2004-Fez conference on Differential Equations and Mechanics
Electronic Journal of Differential Equations, Conference 11, 2004, pp. 41–51.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

NUMERICAL ANALYSIS OF EULER-SUPG MODIFIED METHOD FOR TRANSIENT VISCOELASTIC FLOW

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Abstract. We study a new approximation scheme of transient viscoelastic fluid flow obeying an Oldroyd-B type constitutive law. The approximation stress, velocity, pressure are respectively P_1-continuous, P_2-continuous, P_1-continuous. We use the modified streamline upwinding Petrov-Galerkin method induced by the modified Euler method. We assume that the continuous problem admits a sufficiently smooth and sufficiently small solution. We show that the approximate problem has a solution and we give an error bound.

1. Introduction and Presentation of the problem

In the numerical simulation of the viscoelastic fluid flows, the hyperbolic character of the constitutive equation (when using differential models) has to be taken into account (see [8, 13]). This hyperbolic character implies that some upwinding is needed to avoid oscillations as in the method of characteristics [8, 2], the Lesaint-Raviart discontinuous finite element method [1, 5], the streamline-upwind method (SU) and the streamline-upwind-Petrov-Galerkin method (SUPG) [6, 13]. The numerical analysis of the steady case of the viscoelastics fluids flows is abundant. Although the list is not exhaustive, one may see for example [1, 6, 15]. Moreover the numerical analysis for transient viscoelastic flow remain quite few [11, 16]. For example, some difficulties appear, when we use continuous finite element approximation for (σ, u, p) and the standard SUPG method for the convection of the extra stress tensor. To give some response to this difficulties, we develop in this paper the study of continuous finite element(F.E) approximation of a transient viscoelastic fluid flow obeying an Oldroyd-B model. For the convective term of the constitutive equation we use some modified SUPG method linked to a variant of implicit Euler method (see [3]). Under this condition we are able to show that the approximate problem is stabilized and has a solution and we give an error bound.

The transient viscoelastic fluid flow obeying an Oldroyd-B type constitutive law is considered flowing in a bounded, connected open set Ω in IR^2 with lipshitzian

2000 *Mathematics Subject Classification.* 65N30, 34K25, 74S05.
Key words and phrases. Stabilized method; finite elements; modified SUPG method; transient viscoelastic flow.

Published October 15, 2004.
Supported by CNRST - CNRS, Program STIC01/03, and by CNRST- GRICES Portugal.

boundary Γ; n is the outward unit normal to Γ. The basic set of equations of the Oldroyd-B model with a single relaxation time is given by

$$\begin{gathered}
\lambda(\frac{\partial\sigma}{\partial t} + (u.\nabla)\sigma + \beta(\sigma,\nabla u)) + \sigma - 2\alpha D(u) = 0 \quad \text{in } \Omega\times]0,T[, \\
Re\,\frac{\partial u}{\partial t} - \nabla.\sigma - 2(1-\alpha)\nabla.D(u) + \nabla p = f \quad \text{in } \Omega\times]0,T[, \\
\nabla.u = 0 \quad \text{in } \Omega\times]0,T[, \\
u = 0 \quad \text{on } \Gamma\times]0,T[, \\
u = u_0, \quad \sigma = \sigma_0 \quad \text{in } \Omega\; t = 0.
\end{gathered} \tag{1.1}$$

Where $\lambda > 0$, Re and $0 < \alpha < 1$ are respectively the Weissenberg number, the Reynolds number and the viscosity ratio constant. f is a density of forces. $D(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ the rate of strain tensor, and $\beta(\sigma,\nabla u) = -\nabla u\sigma - \sigma\nabla u^\top$.

Remark 1.1. The boundary condition $u = 0$ on Γ can be replaced by $u = u_d$ on Γ. Regarding σ and the hyperbolic character of the constitutive equation, we have to impose $\sigma = \sigma_d$ on $\Gamma^- = \{x \in \Gamma;\ u_d.n(x) < 0\}$.

Remark 1.2. The inertia term $(u.\nabla)u$ is neglected in the momentum equation in order to make the analysis simpler.

Let us define the following spaces:

$$\begin{gathered}
T = \{\tau = (\tau_{ij})_{1\le i,j\le 2} : \tau_{ij} = \tau_{ji};\ \tau_{ij} \in L^2(\Omega);\ i,j = 1,2\}, \quad X = (H_0^1(\Omega))^2 \\
Q = \{q \in L^2(\Omega)/\int_\Omega q\,dx = 0\}, \quad V = \{v \in X/(q,\nabla.v) = 0; \forall q \in Q\}.
\end{gathered}$$

The norm and scalar product in $L^2(\Omega)$ of functions, vectors and tensors are denoted respectively by $|\cdot|$ and $(\cdot,\cdot)$; ($|\cdot|_\Gamma$ and $(\cdot,\cdot)_\Gamma$ in $L^2(\Gamma)$); $\langle f, v\rangle$ will denote the duality between $f \in (H^{-1}(\Omega))^2$ and $v \in X$.

Remark 1.3. Existence results for problem (1.1) are proved in [9]. In order to make some theoretical analysis of approximate problem of (1.1) we use the regularity imposed in [9].

2. Description of the approximation scheme

FE approximation. We suppose Ω polygonal and we consider a triangulation $\Im_h$ on Ω made of triangles K such that $\overline{\Omega} = \{\bigcup K; K \in \Im_h\}$ uniformly regular, $\exists \nu_0, \nu_1 : \nu_0 h \le h_K \le \nu_1 \varrho_K$ where ϱ_K is the diameter of the greatest ball included in K and $h_{\max} = \max_{K\in\Im_h} h_K$.

We use the Taylor-Hood finite element method for approximations in space of (u,p): P_2-continuous in velocity, P_1-continuous in pressure and we consider P_1-continuous approximation of the stresses. The corresponding FE space are:

$$\begin{gathered}
X_h = \{v \in X \cap C^0(\Omega)^2 : v_{|K} \in P_2(K)^2, \forall K \in \Im_h\} \\
Q_h = \{q \in Q \cap C^0(\Omega) : q_{|K} \in P_1(K), \forall K \in \Im_h\} \\
V_h = \{v \in X_h : (q,\nabla.v) = 0,\ \forall q \in Q_h\}. \\
T_h = \{\tau \in T \cap C^0(\Omega) : \tau|_K \in P_1(K), \forall K \in \Im_h\},
\end{gathered}$$

where $P_m(K)$ denotes the space of polynomials of degrees less or equal to m on $K \in T_h$. The term $((u.\nabla)\sigma, \tau)$ is approximated by means of an operator B on $X_h \times T_h \times T_h$ defined by

$$\begin{aligned} B(u_h, \sigma_h; \tau_h) &= ((u_h(t).\nabla)\sigma_h(t), \tau_h) + \delta_0(h,t)((u_h(t)\nabla)\sigma_h(t), (u_h(t).\nabla)\tau) \\ &\quad + (1/2)((\nabla.u_h)(t)\sigma, \tau). \end{aligned}$$

For the steady case you can see [15].

Numerical method. We propose Euler-SUPG modified scheme, implicit in time, based on the scheme proposed for the transport equation (see [3]). We construct an approximation of the solution at each time step $nk, n = 0, \dots, N$ in the following way. We start with $u_h^0 = \tilde{u}_0$: elliptic projection of u_0 into $V_h, \sigma_h^0 = \tilde{\sigma}_0$: orthogonal projection of σ_0 into T_h. Given $u_h^0, \dots, u_h^n$; $\sigma_h^0, \dots, \sigma_h^n$, because (X_h, Q_h) satisfies the inf sup condition, we look for the solution of the following problem, find $(u_h^{n+1}, \sigma_h^{n+1}) \in V_h \times T_h$, such that

$$\begin{aligned} &\lambda(\frac{\sigma^{n+1}_{hu_h^n,\delta} - \sigma^n_{hu_h^n,\delta}}{k}, \tau_{u_h^n,\lambda}) + (\sigma_h^{n+1}, \tau_{u_h^n,\lambda}) + B(\lambda u_h^n, \sigma_h^{n+1}; \tau) \\ &- 2\alpha(D(u_h^{n+1}), \tau_{u_h^n,\lambda}) + \lambda(\beta(\sigma_h^{n+}, \nabla u_h^{n+1}), \tau_{u_h^n,\lambda}) = 0 \quad \forall \tau \in T_h; \end{aligned} \tag{2.1}$$

$$\begin{aligned} &Re(\frac{u_h^{n+1} - u_h^n}{k}, v) + (\sigma_h^{n+1}, D(v)) + 2(1-\alpha))(D(u_h^{n+1}), D(v)) \\ &= \langle f(t_{n+1}, x), v\rangle \quad \forall v \in V_h, \end{aligned} \tag{2.2}$$

where $\sigma^i_{hu_h^n,\delta} = \sigma_h^i + \delta(h,k)(u_h^n.\nabla)\sigma_h^i$ $(i = n, n+1)$; $\tau_{u_h^n,\lambda} = \tau + \lambda\delta_0(h,k)(u_h^n.\nabla)\tau$ for all $\tau \in T_h$, and $\delta(,)$ (resp. $\delta_0(,)$) will be specified later. In order to show that equation (2.1) – (2.2) defined uniquely $(u_h^{n+1}, \sigma_h^{n+1})$, we multiply equation (2.2) by 2α and add the equation obtained to equation (2.1), we get

$$\begin{aligned} &\lambda(\frac{\sigma^{n+1}_{hu_h^n,\delta} - \sigma^n_{hu_h^n,\delta}}{k}, \tau_{u_h^n,\lambda}) + 2\alpha Re(\frac{u_h^{n+1} - u_h^n}{k}, v_h) + B(\lambda u_h^n, \sigma_h^{n+1}; \tau_h) \\ &+ A(u_h^n; (\sigma_h^{n+1}, u_h^{n+1}), (\tau, v)) + \lambda(\beta(\sigma_h^{n+1}, \nabla u_h^{n+1}), \tau_{u_h^n,\lambda}) \\ &= 2\alpha\langle f(t_{n+1}), v_h\rangle, \quad \forall(\tau_h, v_h) \in T_h \times V_h. \end{aligned}$$

where $A(.,.)$ is a bilinear form on $T_h \times V_h$ defined as

$$A(w; (\sigma, u), (\tau, v)) = (\sigma, \tau_{w,\lambda}) + 2\alpha(\sigma, D(v)) - 2\alpha(D(u), \tau_{w,\lambda}) + 4\alpha(1-\alpha)(D(u), D(v))$$

From this, we can establish some error bound and then the following existence result

Theorem 2.1. *There exists M_0, and h_0 such that if problem* (1.1) *admits a solution (σ, u, p) with,*

$$\begin{aligned} M = \max\big\{&\|\sigma\|_{C^1([t_n,t_{n+1}];H^2)}, \|u\|_{C^1([t_n,t_{n+1}];H^3)}, \|p\|_{C^0([t_n,t_{n+1}];H^2)}, \\ &\|\sigma\|_{C^2([t_n,t_{n+1}];L^2)}, \|u\|_{C^2([t_n,t_{n+1}],T;L^2)}\big\} < M_0, \end{aligned}$$

then if $\delta(,)$ satisfies $\delta(,) = \lambda\delta_0(,)$ and $h \le h_0$ there exists a unique solution in $T_h \times V_h$ of problem (2.1)- (2.2).

Proof. For this purpose we define a mapping $\Phi : T_h \times V_h \longmapsto T_h \times V_h$, which to (σ_1, u_1) associates $(\sigma_2, u_2) = \Phi(\sigma_1, u_1)$, where $(\sigma_2, u_2) \in T_h \times V_h$ satisfies

$$\lambda(\frac{\sigma_{2u_h^n,\delta} - \sigma^n_{hu_h^n,\delta}}{k}, \tau_{u_h^n,\lambda}) + 2\alpha Re(\frac{u_2 - u_h^n}{k}, v_h) + B(\lambda u_h^n, \sigma_h^{n+1}; \tau_h)$$

$$
\begin{aligned}
&+ A(u_h^n; (\sigma_2, u_2), (\tau, v)) \\
&= -\lambda(\beta(\sigma_1, \nabla u_1), \tau_{u_h^n,\lambda}) + 2\alpha\langle f(t_{n+1}), v_h\rangle\,, \quad \forall(\tau_h, v_h) \in T_h \times V_h.
\end{aligned}
$$

We define a ball $B_h^{(n+1)}$ as follows: let C^* be given. Then we define

$$
\begin{aligned}
B_h^{(n+1)} = \big\{(\tau_h, v_h) \in T_h \times V_h : [&\frac{\alpha Re}{k}\|v_h - u(t_{n+1})\|_{0,\Omega}^2 \\
&+ \frac{\lambda}{2k}\|(\tau_h - \sigma(t_{n+1}))_{u_h^n,\lambda}\|_{0,\Omega}^2]^{1/2} \le C^*(k + \delta + h\sqrt{\delta_0} + \frac{h^2}{\sqrt{\delta_0}} + h^{\frac{3}{2}}), \\
&[\frac{1}{4}\{4\alpha(1-\alpha)\|D(v_h - u(t_{n+1})\|_{0,\Omega}^2 + \|\tau_h - \sigma(t_{n+1})\|_{0,\Omega}^2]^{1/2} \\
&\le C^*(k + \delta + h\sqrt{\delta_0} + \frac{h^2}{\sqrt{\delta_0}} + h^{\frac{3}{2}})\big\}.
\end{aligned}
$$

The proof is decomposed in five parts:
(a) Φ is well defined for $h \le h_1 = h_1(h_{\max}, \alpha)$ and bounded on bounded sets,
(b) let C_0 be a positive constant independent of h, k and λ, α, Re. If $h \le h_0 = \min\big\{h_1, \sqrt{k}\big(\frac{C^*}{C_0 M\sqrt{\alpha Re}}\big)^{2/3}, \frac{1}{C_0 M}\sqrt{k/\lambda}\big\}$ and $\sqrt{\delta/k} \le \frac{1}{C_0 M}$, we have $B_h^{(n+1)}$ is non-empty. On the other hand, if $M_0 = C^*\frac{\gamma}{\lambda+\gamma}$, $\gamma = \sqrt{\alpha(1-\alpha)}$, we can prove that $\Phi(\overbrace{\overset{\circ}{B}}{}_h^{(n+1)}) \subset \overbrace{\overset{\circ}{B}}{}_h^{(n+1)}$,
(c) Φ is continuous on $T_h \times V_h$. $\Phi(B_h^{(n+1)}) \subset B_h^{(n+1)}$, Brouwer's theorem then gives the existence of fixed point $(\sigma_h^{n+1}, u_h^{n+1})$ of Φ solution of problem $(1)-(2)$.
(d) Furthermore, if λM and $\lambda M\gamma^{-1}$ is sufficiently small, Φ is a contraction mapping on $B_h^{(n+1)}$.

Then a result of existence and uniqueness follows from the fixed theorem. □

Remark 2.2. *When we use only the classical Euler-scheme in time and SUPG method, we can't point out result of (a) in the above proof.*

3. Main result and error estimates

Suppose that the continuous problem admits a sufficiently smooth and sufficiently small solution, we can show, the following result.

Theorem 3.1. *There exists M_0 and h_0 such that if problem* (1.1) *admits a solution (σ, u, p) with $\sigma \in C^1([0,T], (H^2)^4) \cap C^2([0,T], (L^2)^4)$; $u \in C^1([0,T], (H^3)^2) \cap C^2([0,T], (L^2)^2)$; $p \in L^2([0,T], (H^2) \cap L_0^2) \cap C^0([0,T], H^2)$, satisfying*

$$
\max\{\|\sigma\|_{C^1(0,T;H^2)}, \|u\|_{C^1(0,T;H^3)}, \|p\|_{C^0(0,T;H^2)}, \|\sigma\|_{C^2(0,T;L^2)}, \|u\|_{C^2(0,T;L^2)}\} \le M_0
$$

then for $kh^{-(1+\varepsilon)} (0 < \varepsilon \le 1/2)$ bounded, there exists a constant C independent of h and k such that

$$
\max_{0\le n\le N} |(\sigma_h^n - \sigma(t_n))_{u_h^{n-1}}| + (\sum_{n=0}^N k|\sigma_h^n - \sigma(t_n)|)^2)^{1/2} \le C(k + \delta + h\sqrt{\delta_0} + \frac{h^2}{\sqrt{\delta_0}} + h^{1+\varepsilon})
$$

and

$$
\max_{0\le n\le N} |u_h^n - u(t_n)| + (\sum_{n=0}^N k|D(u_h^n - u(t_n))|^2)^{1/2} \le C(k + \delta + h\sqrt{\delta_0} + \frac{h^2}{\sqrt{\delta_0}} + h^{1+\varepsilon})
$$

where $N \in N^* : Nk = T$, $u_h^{-1} = u_h^0$ *and* $(\sigma_h^i, u_h^i)_{1\le i\le N}$ *are solutions of* $((2.1)-(2.2))_{1\le i\le N}$.

Remark 3.2. The discontinuous stresses approach case with Euler semi-implicit method in time was treated by Baranger and all. [16]. In the proof of the present result we can fund some similar technical like in [16].

Proof of Theorem 3.1. For $0 \le n < N$, let $u(t_n)$, $\sigma(t_n)$, $p(t_n)$ be respectively in $H^3(\Omega)$, ${}^2(\Omega)$, $H^2(\Omega)$ and so, there exists $(\tilde{u}(t_n), \tilde{p}(t_n)) \in V_h \times Q_h$ such that,

$$|(u-\tilde{u})(t_n)| + h\|(u-\tilde{u})(t_n)\|_{1,2} \le C_1 h^3 \|u(t_n)\|_{3,2}, \tag{3.1}$$

$$|(p-\tilde{p})(t_n)| \le C_2 h^2 \|p(t_n)\|_{2,2}, \tag{3.2}$$

(see [10]) and there exists $\tilde{\sigma}(t_n) \in T_h$ such that

$$|(\sigma-\tilde{\sigma})(t_n)| \le C_3 h^2 \|\sigma(t_n)\|_{2,2} \tag{3.3}$$

(see [10]). We remark that we can define $\tilde{u}(.)$ by the elliptic projection of $u(.)$ on V_h such that $a((u-\tilde{u})(.), v_h) = 0, \forall v_h \in V_h$, where $a(u,v) = (d(u), d(v))$; then the following properties are also satisfied

$$d\tilde{u}/dt = (du/dt)^{\sim}$$

and

$$\begin{aligned} |(du/dt)(s) - (du/dt)^{\sim}(s)| &\le C_4 h \|(du/dt)(s) - (du/dt)^{\sim}(s)\|_{1,2} \\ &\le C_5 h^3 \|(du/dt)(s)\|_{3,2} \end{aligned} \tag{3.4}$$

(see([14])), u being in $C^1([0,T], H^3)$; same properties are satisfied for σ:

$$d\tilde{\sigma}/dt = (d\sigma/dt)^{\sim}$$

and

$$|\frac{d\sigma}{dt}(s) - \tilde{\frac{d\sigma}{dt}}(s)| \le C_6 h \|\frac{d\sigma}{dt}(s) - \tilde{\frac{d\sigma}{dt}}(s)\|_{1,2} \le C_7 h^2 \|\frac{d\sigma}{dt}(s)\|_{2,2}, \tag{3.5}$$

(see([14])). □

In the sequel we shall use the following inverse inequalities (see [4]).

Lemma 3.3. *Let* $k \ge 0$ *be an integer and* $W_h = \{v, v_{|_K} \in P_k(K)\, \forall K \in \Im_h\}$. *Let* r *and* p *be reals with* $1 \le r, p \le \infty$ *and let* $l \ge 0$ *and* $m \ge 0$ *be integers such that* $l \le m$. *Then there exists a constant* $C = C(\nu_0, \nu_1, l, r, m, p, k)$ *such that* $\forall v \in W_h \cap W^{l,r}(\Omega) \cap W^{m,p}(\Omega)$, $|v|_{m,p} \le C h^{l-m-2\max\{0,1/r-1/p\}} |v|_{l,r}$.

We shall also use the following Sobolev's imbedding theorems.

Lemma 3.4. *Let* $m \ge 0$ *be an integer. The following embedding hold algebraically and topologically:*

$$W^{m+1,2}(\Omega) \subset W^{m,q}(\Omega)\ \forall q \in [1,\infty[, \quad \text{and} \quad W^{m,p}(\Omega) \subset C^0(\bar{\Omega}).$$

Now, let us denote $e_h^n = u_h^n - \tilde{u}(t_n)$, $\varepsilon_h^n = \sigma_h^n - \tilde{\sigma}(t_n)$. From equations (2.1)-(2.2) we have for $(\tau, v) \in T_h \times V_h$

$$\begin{aligned}
&2\alpha Re(\frac{e_h^{n+1} - e_h^n}{k}, v) + \lambda(\frac{\varepsilon_{hu_h^n,\delta}^{n+1} - \varepsilon_{hu_h^n,\delta}^n}{k}, \tau_{u_h^n,\lambda}) + B(\lambda u_h^n, \varepsilon_h^{n+1}; \tau)\\
&+ A(u_h^n; (\varepsilon_h^{n+1}, e_h^{n+1}), (\tau, v))\\
&= -\lambda(\frac{\tilde{\sigma}_{hu_h^n,\delta}(t_{n+1}) - \tilde{\sigma}_{hu_h^n,\delta}(t_n)}{k}, \tau_{u_h^n,\lambda}) - 2\alpha e(\frac{\tilde{u}(t_{n+1}) - \tilde{u}(t_n)}{k}, v)\\
&\quad - B(\lambda u_h^n, \tilde{\sigma}(t_{n+1}); \tau) + A(u_h^n; (-\tilde{\sigma}(t_{n+1}), -\tilde{u}(t_{n+1})), (\tau, v))\\
&\quad - \lambda(\beta(\sigma_h^n, \nabla u_h^n), \tau_{u_h^n,\lambda}) + 2\alpha\langle f(t_{n+1}, x), v\rangle, \quad \forall(\tau, v) \in T_h \times V_h.
\end{aligned} \tag{3.6}$$

But (σ, u, p) being the exact solution of problem (1.1), it satisfies the following consistency equation

$$\begin{aligned}
&2\alpha Re(\frac{du}{dt}(t), v) + \lambda(\frac{d\sigma}{dt}(t), \tau_{u_h^n,\lambda}) + B(\lambda u(t), \lambda u_h^n, \sigma(t); \tau)\\
&+ A(u_h^n; (\sigma(t), u(t)), (\tau, v))\\
&= 2\alpha(p(t), \nabla.v) + 2\alpha\langle f(t), v\rangle - \lambda(\beta(\sigma(t), \nabla u(t)), \tau_{u_h^n,\lambda}) \quad \forall(\tau, v) \in T_h \times V_h
\end{aligned}$$

Inserting the value of $\langle f(t_{n+1}), .\rangle$ in equation (3.6) we obtain

$$\begin{aligned}
&Re(\frac{e_h^{n+1} - e_h^n}{k}, v) + \lambda(\frac{\varepsilon_{hu_h^n,\delta}^{n+1} - \varepsilon_{hu_h^n,\delta}^n}{k}, \tau_{u_h^n,\lambda}) + B(\lambda u_h^n, \varepsilon_h^{n+1}; \tau)\\
&+ A(u_h^n; (\varepsilon_h^{n+1}, e_h^{n+1}), (\tau, v))\\
&= \lambda(g_a(\sigma(t_{n+1}), \nabla u(t_{n+1})) - g_a(\sigma_h^n, \nabla u_h^n), \tau)\\
&\quad + 2\alpha Re(\frac{du}{dt}(t_{n+1}) - \frac{\tilde{u}(t_{n+1}) - \tilde{u}(t_n)}{k}, v)\\
&\quad + \lambda(\frac{d\sigma}{dt}(t_{n+1}) - \frac{\tilde{\sigma}_{u_h^n}(t_{n+1}) - \tilde{\sigma}_{u_h^n}(t_n)}{k}, \tau_{u_h^n,\lambda})\\
&\quad + A(u_h^n, ((\sigma - \tilde{\sigma})(t_{n+1}), (u - \tilde{u})(t_{n+1})), (\tau, v))\\
&\quad + \lambda[B(\lambda u(t_{n+1}), \lambda u_h^n, \sigma(t_{n+1}); \tau) - B(\lambda u_h^n, \tilde{\sigma}(t_{n+1}); \tau)] + 2\alpha(p(t_{n+1}), \nabla.v).
\end{aligned} \tag{3.7}$$

Taking $v = e_h^{n+1}$ and $\tau = \varepsilon_h^{n+1}$ in equation (3.7) and using the identity $(a - b, a) = \frac{1}{2}(|a|^2 - |b|^2 + |a - b|^2)$, and coercivity, we obtain

$$\begin{aligned}
&\frac{\alpha Re}{k}\{|e_h^{n+1}|^2 - |e_h^n|^2 + |e_h^{n+1} - e_h^n|^2\} + |\varepsilon_h^{n+1}|^2\\
&+ \frac{\lambda}{2k}\{|\varepsilon_{hu_h^n,\lambda}^{n+1}|^2 - |\varepsilon_{hu_h^n,\lambda}^n|^2 + |\varepsilon_{hu_h^n,\lambda}^{n+1} - \varepsilon_{hu_h^n,\lambda}^n|^2\}\\
&+ 2\alpha(1 - \alpha)|D(e_h^{n+1})|^2 + (1/2)|\varepsilon_h^{n+1}|^2 + (\delta_0/4)|\lambda u_h^n.\nabla\varepsilon_h^{n+1}|^2
\end{aligned} \tag{3.8}$$

which is less than ro equal to the right-hand side of (3.7). To bound each term of the second member of inequality (3.8), let us define for $C_0 > 0$, the ball $B_{h,k}^m$ for $0 \leq m \leq N$, by

$$\begin{aligned}
&B_{h,k}^m\\
&= \big\{(\tau_i, v_i)_{i=0,..,m} \in (T_h \times V_h)^{m+1} : \max_{0\leq i\leq m}\{|(\tau_i - \sigma(t_i))_{v_{i-1}}|^2 + |v_i - u(t_i)|^2\}^{1/2}\\
&\quad \leq C_0(k + \delta + h\sqrt{\delta_0} + \frac{h^2}{\sqrt{\delta_0}} + h^{1+\varepsilon}) \text{ and}
\end{aligned}$$

$$[\sum_{n=0}^{m} k\{|\tau_i - \sigma(t_i)|^2 + |D(v_i - u(t_i))|^2\}]^{1/2} \le C_0(k + \delta + h\sqrt{\delta_0} + \frac{h^2}{\sqrt{\delta_0}} + h^{1+\varepsilon})\}.$$

Our aim is to prove that we can choose M_0, h_0, C_0 such that for $M \le M_0$, $h \le h_0$ if $(\sigma_h^n, u_h^n)_{0\le n\le m-1} \in B_{h,k}^{m-1}$ for a $C_0 = C_0(M_0, h_0, C_i)$ then $(\sigma_h^n, u_h^n)_{0\le n\le m} \in B_{h,k}^m$ for the same C_0, thus for all m such $mk \le T$. Firstly, by equations (3.1) to (3.5) we have

$$|(\sigma_h^0 - \sigma(0))_{u_h^0}| + |u_h^0 - u(0)| \le Mh^2\{C_3(1 + \lambda M\delta_0 h^{-1}) + C_1\}$$

and

$$[k\{|\sigma_h^0 - \sigma(0)|^2 + |D(u_h^0 - u(0)|^2\}]^{1/2} \le \sqrt{2k}(C_1 + C_3)Mh^2.$$

To ensure that $(\sigma_h^0, u_h^0) = (\tilde\sigma_0, \tilde u_0) \in B_{h,k}^0$, it suffices to impose, for $h < h_1$,

$$Mh^{1/2}\{C_3(1 + \lambda M\delta_0 h^{-1}) + C_1\} \le C_0$$

and for $h \le h_2$,

$$(C_1 + C_3)\sqrt{2\bar C}Mh^{\frac{3-\varepsilon}{2}} \le C_0.$$

So, if we take $h_0 \le \min\{h_1, h_2\}$ we have for $h \le h_0$: $B_h^0 \ne \emptyset$. Now, let us suppose that $(\sigma_h^n, u_h^n)_{0\le n\le m-1} \in B_{h,k}^{(m-1)}$. We multiply inequality (3.8) by k and sum it for $n = 0$ to $n = m - 1$,

$$\begin{aligned}
&\alpha Re \sum_{n=0}^{m-1}[|e_h^{n+1}|^2 - |e_h^n|^2 + |e_h^{n+1} - e_h^n|^2] + (1/2)\sum_{n=0}^{m-1} k|\varepsilon_h^{n+1}|^2 \\
&+ \frac{\lambda}{2}\sum_{n=0}^{m-1}[|\varepsilon_{hu_h^n,\lambda}^{n+1}|^2 - |\varepsilon_{hu_h^n,\lambda}^n| + |\varepsilon_{hu_h^n,\lambda}^{n+1} - \varepsilon_{hu_h^n,\lambda}^n|^2] + 2\alpha(1-\alpha)\sum_{n=0}^{m-1} k|D(e_h^{n+1})|^2 \\
&+ (\delta_0/4)\sum_{n=0}^{m-1} k|\lambda u_h^n.\nabla\varepsilon_h^{n+1}|^2 \\
&\le \lambda\sum_{n=0}^{m-1} k(g_a(\sigma(t_{n+1}), \nabla u(t_{n+1})) - g_a(\sigma_h^{n+1}, \nabla u_h^{n+1}), \varepsilon_{hu_h^n,\lambda}^{n+1}) && (3.9)\\
&\quad + \lambda\sum_{n=0}^{m-1} k(\frac{d\sigma}{dt}(t_{n+1}) - \frac{\tilde\sigma_{u_h^n,\delta}(t_{n+1}) - \tilde\sigma_{u_h^n,\delta}(t_n)}{k}, \varepsilon_{hu_h^n,\lambda}^{n+1}) && (3.10)\\
&\quad + 2\alpha Re\sum_{n=0}^{m-1} k(\frac{du}{dt}(t_{n+1}) - \frac{\tilde u(t_{n+1}) - \tilde u(t_n)}{k}, e_h^{n+1}) && (3.11)\\
&\quad + \sum_{n=0}^{m-1} kA(u_h^n; (\sigma - \tilde\sigma)(t_{n+1}), (u - \tilde u)(t_{n+1}), (\varepsilon_h^{n+1}, e_h^{n+1})) && (3.12)\\
&\quad + \lambda\sum_{n=0}^{m-1} k\{B(\lambda u(t_{n+1}), \lambda u_h^n, \sigma(t_{n+1}); \varepsilon_h^{n+1}) - B(\lambda u_h^n, \tilde\sigma(t_{n+1}); \varepsilon_h^{n+1})\} && (3.13)\\
&\quad + 2\alpha\sum_{n=0}^{m-1} k(p(t_{n+1}), \nabla.e_h^{n+1}). && (3.14)
\end{aligned}$$

We use for the estimate of terms (3.11)–(3.14), the results (3.1)–(3.5). However, for the estimation of the term (3.9) we choose the fixed point (σ_h^m, u_h^m) of the mapping

Φ defined in section 2. This choice is possible because we have, $B_h^{(0)} \cap B_{h,\Delta t}^0 \neq \emptyset$ and by construction:

$$(\sigma_h^l, u_h^l)_{l=0;..;m-1} \in B_{h,\Delta t}^{m-1} \Rightarrow (\sigma_h^{m-1}, u_h^{m-1}) \in B_h^{(m-1)}.$$

On the other hand for estimate of the term (3.10) we prepare the following lemma:

Lemma 3.5. *Let* (3.1)–(3.5) *hold. Then*

$$\begin{aligned}
&|(\frac{d\sigma}{dt}(t_{n+1}) - \frac{\tilde{\sigma}_{u_h^n,\delta}(t_{n+1}) - \tilde{\sigma}_{u_h^n,\delta}(t_n)}{k}, \varepsilon_{hu_h^n,\lambda}^{n+1})| \\
&\leq \lambda M[C_7h^2 + k + \delta M C_{10}(M + hM + C_1 M h^{\frac{3}{2}})](\sum_{n=0}^{m-1} k)^{1/2} \max_{0\leq n\leq m-1} |\varepsilon_{hu_h^n,\lambda}^{n+1})| \\
&\quad + \big\{\delta M[C_7C_8kh(\sum_{n=0}^{m-1} k|\nabla(u_h^n - u(t_n))|^2)^{1/2} \\
&\quad + (C_7\lambda^{-1}h^2 + \frac{C7}{C_6}C_9\delta_0(M + \max_{0\leq n\leq m-1}|u(t_n) - u_h^n|))](\sum_{n=0}^{m-1} k)^{1/2} \\
&\quad + \lambda(1+\sqrt{\delta_0})\delta h^{-1}(\sum_{n=0}^{m-1} k|\nabla(u_h^n - u(t_n))|^2\big)^{1/2}\big\} \\
&\quad \times \big(\sum_{n=0}^{m-1} k\{|\varepsilon_{hu_h^n,\lambda}^{n+1}|^2 + \delta_0|\lambda(u_h^n.\nabla)\varepsilon_h^{n+1}|^2\}\big)^{1/2}.
\end{aligned}$$

Proof. To proof the inequality in this lemma, we write each term as follows:

$$\begin{aligned}
&\frac{d\sigma}{dt}(t_{n+1}) - \frac{\tilde{\sigma}_{u_h^n,\delta}(t_{n+1}) - \tilde{\sigma}_{u_h^n,\delta}(t_n)}{k} \\
&= \big(\frac{d\sigma}{dt}(t_{n+1}) - \frac{\sigma_{u_h^n,\delta}(t_{n+1}) - \sigma_{u_h^n,\delta}(t_n)}{k}\big) \\
&\quad + \big(\frac{\sigma_{u_h^n,\delta}(t_{n+1}) - \sigma_{u_h^n,\delta}(t_n)}{k} - \frac{\tilde{\sigma}_{u_h^n,\delta}(t_{n+1}) - \tilde{\sigma}_{u_h^n,\delta}(t_n)}{k}\big)
\end{aligned}$$

we can write the second term in the form,

$$\frac{1}{k}(\int_{t_n}^{t_{n+1}} (\frac{d\sigma}{dt} - (\frac{\tilde{d\sigma}}{dt}))(s)\,ds, \varepsilon_{hu_h^n,\lambda}^{n+1}) + \frac{\delta}{k}(\int_{t_n}^{t_{n+1}} u_h^n.\nabla(\frac{d\sigma}{dt} - (\frac{\tilde{d\sigma}}{dt}))(s)\,ds, \varepsilon_{hu_h^n,\lambda}^{n+1})$$

so,

$$\begin{aligned}
&|k(\frac{\sigma_{u_h^n,\delta}(t_{n+1}) - \sigma_{u_h^n,\delta}(t_{n+1})}{k} - \frac{\tilde{\sigma}_{u_h^n,\delta}(t_{n+1}) - \tilde{\sigma}_{u_h^n,\delta}(t_{n+1})}{k}, \varepsilon_{hu_h^n,\lambda}^{n+1})| \\
&\leq |(\int_{t_n}^{t_{n+1}} (\frac{d\sigma}{dt} - (\frac{\tilde{d\sigma}}{dt}))(s)\,ds, \varepsilon_{hu_h^n,\lambda}^{n+1})| + \delta|(\int_{t_n}^{t_{n+1}} u_h^n.\nabla(\frac{d\sigma}{dt} - (\frac{\tilde{d\sigma}}{dt}))(s)\,ds, \varepsilon_{hu_h^n,\lambda}^{n+1})|
\end{aligned}$$

we estimate the first term in this inequality using (3.5):

$$\begin{aligned}
|(\int_{t_n}^{t_{n+1}} (\frac{d\sigma}{dt} - (\frac{\tilde{d\sigma}}{dt}))(s)\,ds, \varepsilon_{hu_h^n,\lambda}^{n+1})| &\leq k \max_{t_n\leq s\leq t_{n+1}} |(\frac{d\sigma}{dt} - (\frac{\tilde{d\sigma}}{dt}))(s)||\varepsilon_{hu_h^n,\lambda}^{n+1}| \\
&\leq C_7Mkh^2|\varepsilon_{hu_h^n,\lambda}^{n+1}|.
\end{aligned}$$

To study the second term, we write

$$
\begin{aligned}
&(\int_{t_n}^{t_{n+1}} u_h^n.\nabla(\frac{d\sigma}{dt} - (\tilde{\frac{d\sigma}{dt}}))(s)\,ds, \varepsilon_{hu_h^n,\lambda}^{n+1}) \\
&= -(\int_{t_n}^{t_{n+1}} (\frac{d\sigma}{dt} - (\tilde{\frac{d\sigma}{dt}}))(s)\,ds, (u_h^n.\nabla)\varepsilon_h^{n+1}) \\
&\quad + ((\nabla.u_h^n)\int_{t_n}^{t_{n+1}} (\frac{d\sigma}{dt} - (\tilde{\frac{d\sigma}{dt}}))(s)\,ds, \varepsilon_h^{n+1}) \\
&\quad + \delta_0(\int_{t_n}^{t_{n+1}} u_h^n.\nabla(\frac{d\sigma}{dt} - (\tilde{\frac{d\sigma}{dt}}))(s)\,ds, \lambda(u_h^n.\nabla)\varepsilon_{hu_h^n,\lambda}^{n+1})
\end{aligned}
$$

and

$$
\begin{aligned}
&\delta|(\int_{t_n}^{t_{n+1}} u_h^n.\nabla(\frac{d\sigma}{dt} - (\tilde{\frac{d\sigma}{dt}}))(s)\,ds, \varepsilon_{hu_h^n,\lambda}^{n+1})| \\
&\le \delta\lambda^{-1}k \max_{t_n\le s\le t_{n+1}} |(\frac{d\sigma}{dt} - (\tilde{\frac{d\sigma}{dt}}))(s)||\lambda(u_h^n.\nabla)\varepsilon_h^{n+1}| \\
&\quad + \delta k \max_{t_n\le s\le t_{n+1}} |(\frac{d\sigma}{dt} - (\tilde{\frac{d\sigma}{dt}}))(s)||\nabla(u_h^n - u(t_n))||\varepsilon_h^{n+1}|_{0,\infty} \\
&\quad + k\delta_0\delta|u_h^n|_{0,\infty} \max_{t_n\le s\le t_{n+1}} |(\frac{d\sigma}{dt} - (\tilde{\frac{d\sigma}{dt}}))(s)|_{1,2}|\lambda(u_h^n.\nabla)\varepsilon_h^{n+1}|
\end{aligned}
$$

using (3.5) and the result of Lemma 3.3, we obtain the following estimate

$$
\begin{aligned}
&\delta|(\int_{t_n}^{t_{n+1}} u_h^n.\nabla(\frac{d\sigma}{dt} - (\tilde{\frac{d\sigma}{dt}}))(s)\,ds, \varepsilon_{hu_h^n,\lambda}^{n+1})| \\
&\le \delta C_8 C_7 M\delta kh|\nabla(u_h^n - u(t_n))||\varepsilon_h^{n+1}| \\
&\quad + \{\delta\lambda^{-1}C_7Mh^2 + C_9\frac{C7}{C_6}Mh\delta_0\delta h^{-1}|u_h^n|\}k|\lambda(u_h^n.\nabla)\varepsilon_h^{n+1}| \\
&\le C_7C_8M\delta kh|\nabla(u_h^n - u(t_n))|\varepsilon_h^{n+1}| \\
&\quad + \big\{C_7\delta\lambda^{-1}M\frac{h^2}{\sqrt{\delta_0}} + C_9\frac{C7}{C_6}M\delta\sqrt{\delta_0}(M + \max_{0\le n\le m-1}|u_h^n - u(t_n)|)\big\} \\
&\quad \times\sqrt{\delta_0}|\lambda(u_h^n.\nabla)\varepsilon_h^{n+1}|
\end{aligned}
$$

and finally

$$
\begin{aligned}
&\lambda|\sum_{n=0}^{m-1} k(\frac{\sigma_{u_h^n,\delta}(t_{n+1}) - \sigma_{u_h^n,\delta}(t_{n+1})}{k} - \frac{\sigma_{\widetilde{u_h^n},\delta}(t_{n+1}) - \sigma_{\widetilde{u_h^n},\delta}(t_{n+1})}{k}, \varepsilon_{hu_h^n,\lambda}^{n+1})| \\
&\le \lambda C_7Mh^2(\sum_{n=0}^{m-1} k)\max_{0\le n\le m-1}|\varepsilon_{hu_h^n,\lambda}^{n+1}| \\
&\quad + \lambda C_7C_8M\delta kh(\sum_{n=0}^{m-1} k|\nabla(u_h^n - u(t_n))|^2)^{1/2}(\sum_{n=0}^{m-1} k|\varepsilon_h^{n+1}|^2)^{1/2} \qquad (3.15) \\
&\quad + \delta M\{C_7\lambda^{-1}\frac{h^2}{\sqrt{\delta_0}} + \frac{C7}{C_6}C_9\sqrt{\delta_0}(M + \max_{0\le n\le m-1}|u_h^n - u(t_n)|))\}
\end{aligned}
$$

$$\times (\sum_{n=0}^{m-1} k)^{1/2} (\sum_{n=0}^{m-1} k\delta_0 |\lambda u_h^n.\nabla \varepsilon_h^{n+1}|^2)^{1/2}.$$

□

Now, we return to the first part of term (3.10).

$$\begin{aligned}
&|k(\frac{d\sigma}{dt}(t_{n+1}) - \frac{\sigma_{u_h^n,\delta}(t_{n+1}) - \sigma_{u_h^n,\delta}(t_n)}{k}, \varepsilon^{n+1}_{hu_h^n,\lambda})| \\
&\leq |k(\frac{d\sigma}{dt}(t_{n+1}) - \frac{\sigma(t_{n+1}) - \sigma(t_n)}{k}, \varepsilon^{n+1}_{hu_h^n,\lambda})| + \delta |(u_h^n.\nabla(\sigma(t_{n+1}) - \sigma(t_n)), \varepsilon^{n+1}_{hu_h^n,\lambda})| \\
&\leq |(\int_{t_n}^{t_{n+1}} (s-t) \frac{d^2\sigma}{dt^2}(s)\, ds, \varepsilon^{n+1}_{hu_h^n,\lambda})| + \delta |(\int_{t_n}^{t_{n+1}} u_h^n.\nabla \frac{d\sigma}{dt}(s)\, ds, \varepsilon^{n+1}_{hu_h^n,\lambda})| \\
&\leq k^2 \max_{t_n \leq s \leq t_{n+1}} |\frac{d^2\sigma}{dt^2}(s)| |\varepsilon^{n+1}_{hu_h^n,\lambda}| + \delta k |u_h^n|_{0,\infty} \max_{t_n \leq s \leq t_{n+1}} |\frac{d\sigma}{dt}(s)|_{1,2} |\varepsilon^{n+1}_{hu_h^n,\lambda}|\,.
\end{aligned}$$

Using the regularity of σ, we have

$$|k(\frac{d\sigma}{dt}(t_{n+1}) - \frac{\sigma_{u_h^n,\delta}(t_{n+1}) - \sigma_{u_h^n,\delta}(t_n)}{k}, \varepsilon^{n+1}_{hu_h^n,\lambda})| \leq M(k^2 + k\delta |u_h^n|_{0,\infty}) |\varepsilon^{n+1}_{hu_h^n,\lambda}|.$$

On the other hand, by the imbedding result $W^{1,4} \subset L^\infty$ (see Lemma 3.3) and the inverse inequality result $|\cdot|_{0,\infty} \leq C_{10} h^{\frac{-1}{2}} |\cdot|_{0,4}$ (see Lemma 3.4) we can prove,

$$|u_h^n|_{0,\infty} \leq C_{10}(M + h\|u\|_{3,2} + C_1 M h^{\frac{3}{2}} + h^{-\frac{1}{2}} |\nabla(u(t_n) - u_h^n)|).$$

So we have,

$$\begin{aligned}
&\lambda |\sum_{n=0}^{m-1} k(\frac{d\sigma}{dt}(t_{n+1}) - \frac{\sigma_{u_h^n,\delta}(t_{n+1}) - \sigma_{u_h^n,\delta}(t_n)}{k}, \varepsilon^{n+1}_{hu_h^n,\lambda})| \\
&\leq \lambda M[k + \delta M C_{10}(M + hM + C_1 M h^{\frac{3}{2}})] (\sum_{n=0}^{m-1} k) \times \max_{0 \leq n \leq m-1} |\varepsilon^{n+1}_{hu_h^n,\lambda}| \\
&\quad + \lambda(1 + \sqrt{\delta_0}) \delta h^{-\frac{1}{2}} \big(\sum_{n=0}^{m-1} k |\nabla(u(t_n) - u_h^n)|^2 \big)^{1/2} \\
&\quad \times \big[(\sum_{n=0}^{m-1} k |\varepsilon_h^{n+1}|^2)^{1/2} + (\sum_{n=0}^{m-1} k\delta_0 |\lambda u_h^n.\nabla \varepsilon_h^{n+1}|^2)^{1/2} \big]\,.
\end{aligned}$$

Then Lemma 3.5 follows from the above inequality and (3.15).

Conclusion. We conclude this analysis with some comments. The proof given here can be extended to the more realistic rheological PTT model, to a quadrilateral FE approximation, following [1] and to the higher finite element methods($P_k, k \geq 1$). With a judicious coefficient choice of stabilization δ and δ_0, we find the error bound given respectively by Baranger and al. [16] and Ervin and al. [7].

The use of a decoupled fractional step scheme would be computationally cheaper, following [6]. Numerical analysis of such method is currently in progress.

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