and w_2 are two solutions of (3.1), then we have

$$-\tau\Delta_p w_1+\tau\Delta_p w_2+\tau(\overline{f}(x,w_1)-\overline{f}(x,w_2))+\beta(w_1)-\beta(w_2)=0. \tag{3.2}$$

Multiplying (3.2) by w_1-w_2 and integrating over Ω, gives

$$\begin{aligned}&\langle -\tau\Delta_p w_1+\tau\Delta_p w_2, w_1-w_2\rangle+\tau\int_\Omega \big(\overline{f}(x,w_1)-\overline{f}(x,w_2)\big)(w_1-w_2)dx\\ &+\int_\Omega \big(\beta(w_1)-\beta(w_2)\big)(w_1-w_2)dx=0.\end{aligned} \tag{3.3}$$

Applying (H3) yields

$$\int_\Omega \big(\overline{f}(x,w_1)-\overline{f}(x,w_2)\big)(w_1-w_2)dx\ge -C_6\int_\Omega \big(\beta(w_1)-\beta(w_2)\big)(w_1-w_2)dx. \tag{3.4}$$

Now, (3.4) and the monotonicity condition of the p-Laplacian operator reduce (3.3) to

$$(1-\tau C_6)\int_\Omega \big(\beta(w_1)-\beta(w_2)\big)(w_1-w_2)\,dx\le 0.$$

So by (H1), if $\tau<\frac{1}{C_6}$, we get $w_1=w_2$. □

4. Stability

Theorem 4.1. *Assume* $p \geq \frac{2d}{d+2}$. *Then there exists a positive constant* $C(T, u_0)$ *such that, for all* $n = 1, \dots, N$

$$\int_\Omega \psi^*(\beta(U^n))dx + \tau \sum_{k=1}^n \|U^k\|_{1,p}^p + C\tau \sum_{k=1}^n \|U^k\|_q^q \leq C(T, u_0) \tag{4.1}$$

and

$$\max_{1\leq k\leq n} \|\beta(U^k)\|_2^2 + \sum_{k=1}^n \|\beta(U^k) - \beta(U^{k-1})\|_2^2 \leq C(T, u_0). \tag{4.2}$$

Proof. **(a)** Multiply the first equation of (1.2), with k instead of n, by U^k. Then using (H2) and the relation

$$\int_\Omega \psi^*(\beta(U^k))dx - \int_\Omega \psi^*(\beta(U^{k-1}))dx \leq \int_\Omega \left(\beta(U^k) - \beta(U^{k-1})\right) U^k dx,$$

we get, after summing from $k = 1$ to n,

$$\int_\Omega \psi^*(\beta(U^n))dx + \tau \sum_{k=1}^n \|U^k\|_{1,p}^p + C\tau \sum_{k=1}^n \|U^k\|_q^q \leq C\tau \sum_{k=1}^n \|U^k\|_1 + \int_\Omega \psi^*(\beta(u_0))dx.$$

Thanks to Young's inequality, for all $\varepsilon > 0$ there exists $C_\varepsilon(T, u_0)$ such that

$$\int_\Omega \psi^*(\beta(U^n))dx + \tau \sum_{k=1}^n \|U^k\|_{1,p}^p + C\tau \sum_{k=1}^n \|U^k\|_q^q \leq \varepsilon\tau \sum_{k=1}^n \|U^k\|_p^p + C_\varepsilon(T, u_0).$$

Now for a suitable choice of ε, we have

$$\varepsilon\tau \sum_{k=1}^n \|U^k\|_p^p \leq C_\varepsilon(T, u_0).$$

That is, (4.1) is satisfied.

(b) Multiplying the first equation of (1.2), with k instead of n, by $\beta(U^k)$. Then using (H2), we have

$$\int_\Omega \left(\beta(U^k) - \beta(U^{k-1})\right)\beta(U^k)dx + \tau\langle -\Delta_p U^k, \beta(U^k)\rangle \leq C\tau \int_\Omega |\beta(U^k)|dx. \tag{4.3}$$

With the aid of the identity $2a(a - b) = a^2 - b^2 + (a - b)^2$, we obtain from (4.3),

$$\|\beta(U^k)\|_2^2 - \|\beta(U^{k-1})\|_2^2 + \|\beta(U^k) - \beta(U^{k-1})\|_2^2 \leq C\tau\|\beta(U^k)\|_1. \tag{4.4}$$

Summing (4.4) from $k = 1$ to n, yields

$$\|\beta(U^n)\|_2^2 + \sum_{k=1}^n \|\beta(U^k) - \beta(U^{k-1})\|_2^2 \leq \|\beta(u_0)\|_2^2 + C\tau \sum_{k=1}^n \|\beta(U^k)\|_1. \tag{4.5}$$

As in (a), we conclude to (4.2). □

5. The semi-discrete dynamical system

We fix τ such that $0 < \tau < \min(1, \frac{1}{C_6})$, and assume that $p > \frac{2d}{d+2}$. Theorem 3.1 allows us to define a map S_τ on $L^2(\Omega)$ by setting

$$S_\tau U^{n-1} = U^n.$$

As S_τ is continuous, we have $S_\tau^n U^0 = U^n$.

Our aim is to study the discrete dynamical system associated with (1.2) . We begin by showing the existence of absorbing balls in $L^\infty(\Omega)$. (We refer to [13] for the definition of absorbing sets and global attractor).

Absorbing sets in $L^\infty(\Omega)$.

Lemma 5.1. *If $p > \frac{2d}{d+1}$, then there exists $n(d,p) \in \mathbb{N}^*$ depending on d and p, and $C > 0$ depending on d, Ω and the constants in (H1)–(H3) such that*

$$U^n \in L^\infty(\Omega) \text{ for all } n \geq n(d,p) \tag{5.1}$$

and

$$\|U^{n(d,p)}\|_\infty \leq \frac{C}{\tau^{\alpha+\alpha^2+\dots+\alpha^{n(d,p)}}}\big(\|u_0\|_2^{\alpha^{n(d,p)}} + 1\big), \tag{5.2}$$

where $\alpha = p'/p$. Moreover, if $d = 1, d = 2$ or $d < 2p$, then $n(d,p) = 1$.

The proof of the lemma above follows from a repeated application of the following lemma (cf. [11])

Lemma 5.2. *If $u \in W_0^{1,p}(\Omega)$ is a solution to the equation*

$$-\tau\Delta_p u + F(x,u) = T,$$

where $T \in W^{-1,r}(\Omega)$ and F satisfies $\xi F(x,\xi) \geq 0$ in $\Omega \times \mathbb{R}$ then we have the following estimates

(a) *If $r > \frac{d}{p-1}$, then $u \in L^\infty(\Omega)$ and*

$$\|u\|_\infty \leq C\big(\frac{\|T\|_{-1,r}}{\tau}\big)^{p'/p}$$

(b) *If $p' \leq r < \frac{d}{p-1}$, then $u \in L^{r^*}(\Omega)$ and*

$$\|u\|_{r^*} \leq C\big(\frac{\|T\|_{-1,r}}{\tau}\big)^{p'/p},$$

where $\frac{1}{r^*} = \frac{1}{(p-1)r} - \frac{1}{d}$

(c) *If $r = \frac{d}{p-1}$ and $r \geq p'$ then $u \in L^q(\Omega)$ for any q, $1 \leq q < \infty$ and*

$$\|u\|_q \leq C\big(\frac{\|T\|_{-1,r}}{\tau}\big)^{p'/p}.$$

We can write (1.2) as

$$\begin{gathered} -\tau\Delta_p U^m + F_m(x,U^m) = \beta(U^{m-1}) + C_4 sign(U^m) = T_m \quad \text{in } \Omega, \\ U^m = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where

$$F_m(x,\xi) = \tau f(x, m\tau, \xi) + \beta(\xi) + C_4 sign(\xi).$$

Note that by (H1) and (H2) we have

$$\xi F_m(x,\xi) \geq 0 \text{ for all } \xi,$$
$$T_m \in W^{-1,p'}(\Omega).$$

Now, applying lemma 5.2, we can find an increasing sequence $\big(\alpha(m)\big)_{m\geq 1}$ such that

$$\alpha(m) \geq p', \quad \frac{1}{\alpha(m+1)} = \frac{1}{(p-1)\alpha(m)} - \frac{1}{d}, \tag{5.3}$$

and

$$\|U^m\|_{\alpha(m)} \leq \frac{C_m}{\tau^{\alpha+\alpha^2+\dots+\alpha^m}}\big(\|u_0\|_2^{\alpha^m}+1\big) \tag{5.4}$$

We shall stop the iteration on m once we have $\alpha(m-1) > \frac{d}{p}$. Indeed, if $q > \frac{d}{p}$, then there exists $r > \frac{d}{p-1}$ such that $L^q(\Omega) \subset W^{-1,r}(\Omega)$. Then we have $T_m \in W^{-1,r}(\Omega)$ and thus $U^m \in L^\infty(\Omega)$. $n(d,p)$ will be the first integer m such that $\alpha(m-1) > \frac{d}{p}$. Finally (5.2) will follow from (5.4) and lemma 5.2.

Remark 5.3. (i) If $d = 1$ or $d = 2$, then for all $q > 1$, we have $L^2(\Omega) \subset W^{-1,q}(\Omega)$, in particular for $q > \frac{d}{p-1}$. If $d \geq 3$ and $d < 2p$, we can choose $q > 1$ to be such that $\frac{d}{p-1} < q < \frac{2d}{d-2}$. In the two cases, $T_1 \in W^{-1,q}(\Omega)$ for some $q > \frac{d}{p-1}$ and, from lemma 5.2, $U^1 \in L^\infty(\Omega)$. We have then $n(d,p) = 1$.

(ii) If $\alpha(m) \leq \frac{d}{p}$ for all m , then $l = \lim_{m\to\infty} \alpha(m)$ exists and equals $\frac{2-p}{p-1}d$.

Consequently, for $p > \frac{2d}{d+1}$, we have $l < p'$, which contradicts the fact that $\alpha(m) \geq p'$. Hence, the existence of $n(d,p)$ is justified.

(iii) If $p = \frac{2d}{d+1}$, then $\alpha(m) = p'$ for all m, and $T_m \in W^{-1,p'}(\Omega)$. Then we cannot necessarily expect an L^∞ solution.This is due to results in [1], [2] and [11].

For the remaining of this article, we set $n_0 = n(d,p)$ and $C_1 = C\big(\|u_0\|_2^{\alpha^{n_0}} + 1\big)$.

Lemma 5.4. *Let k be such that $1 < k < q-1$ and $k \leq 1 + \frac{1}{n_0}$. Then, there exist $\gamma > 0, \delta > 0$ depending on the data of (H1)–(H3) and $\mu > 0$ depending on $n_0, q, \gamma, \delta, k$ such that, for all $n \geq n_0$, we have*

$$\|\beta(U^n)\|_\infty \leq \Big(\frac{\delta}{\gamma}\Big)^{\frac{1}{q-1}} + \frac{C_1+\mu}{\big(\tau^\beta(n-n_0+1)\big)^{\frac{1}{k-1}}} \quad ,$$

where $\beta = \begin{cases} 1 & \text{if } \alpha \leq 1, \\ \alpha^{n_0} & \text{if } \alpha \geq 1. \end{cases}$

Proof. From lemma 5.1, for $n \geq n_0$ we have

$$U^n \in L^\infty(\Omega) \quad \text{and} \quad \|U^{n_0}\|_\infty \leq \frac{C_1}{\tau^{\alpha+\alpha^2+\dots+\alpha^{n_0}}}.$$

Next, multiplying the first equation of (1.2) by $|\beta(U^n)|^m\beta(U^n)$ for some positive integer m, we derive from (H_1) and (H_2), after dropping some positive terms, that

$$\|\beta(U^n)\|_{m+2}^{m+2} \leq \int_\Omega |\beta(U^n)|^{m+1}\beta(U^{n-1})dx + C\tau\ |\beta(U^n)\|_{m+1}^{m+1} - C\tau\|\beta(U^n)\|_{m+q}^{m+q}.$$

By setting

$$y_m^n = \|\beta(U^n)\|_{m+2} \text{ and } z_n = \|\beta(U^n)\|_\infty,$$

and using Hölder's inequality, we deduce the existence of two constants $\gamma > 0, \delta > 0$ (not depending on m nor on U^n) such that

$$y_m^n + \gamma\tau(y_m^n)^{q-1} \leq \delta\tau + y_m^{n-1}.$$

As m approaches infinity, we then obtain

$$z_n + \gamma\tau z_n^{q-1} \leq \delta\tau + z_{n-1},$$

with

$$z_{n_0} \leq \frac{C_1}{\tau^{\alpha+\alpha^2+\dots+\alpha^{n_0}}}.$$

(i) If $\alpha \leq 1$, then $\alpha + \alpha^2 + \dots + \alpha^{n_0} \leq n_0$. So, we have

$$\begin{gathered} z_{n_0} \leq C_1/\tau^{n_0}, \\ z_n + \gamma\tau z_n^{q-1} \leq \delta\tau + z_{n-1}. \end{gathered}$$

Then we can apply lemma 7.1 in [4] to get

$$z_n \leq \big(\frac{\delta}{\gamma}\big)^{\frac{1}{q-1}} + \frac{C_1+\mu}{\big(\tau(n-n_0+1)\big)^{\frac{1}{k-1}}} \equiv c_\alpha(n).$$

(ii) If $\alpha \geq 1$, then $\alpha + \alpha^2 + \dots + \alpha^{n_0} \leq n_0\alpha^{n_0}$. By setting $\tau_1 = \tau^{\alpha^{n_0}}$, we have

$$\begin{gathered} z_{n_0} \leq C_1/\tau_1^{n_0}, \\ z_n + \gamma'\tau_1 z_n^{q-1} \leq \delta'\tau_1 + z_{n-1}, \end{gathered}$$

where $\gamma' = \tau^{1-\alpha^{n_0}}\gamma$ and $\delta' = \tau^{1-\alpha^{n_0}}\delta$. Then, once again, we can apply lemma 7.1 in [4] to get

$$z_n \leq \big(\frac{\delta}{\gamma}\big)^{\frac{1}{q-1}} + \frac{C_1+\mu}{\big(\tau_1(n-n_0+1)\big)^{\frac{1}{k-1}}} \equiv c_\alpha(n).$$

□

Remark 5.5. In the case $\alpha \geq 1$, a slight modification has to be introduced in the proof of lemma 7.1 in [4], since μ is depending on δ' and γ' and hence on τ. In fact, it suffices to choose in that proof, with the same notation, μ such that

$$\gamma\big(\frac{\delta}{\gamma}\big)^{1-\frac{k}{q-1}}\mu^{k-1} \geq 2^{\frac{1}{k-1}}/(k-1).$$

and to remark that $\gamma' \geq \gamma$.

Consequently, lemma 5.4 implies that there exist absorbing sets in $L^q(\Omega)$ for all $q \in [1,\infty]$. Indeed, this is due to the fact that

$$\|U^n\|_\infty \leq \max\big(\beta^{-1}(c_\alpha(n)), |\beta^{-1}(-c_\alpha(n))|\big),$$

for all $n \geq n_0$, with $c_\alpha(n) \to \big(\frac{\delta}{\gamma}\big)^{\frac{1}{q-1}}$ as $n \to \infty$.

Absorbing sets in $W_0^{1,p}(\Omega)$, existence of the the global attractor. Multiplying the equation in (1.2) by $\delta_n = U^n - U^{n-1}$, we get

$$\begin{aligned}&\langle \frac{\beta(U^n)-\beta(U^{n-1})}{\tau}, \delta_n\rangle + \int_\Omega |\nabla U^n|^{p-2}\nabla U^n.(\nabla U^n - \nabla U^{n-1})dx \\ &+ \langle f(x,n\tau,U^n),\delta_n\rangle = 0.\end{aligned} \tag{5.5}$$

By setting

$$F_\beta(u) = \int_0^u \big(f(x,n\tau,w) + C_6\beta(w)\big)dw,$$

we deduce from (H3) that

$$F'_\beta(u)(u-v) \geq F_\beta(u) - F_\beta(v),$$

and then

$$\begin{aligned}\langle f(x,n\tau,U^n),\delta_n\rangle &= \langle f(x,n\tau,U^n) + C_6\beta(U^n),\delta_n\rangle - C_6\langle \beta(U^n),\delta_n\rangle \\ &\geq \int_\Omega \big(F_\beta(U^n) - F_\beta(U^{n-1})\big)\,dx - C_6\langle\beta(U^n),\delta_n\rangle.\end{aligned}$$

Now, using (H1), we get $\psi'(v)(u-v) \leq \psi(u) - \psi(v)$. Therefore, we have

$$\begin{aligned}&\int_\Omega \beta(U^n)(U^n - U^{n-1})dx \\ &= \int_\Omega \big(\beta(U^n) - \beta(U^{n-1})\big)(U^n - U^{n-1})dx + \int_\Omega \beta(U^{n-1})(U^n - U^{n-1})dx \\ &\leq \int_\Omega \big(\beta(U^n) - \beta(U^{n-1})\big)(U^n - U^{n-1})dx + \int_\Omega \big(\psi(U^n) - \psi(U^{n-1})\big)\,dx.\end{aligned}$$

With the aid of the elementary identity

$$|a|^{p-2}a.(a-b) \geq \frac{1}{p}|a|^p - \frac{1}{p}|b|^p, \tag{5.6}$$

we obtain

$$\int_\Omega |\nabla U^n|^{p-2}\nabla U^n.(\nabla U^n - \nabla U^{n-1})dx \geq \frac{1}{p}\|U^n\|_{1,p}^p - \frac{1}{p}\|U^{n-1}\|_{1,p}^p. \tag{5.7}$$

Since $\tau < \frac{1}{C_6}$, from (5.5) we obtain

$$\frac{1}{p}\|U^n\|_{1,p}^p + \int_\Omega F_\beta(U^n)dx \leq C_6 \int_\Omega \big(\psi(U^n) - \psi(U^{n-1})\big)\,dx + \int_\Omega F_\beta(U^{n-1})dx. \tag{5.8}$$

Now, setting

$$F(u) = \int_0^u f(x,n\tau,w)dw,$$

yields

$$\int_\Omega F_\beta(u)dx = \int_\Omega F(u)dx + C_6\int_\Omega \psi(u)dx.$$

Hence, from (5.8), we get

$$\frac{1}{p}\|U^n\|_{1,p}^p + \int_\Omega F(U^n)dx \leq \frac{1}{p}\|U^{n-1}\|_{1,p}^p + \int_\Omega F(U^{n-1})dx.$$

By setting

$$y_n = \frac{1}{p}\|U^n\|_{1,p}^p + \int_\Omega F(U^n)dx,$$

we get $y_n \leq y_{n-1}$. And by choosing $N\tau = 1$, using the boundedness of U^n and the stability analysis, there exists $n_\tau > 0$ such that

$$\tau \sum_{n=n_0}^{n_0+N} y_n \leq a_1, \quad \text{for all } n \geq n_\tau.$$

Hence we can apply the discrete version of the uniform Gronwall lemma (cf. [4, Lemma 7.5]) with $h_n = 0$ to obtain

$$\frac{1}{p}\|U^n\|_{1,p}^p + \int_\Omega F(U^n)dx \leq C \quad \text{for all } n \geq n_\tau.$$

On the other hand, since U^n is bounded, we deduce that

$$\|U^n\|_{1,p} \leq C.$$

We have then proved the following result.

Proposition 5.6. *If $\tau < \frac{1}{C_6}$, there exist absorbing sets in $L^\infty(\Omega) \cap W^{-1,p}(\Omega)$. More precisely, for any $u_0 \in L^2(\Omega)$, there exists a positive integer n_τ such that*

$$\|U^n\|_\infty + \|U^n\|_{1,p} \leq C, \quad \forall n \geq n_\tau, \tag{5.9}$$

where C does not depend on τ.

In order for the nonlinear map S_τ to satisfy the properties of a semi-group, namely $S_\tau^{n+p} = S_\tau^n o S_\tau^p$, we need (1.2) to be autonomous. So, we assume that $f(x,t,\xi) \equiv f(x,\xi)$. Thus, S_τ defines a semi-group from $L^2(\Omega)$ into itself and possesses an absorbing ball B in $L^\infty(\Omega) \cap W^{-1,p}(\Omega)$. Setting

$$\mathcal{A}_\tau = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m\geq n} S_\tau^m(B)},$$

we have a compact subset of $L^2(\Omega)$ which attracts all solutions. That implies that for all $u_0 \in L^2(\Omega)$,

$$\operatorname{dist}\big(\mathcal{A}_\tau, S_\tau^n u_0\big) \mapsto 0 \quad \text{as } n \mapsto \infty.$$

Therefore, we have proved the following result.

Theorem 5.7. *Assuming $u_0 \in L^2(\Omega)$ and (H1)–(H3), the discrete problem* (1.2) *has an associated solution semi-group S_τ that maps $L^2(\Omega)$ into $L^\infty(\Omega) \cap W^{-1,p}(\Omega)$. This semi-group has a compact attractor which is bounded in $L^\infty(\Omega) \cap W^{-1,p}(\Omega)$.*

6. More regularity for the attractor

In this section, we shall show supplementary regularity estimates on the solutions of problem (1.2) in the particular case where $\beta(\xi) = \xi$. We obtain therefore more regularity for the attractor obtained in section 5. The assumptions are similar to those used for the continuous problem in [7]; namely $u_o \in L^2(\Omega)$ and f verifying the following assumption

(H4) $f(x,t,\xi) = g(\xi) - h(x)$ where $h \in L^\infty(\Omega)$ and g satisfying the conditions (H1)–(H3).

The problem (1.2) becomes

$$\delta_n - \Delta_p U^n + g(U^n) = f, \tag{6.1}$$

where $\delta_n = \frac{U^n - U^{n-1}}{\tau}$.

First, we state the following lemma which we shall use to prove the principal result of this section.

Lemma 6.1. *There exists a positive real C such that for all $n_0 \geq n_\tau$, and all N in $\mathbb{N}$, we have*

$$\tau \sum_{n=n_0}^{n_0+N} \|\delta_n\|_2^2 \leq C. \tag{6.2}$$

Proof. Multiplying (6.1) by δ_n, using (5.7), (5.9), (H4) and Young's inequality, we get after some simple calculations

$$\frac{1}{4}\tau\|\delta_n\|_2^2 + \frac{1}{p}\|U^n\|_{1,p}^p - \frac{1}{p}\|U^{n-1}\|_{1,p}^p \leq C\tau. \tag{6.3}$$

Summing (6.3) from $n = n_0$ to $n = n_0 + N$, yields

$$\frac{1}{4}\tau \sum_{n=n_0}^{n_0+N} \|\delta_n\|_2^2 + \frac{1}{p}\|U^{n_0+N}\|_{1,p}^p \leq \frac{1}{p}\|U^{n_0}\|_{1,p}^p + CN\tau. \tag{6.4}$$

Now choosing $n_0 \geq n_\tau$ shows that U^{n_0} is in an $W^{-1,p}(\Omega)$–absorbing ball. As $\tau N = 1$, we therefore obtain (6.2) from (6.4). □

Theorem 6.2. *For all $n \geq n_\tau$, we have $\|\delta_n\|_2 \leq C$, where C is a positive constant.*

Proof. By subtracting equation (6.1), with $n-1$ instead of n, from (6.1) and multiplying the difference by δ_n, we deduce from the monotonicity of the p-Laplacian operator, Young's inequality and (H3) that

$$\frac{1}{2}\|\delta_n\|_2^2 \leq \frac{1}{2}\|\delta_{n-1}\|_2^2 + C\tau\|\delta_n\|_2^2.$$

Setting

$$y_n = \frac{1}{2}\|\delta_n\|_2^2 \quad \text{and} \quad h_n = C\|\delta_n\|_2^2.$$

and using [4, Lemma 7.5] and Lemma 6.1, we deduce that

$$y_{n+N} \leq \frac{C}{N\tau} + C.$$

If $n \geq n_\tau$ and $N\tau = 1$, then we get the desired estimate. □

Using this theorem, we have the following regularizing estimates.

Corollary 6.3. *If $p > \frac{2d}{d+2}$ and $p \neq 2$, then there exists some σ, $0 < \sigma < 1$, such that*

$$\|U^n\|_{B_\infty^{1+\sigma,p}(\Omega)} \leq C \text{ for all } n \geq n_\tau,$$

where $B_\infty^{\alpha,p}(\Omega)$ denotes a Besov space defined by real interpolation method.
If $p = 2$, then

$$\|U^n\|_{W^{2,2}(\Omega)} \leq C \text{ for all } n \geq n_\tau.$$

Proof. (i) If $\frac{2d}{d+2} < p < 2$ then there exists some $\sigma', 0 < \sigma' < 1$ such that

$$L^2(\Omega) \hookrightarrow W^{-\sigma',p'}(\Omega) \tag{6.5}$$

By $(6)_n$, (6.5), (H4) and theorem 6.2 we get

$$\|-\Delta_p U^n\|_{B_\infty^{-\sigma',p'}(\Omega)} \leq C \quad \text{for all } n \geq n_\tau.$$

Therefore, Simon's regularity result in [12] yields

$$\|U^n\|_{B_\infty^{1+(1-\sigma')(p-1)^2,p}(\Omega)} \leq C \quad \text{for all } n \geq n_\tau.$$

(ii) If $p > 2$, then, by (6.1), (H4) and theorem 6.2, we get

$$\| -\Delta_p U^n\|_{p'} \leq C \quad \text{for all } n \geq n_\tau.$$

Therefore, Simon's regularity result in [12] yields

$$\|U^n\|_{B_\infty^{1+\frac{1}{(p-1)^2},p}(\Omega)} \leq C \quad \text{for all } n \geq n_\tau.$$

(iii) For $p = 2$, see [4]. □

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