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ERROR ESTIMATE FOR THE CHARACTERISTIC METHOD INVOLVING OLDROYD DERIVATIVE IN A TENSORIAL TRANSPORT PROBLEM

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Abstract. An optimal *a priori* error estimate $O(h^{k+1}+\Delta t)$, result is presented for a tensor problem involving Oldroyd derivative when using a suitable characteristic method and a finite element method. To conclude, we present results of numerical tests which confirm the previous estimates. Our long time goal is to deal with the viscoelastic fluid flow problem.

1. Introduction

The Oldroyd derivative of a symmetric tensor σ is defined by

$$\frac{\mathscr{D}_a\sigma}{\mathscr{D}t}=\frac{\partial\sigma}{\partial t}+(\mathbf{u}.\nabla)\sigma+\sigma M_a(\mathbf{u})+M_a^T(\mathbf{u})\sigma, \tag{1.1}$$

where $\mathbf{u}$ is a given velocity field, $M_a(\mathbf{u})=\left((1-a)\boldsymbol{\nabla}\mathbf{u}-(1+a)\boldsymbol{\nabla}\mathbf{u}^T\right)/2$ and $a\in[-1,1]$ is the parameter of the Oldroyd derivative. In this paper, we suppose also, for simplicity, that $\mathbf{u}=0$ on $\partial\Omega$. Problems involving the Oldroyd derivative appear in viscoelasticity (non-Newtonian polymer melt flow problems, see e.g. [6]), in turbulence modelling ($R_{ij}-\epsilon$ models) or in liquid crystals modelling. Let Ω be a bounded polygonal subset of $\mathbb{R}^d$, $d=1,2$ or 3, $T>0$ be a time constant, γ a given symmetric tensor defined in $\Omega\times]0,T[$, σ_0 given in Ω. and $\lambda>0$. The aim of this paper is to study, as a preliminary result, the approximation of the linear transport equation involving the Oldroyd derivative: *find σ defined in $\Omega\times]0,T[$ such that*

$$\begin{gathered}\lambda\frac{\mathscr{D}_a\sigma}{\mathscr{D}t}+\sigma=\gamma\quad\text{in }\Omega\times]0,T[,\\ \sigma(0)=\sigma_0\quad\text{in }\Omega.\end{gathered}\tag{1.2}$$

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and its discrete counterpart: *find* $\sigma_h^{(n)} \in T_h$, $1 \leq n \leq N$, *such that, for all* $\tau_h \in T_h$,

$$\Big(\lambda \frac{\sigma_h^{(n+1)} - R_{\Delta t}^{(n)} \times \sigma_h^{(n)} \circ X_{\Delta t}^{(n)} \times \left(R_{\Delta t}^{(n)}\right)^T}{\Delta t} + \sigma_h^{(n+1)}, \tau_h\Big) = (\gamma(t_{n+1}), \tau_h)\,, \quad (1.3)$$

and $\sigma_h^{(0)} = \pi_h \sigma_0$, where $N \geq 1$, $\Delta t = T/N$, $t_n = n\Delta t$, $X_{\Delta t}^{(n)}(x) = x - \Delta t\, \mathbf{u}(x, t_{n+1})$, $R_{\Delta t}^{(n)}(x) = I - \Delta t\, M_a^T(\mathbf{u})(x, t_{n+1})$, and T_h denotes the space of continuous piecewise polynomial symmetric tensors $T_h = \{\tau \in \left(C^0(\overline{\Omega}) \cap L^2(\Omega)\right)^{d\times d};\ \tau_{|K} \in P_k,\ \forall K \in \mathscr{T}_h\}$.

Here, $(\mathscr{T}_h)_{h>0}$ denotes a suitable family of regular triangulations of $\overline{\Omega}$. The Lagrange interpolation operator from L^2 tensors into T_h is denoted by π_h, $k \geq 1$.

The characteristic method [1] has been proposed for the numerical treatment of convected-dominated flows and transport equations. It is based on an approximation of the material derivative $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u}.\nabla)$:

$$\frac{D\varphi}{Dt}(x,t) \approx \frac{\varphi(x,t) - \varphi(X(x,t;s),s)}{t-s} \quad (1.4)$$

The trajectory $X(x,s;.)$ is defined for all $(x,s) \in \overline{\Omega} \times [0,T]$ by

$$\begin{gathered} \frac{\partial X}{\partial t}(x,s;t) = \mathbf{u}(X(x,s;t),t), \quad t \in]0,T[, \\ X(x,s;s) = x. \end{gathered} \quad (1.5)$$

In 1987, D. Esselaoui and M. Fortin [3] extended the characteristic method for the approximation of the Oldroyd derivative of a symmetric tensor. For all fixed $s \in [0,T]$, these authors considered the following transformed problem: *find* $\widehat{\sigma}(.,.;s)$ *defined in* $\Omega\times]0,T[$ *such that*

$$\begin{gathered} \lambda \frac{D\widehat{\sigma}}{Dt}(x,t;s) + \widehat{\sigma}(x,t;s) = R(x,t;s)\gamma(x,t)R^T(x,t;s), \quad (x,t) \in \Omega\times]0,T[, \\ \widehat{\sigma}(x,0;s) = R(x,0;s)\sigma_0(x)R^T(x,0;s), \quad x \in \Omega, \end{gathered} \quad (1.6)$$

where $\widehat{\sigma}(x,t;s) = R(x,t;s)\sigma(x,t)R(x,t;s)^T$ and the tensor flow $R(x,.;s)$ is defined for all $(x,s) \in \overline{\Omega} \times [0,T]$ by

$$\begin{gathered} \frac{DR}{Dt}(x,t;s) = R(x,t;s)M_a^T(\mathbf{u})(x,t), \quad t \in]0,T[, \\ R(x,s;s) = I. \end{gathered} \quad (1.7)$$

The Oldroyd derivative has been replaced by a material derivative of a tensor, suitable for the characteristic method, as (1.3). A short computation shows that

$$\frac{D^m \widehat{\sigma}}{Dt^m}(x,t;s) = R^m(x,t;s)\frac{\mathscr{D}_a^m \sigma}{\mathscr{D}t^m}(x,t)(R^m)^T(x,t;s), \quad \forall m \in \mathbb{N}, \quad (1.8)$$

and thus problems (1.2) and (1.6) are equivalent. Numerical computations involving the scheme (1.6) has been already performed in [3] while the corresponding numerical analysis was not yet available. We show a $\mathscr{O}\left(h^{k+1} + \Delta t\right)$ optimal estimate. Moreover, since the trajectories and the transformation are both approximated, the scheme (1.6) is of practical interest. To conclude, we present results of numerical tests that confirm the previous estimates.

2. Error estimate

Let $\|\cdot\|$ the $L^2(\Omega)$ norm, $\|\cdot\|_\infty$ the $L^\infty(\Omega)$ one and $|\cdot|_{m,p,\Omega}$ the $W^{m,p}(\Omega)$ semi-norm, for $m \geq 0$ et $p \in [1, \infty]$. Also, $C^{0,1}(\overline{\Omega})$ denotes the space of lipschitzian functions $\overline{\Omega}$. For a Banach space Y, let us denote $C(Y)$ the space $C([0,T], Y)$. We suppose also that the data $\mathbf{u} \in C(C^{0,1}(\overline{\Omega}))$: the existence and the continuity of $x \to X(x, s; t)$ follow then from the Cauchy-Lipschitz theorem. Let us denote finally $X^{(n)}(x) = X(x, t_{n+1}; t_n)$ and $R^{(n)}(x) = R(X^{(n)}(x), t_n; t_{n+1})$. In this paper, C_i, $i \in \mathbb{N}$ is a positive constant, independent of h and Δt.

For a Banach space Y, with norm $\|.\|_Y$ and $1 \leq p < \infty$ we introduce

$$l^p(0,T;Y) = \Big\{\varphi : (t_1, \dots, t_N) \to Y; \|\varphi\|_{l^p(0,T;Y)} = \Big(\sum_{n=1}^{N} \|\varphi(t_i)\|_Y^p \Delta t\Big)^{1/p} < \infty\Big\},$$

$$l^\infty(0,T;Y) = \Big\{\varphi : (t_1, \dots, t_N) \to Y; \|\varphi\|_{l^\infty(0,T;Y)} = \max_{1 \leq i \leq N} \|\varphi(t_i)\|_Y < \infty\Big\}.$$

Theorem 2.1 (Error estimate). *Let σ and σ_h the solutions of (1.2) and (1.3), respectively. Suppose that $\mathbf{u}$ is in $\big(C(C^{0,1}(\overline{\Omega})) \cap W^{1,\infty}(W^{1,\infty}(\Omega)\big)^d$ and σ is in $\big(W^{1,\infty}(H^{m+1}(\Omega)) \cap W^{2,\infty}(L^2(\Omega))\big)^{d \times d}$, with $m \geq 0$. Then, there exist three positive constants Δt_0, h_0 and c, independent of h and Δt, such that, if $\Delta t \leq \Delta t_0$ and $h \leq h_0$, we have*

$$\|\sigma - \sigma_h\|_{l^\infty(0,T;L^2(\Omega))} \leq c\left(h^{r+1} + \Delta t\right), \tag{2.1}$$

$$\|\sigma - \sigma_h\|_{l^2(0,T;L^2(\Omega))} \leq c\left(h^{r+1} + \Delta t\right), \tag{2.2}$$

where $r = \min(k, m)$.

Proof. Let us introduce $\tilde{\sigma}_h(t) = \pi_h \sigma(t)$, $t \in [0, T]$, $\tilde{\sigma}_h^{(n)} = \tilde{\sigma}_h(t_n)$ and $\varepsilon_h^{(n)} = \sigma_h^{(n)} - \tilde{\sigma}_h^{(n)}$. By a development and using (1.3), we get

$$\left(\lambda \frac{\varepsilon_h^{(n+1)} - R_{\Delta t}^{(n)} \times \varepsilon_h^{(n)} \circ X_{\Delta t}^{(n)} \times \left(R_{\Delta t}^{(n)}\right)^T}{\Delta t} + \varepsilon_h^{(n+1)}, \varepsilon_h^{(n+1)}\right) = (\rho_h^{(n+1)}, \varepsilon_h^{(n+1)}), \tag{2.3}$$

where

$$\rho_h^{(n+1)} = \gamma(t_{n+1}) - \tilde{\sigma}_h^{(n+1)} - \lambda \frac{\tilde{\sigma}_h^{(n+1)} - R_{\Delta t}^{(n)} \times \tilde{\sigma}_h^{(n)} \circ X_{\Delta t}^{(n)} \times \left(R_{\Delta t}^{(n)}\right)^T}{\Delta t}. \tag{2.4}$$

From the Cauchy-Schwartz inequality and using the identity $ab \leq (a^2 + b^2)/2$,

$$\begin{aligned}
&\left(R_{\Delta t}^{(n)} \varepsilon_h^{(n)} \circ X_{\Delta t}^{(n)} \big(R_{\Delta t}^{(n)}\big)^T, \varepsilon_h^{(n+1)}\right) \\
&\leq \frac{1}{2}\left(\left\|R_{\Delta t}^{(n)} \varepsilon_h^{(n)} \circ X_{\Delta t}^{(n)} \left(R_{\Delta t}^{(n)}\right)^T\right\|^2 + \left\|\varepsilon_h^{(n+1)}\right\|^2\right)
\end{aligned}$$

and

$$\left(\rho_h^{(n+1)}, \varepsilon_h^{(n+1)}\right) \leq \frac{1}{2}\left(\frac{1}{2}\left\|\rho_h^{(n+1)}\right\|^2 + 2\left\|\varepsilon_h^{(n+1)}\right\|^2\right).$$

Then, (2.3) becomes

$$\left\|\varepsilon_h^{(n+1)}\right\|^2 \leq \left\|R_{\Delta t}^{(n)} \times \varepsilon_h^{(n)} \circ X_{\Delta t}^{(n)} \times (R_{\Delta t}^{(n)})^T\right\|^2 + \frac{\Delta t}{2\lambda}\left\|\rho_h^{(n+1)}\right\|^2.$$

From lemma 3.1 (paragraph 2), for Δt small enough, we get

$$\left\|\varepsilon_h^{(n+1)}\right\|^2 \le (1+C_0\Delta t)\left\|\varepsilon_h^{(n)}\right\|^2 + \frac{\Delta t}{2\lambda}\left\|\rho_h^{(n+1)}\right\|^2.$$

From the discrete Gronwall lemma, and using $n \le N = T/\Delta t$,

$$\left\|\varepsilon_h^{(n)}\right\|^2 \le \frac{e^{C_0 T}}{2\lambda}\|\rho_h\|^2_{l^2(L^2)} \tag{2.5}$$

where $\rho_h = (\rho_h^{(1)}, \dots, \rho_h^{(N)})$. It still remains to bound the right-hand side. From (1.6) with $s = t = t_{n+1}$, we get

$$\lambda \frac{D\widehat{\sigma}}{Dt}(x, t_{n+1}; t_{n+1}) + \sigma(x, t_{n+1}) - \gamma(x, t_{n+1}) = 0. \tag{2.6}$$

Adding (2.6) to the expression (2.4) of $\rho_h^{(n+1)}$,

$$\begin{aligned}\rho_h^{(n+1)} &= (\sigma - \tilde{\sigma}_h)(t_{n+1}) \\ &\quad + \lambda\Big\{\frac{D\widehat{\sigma}}{Dt}(x, t_{n+1}; t_{n+1}) - \frac{\tilde{\sigma}_h^{(n+1)} - R_{\Delta t}^{(n)} \times \tilde{\sigma}_h^{(n)} \circ X_{\Delta t}^{(n)} \times \left(R_{\Delta t}^{(n)}\right)^T}{\Delta t}\Big\}.\end{aligned}$$

Let us introduce the splitting $\rho_h^{(n+1)} = \zeta_h^{(n+1)} + \eta_h^{(n+1)}$, where

$$\begin{aligned}&\zeta_h^{(n+1)} \\ &= (\sigma - \tilde{\sigma}_h)(t_{n+1}) + \frac{\lambda}{\Delta t}\left\{(\sigma - \tilde{\sigma}_h)(t_{n+1}) - R^{(n)} \times (\sigma - \tilde{\sigma}_h)(X^{(n)}, t_n) \times (R^{(n)})^T\right\},\end{aligned}$$

and

$$\begin{aligned}&\eta_h^{(n+1)} \\ &= \lambda\Big\{\frac{D\widehat{\sigma}}{Dt}(x, t_{n+1}; t_{n+1}) - \frac{\sigma(t_{n+1}) - R^{(n)} \times \sigma\left(X^{(n)}, t_n\right) \times \left(R^{(n)}\right)^T}{\Delta t}\Big\} \\ &\quad - \frac{\lambda}{\Delta t}\Big\{R^{(n)} \times \tilde{\sigma}_h\left(X^{(n)}, t_n\right) \times \left(R^{(n)}\right)^T - R_{\Delta t}^{(n)} \times \tilde{\sigma}_h\left(X_{\Delta t}^{(n)}, t_n\right) \times \left(R_{\Delta t}^{(n)}\right)^T\Big\}.\end{aligned}$$

On the one hand, using a classical interpolation result [2], we have $\|\varphi - \pi_h\varphi\| \le C_4 h^{r+1}|\varphi|_{r+1,2,\Omega}$, for all $\varphi \in H^{r+1}(\Omega)$. Then, lemma 3.2 (paragraph 2) yields $\|\zeta_h^{(n+1)}\| = \mathscr{O}(h^{r+1})$. On the other hand, the lemma 3.2 and 3.3 (paragraph 2) and the continuity of π_h in $H^1(\Omega)$ give $\|\eta_h^{(n+1)}\| = \mathscr{O}(\Delta t)$. Thus, the result yields by reporting $\|\rho_h^{(n+1)}\| = \mathscr{O}(h^{r+1} + \Delta t)$ in (2.5). □

3. Auxiliary lemma

Lemma 3.1 (Estimate on the approximate transformation)**.** *There exist two positive constants C_0 and Δt_0 such that, if $\Delta t \le \Delta t_0$, then*

$$\left\|R_{\Delta t}^{(n)} \times \tau \circ X_{\Delta t}^{(n)} \times \left(R_{\Delta t}^{(n)}\right)^T\right\|^2 \le (1 + C_0\Delta t)\|\tau\|^2, \quad \forall \tau \in L^2(\Omega).$$

Proof. Let $d(x) = \mathrm{dist}(x, \partial\Omega)$ for $x \in \Omega$. Since $\mathbf{u} = 0$ on $\partial\Omega$, we have

$$|X_{\Delta t}^{(n)}(x) - x| = |\mathbf{u}(x, t_{n+1})|\,\Delta t \le \|\mathbf{u}\|_{L^\infty(W^{1,\infty})}\, d(x)\Delta t.$$

This leads to $X_{\Delta t}^{(n)}(x) \in \Omega$ for $\Delta t < 1/|\mathbf{u}|_{L^\infty(W^{1,\infty})}$. Since $X_{\Delta t}^{(n)}(\partial\Omega) = \partial\Omega$, and from the continuity of $\mathbf{u}(.,t_{n+1})$, we get $X_{\Delta t}^{(n)}(\Omega) = \Omega$. Let $J_{\Delta t}^{(n)} = \det\left(R_{\Delta t}^{(n)}\right)$. Let $|.|$ denotes the matrix norm in $\mathbb{R}^{d\times d}$. We get

$$\begin{aligned}
&\int_\Omega \left|R_{\Delta t}^{(n)} \times \tau \circ X_{\Delta t}^{(n)} \times \left(R_{\Delta t}^{(n)}\right)^T\right|^2 \mathrm{d}x \\
&\le \|R_{\Delta t}^{(n)}\|_\infty^4 \int_\Omega \left|\tau \circ X_{\Delta t}^{(n)}\right|^2 \mathrm{d}x \\
&\le \|R_{\Delta t}^{(n)}\|_\infty^4 \int_{X_{\Delta t}^{(n)}(\Omega)} |\tau(y)|^2 \, J_{\Delta t}^{(n)}(y)\,\mathrm{d}y \le \|R_{\Delta t}^{(n)}\|_\infty^4 \|J_{\Delta t}^{(n)}\|_\infty^2 \|\tau\|^2.
\end{aligned}$$

From the definition of $R_{\Delta t}^{(n)}$, we get $\|R_{\Delta t}^{(n)}\|_\infty \le 1 + \Delta t\|\mathbf{u}\|_{L^\infty(W^{1,\infty})}$ and, for Δt small enough, there exists a positive constant c depending only of $\mathbf{u}$ such that $\|J_{\Delta t}^{(n)}\|_\infty \le 1 + c\Delta t$. Then we obtain the result. □

Lemma 3.2 (Time approximation). *There exists K_1, C_1 and C_2 such that, if $\Delta t \le K_1/\|\mathbf{u}\|_{L^\infty(W^{1,\infty})}$, then*
(i) for all $\tau \in W^{1,\infty}(L^2(\Omega))^{d\times d}$

$$\left\|\tau(.,t_{n+1}) - R^{(n)} \times \tau(X^{(n)},t_n) \times \left(R^{(n)}\right)^T\right\| \le C_1\Delta t \left\|\frac{\mathscr{D}_a\tau}{\mathscr{D}t}\right\|_{L^\infty(L^2)}$$

(ii) for all $\tau \in W^{2,\infty}(L^2(\Omega))^{d\times d}$ we have

$$\begin{aligned}
&\left\|\frac{D\widehat{\tau}}{Dt}(.,t_{n+1};t_{n+1}) - \frac{\tau(.,t_{n+1}) - R^{(n)} \times \tau(X^{(n)},t_n) \times \left(R^{(n)}\right)^T}{\Delta t}\right\| \\
&\le C_2\Delta t \left\|\frac{\mathscr{D}_a^2\tau}{\mathscr{D}t^2}\right\|_{L^\infty(L^2)}
\end{aligned}$$

Proof. Let $f(t) = \widehat{\tau}(X(x,t_{n+1},t),t;t_{n+1})$. From one hand, remark that $f(t_{n+1}) = \tau(x,t_{n+1})$ and $f(t_n) = R^{(n)} \times \tau(X^{(n)},t_n) \times \left(R^{(n)}\right)^T$. From other hand, by the property of the material derivative $f^{(m)}(t) = \frac{D^m\widehat{\tau}}{Dt^m}(X(x,t_{n+1},t),t;t_{n+1})$, $m \ge 0$. Then, the result yields from the error estimate for the Taylor interpolation polynoms, from (1.8) and from the properties of the solution $R(x,.;t_{n+1})$ of the linear differential equation (1.7). □

Lemma 3.3 (Trajectories and transformation approximations). *If $\mathbf{u}$ is in the space $W^{1,\infty}(W^{1,\infty})^d$ and τ in $H^1(\Omega)^{d\times d}$ then there exists C_3 such that*

$$\begin{aligned}
&\left\|R^{(n)} \times \tau \circ X^{(n)} \times \left(R^{(n)}\right)^T - R_{\Delta t}^{(n)} \times \tau \circ X_{\Delta t}^{(n)} \times \left(R_{\Delta t}^{(n)}\right)^T\right\| \\
&\le C_3\Delta t^2 \|\mathbf{u}\|_{W^{1,\infty}(W^{1,\infty})} \|\tau\|_{1,2,\Omega}
\end{aligned}$$

Proof. Let us consider the following splitting

$$\begin{aligned}
&R^{(n)} \times \tau \circ X^{(n)} \times \left(R^{(n)}\right)^T - R_{\Delta t}^{(n)} \times \tau \circ X_{\Delta t}^{(n)} \times \left(R_{\Delta t}^{(n)}\right)^T \\
&= \left(R^{(n)} - R_{\Delta t}^{(n)}\right) \times \tau \circ X^{(n)} \times \left(R^{(n)}\right)^T \\
&\quad + R_{\Delta t}^{(n)} \times \tau \circ X^{(n)} \times \left(R^{(n)} - R_{\Delta t}^{(n)}\right)^T + R_{\Delta t}^{(n)} \times \left(\tau \circ X^{(n)} - \tau \circ X_{\Delta t}^{(n)}\right) \times \left(R_{\Delta t}^{(n)}\right)^T.
\end{aligned}$$

From the Taylor expansion of $f_1(t) = R(X(x, t_{n+1}, t), t, t_{n+1})$, we get $R^{(n)} - R_{\Delta t}^{(n)} = \mathscr{O}(\Delta t^2)$. Then, from the expansion of $f_2(t) = X(x, t_{n+1}, t)$, we get $X^{(n)} - X_{\Delta t}^{(n)} = \mathscr{O}(\Delta t^2)$. Finally, from the development of

$$g(\theta) = \tau\left(\theta X^{(n)}(x) + (1-\theta) X_{\Delta t}^{(n)}(x)\right),$$

we have $\tau \circ X^{(n)} - \tau \circ X_{\Delta t}^{(n)} = \mathscr{O}(\Delta t^2)$. □

4. Numerical experiments

In this paragraph, we present numerical results for $d = 2$ and $k = 1$. We choose $\mathbf{u} = (-x_2, x_1)$ $\Omega =]-1/2, 1/2[^2$, $T = 2\pi$, $\gamma = 0$ and the initial condition σ_0 is chosen such that the solution of (1.2) is given by

$$\begin{aligned}\sigma(x,t) &= \frac{1}{2}\exp\left\{-\frac{t}{\lambda} - \frac{(x_1 - x_{1,c}(t))^2 + (x_2 - x_{2,c}(t))^2}{r_0^2}\right\} \\ &\quad \times \begin{pmatrix} 1+\cos(2t) & \sin(2t) \\ \sin(2t) & 1-\cos(2t)\end{pmatrix}\end{aligned}$$

where $x_{1,c}(t) = \bar{x}_{1,c}\cos(t) - \bar{x}_{2,c}\sin(t)$ et $x_{2,c}(t) = \bar{x}_{1,c}\sin(t) + \bar{x}_{2,c}\cos(t)$ with $r_0 > 0$ and $(\bar{x}_{1,c}, \bar{x}_{2,c}) \in \mathbb{R}^2$. The numerical tests are performed for $\lambda = 1$, $r_0 = 1/10$ and $(\bar{x}_{1,c}, \bar{x}_{2,c}) = (1/4, 0)$.

The scheme (1.3) is based on the free software `rheolef` [7]. The computation of the scalar product $\left(R_\Delta^{(n)}\sigma_h^{(n)} \circ X_\Delta^{(n)}\left(R_\Delta^{(n)}\right)^T, \tau_h\right)$ in the approximate problem (1.3) is not an obvious task (see also [4, 5]). In each triangle, we use the six point fourth order Gauss quadrature formulae.

Since our domain is a square, we split each edge in M segments, and obtain M^2 small squares of length $1/M$. Then we split each of them in two triangles: the step of the mesh is then $h = \sqrt{2}/M$. Fig. 1.a represents $\|\sigma_h - \pi_h\sigma\|_{l^2(L^2)}$ as a function of Δt for four triangulations given by $h = \sqrt{2}/2^i$, $3 \le i \le 6$. We observe that when $\Delta t \to 0$, the error starts to decrease and then tends to a constant that behaves as h^2. A simultaneous choice of $(h, \Delta t)$ such that $\Delta t = \mathscr{O}(h^2)$ is convenient : let us choose $(h, \Delta t) = (\sqrt{2}/2^i, 2\pi/4^i)$, $3 \le i \le 6$. Fig. 1.b and 1.c presents the corresponding error for the $l^2(L^2)$ and $l^\infty(L^2)$ norms as a function of h and Δt, respectively. These tests confirm the optimality of our estimates (2.1)-(2.2).

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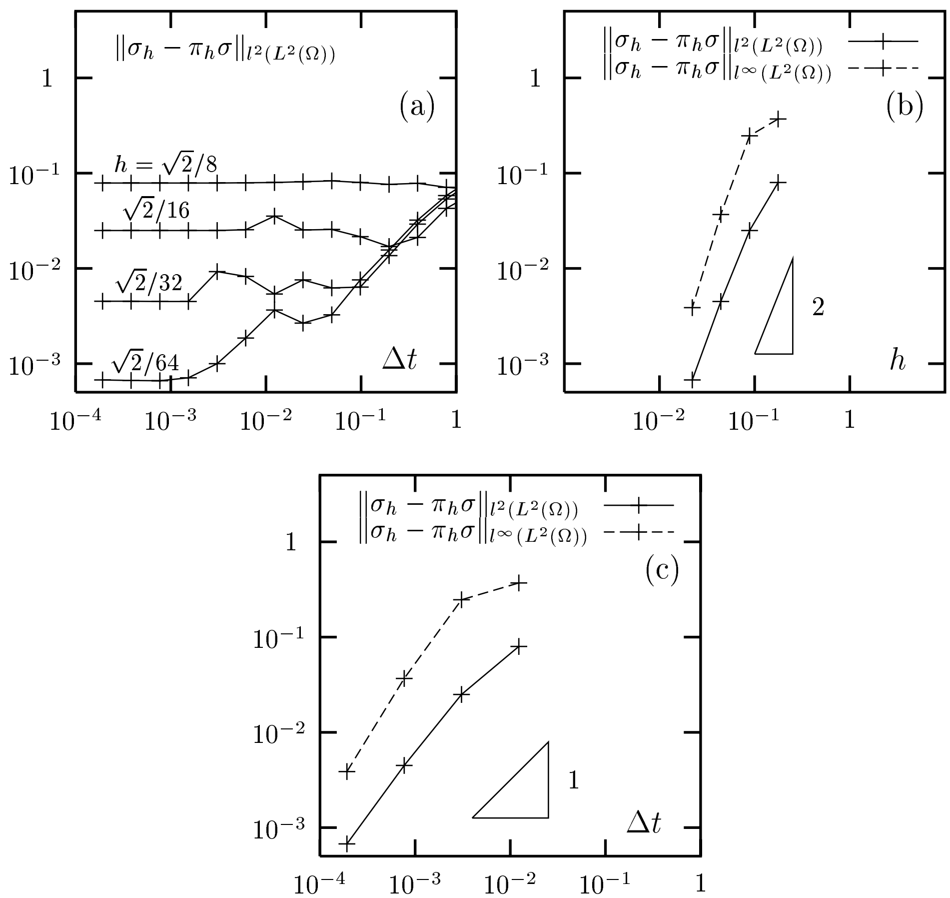

FIGURE 1. Method convergence versus h and Δt.

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