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ON THE WELL-POSEDNESS OF THE HEAT EQUATION ON UNBOUNDED DOMAINS

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Abstract. This work concerns the well-posedness of the heat equation in an unbounded open domain, under basic regularity assumptions on this domain.

1. Introduction

Let Ω be an open set of $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$ and consider the problem

$$\begin{gathered} u'(t) = \Delta u(t), \quad t \in [0,\tau] \\ u(t)\big|_\Gamma = \varphi(t), \quad t \in [0,\tau] \\ u(0) = u_0, \end{gathered} \tag{1.1}$$

where $u_0 \in C(\overline{\Omega})$, $\varphi \in C([0,\tau]; C(\Gamma))$, $\tau > 0$.

The aim of this work is to study the well-posedness of (1.1) when Ω is unbounded. The case where Ω is bounded has been studied in [2, Chapter 6], and sufficient conditions on the initial data u_0 and the boundary condition φ are given to show that the problem (1.1) is well-posed in $C(([0,\tau]; C(\overline{\Omega}))$ whenever Ω is regular (See definition 2.1).

We point out here that the regularity assumption is equivalent when Ω is bounded to that the Dirichlet problem, (1.2),

$$\begin{gathered} u \in C(\overline{\Omega}) \\ \Delta u = 0 \quad \text{in } \mathcal{D}(\Omega)' \\ u\big|_\Gamma = \phi, \end{gathered} \tag{1.2}$$

has for all $\phi \in C(\Gamma)$ a classical solution u, that means that u is a solution of (1.2) and $u \in C^2(\Omega)$. (See [5], [9] for instance). The situation is more complicated when Ω is unbounded since one must take into account the condition at infinity that the solution of (1.2) satisfies (see Theorem 2.2), and the choice of the space $X \subset C(\overline{\Omega})$ in which the solution $u(t)$ of (1.1) belongs will be imposed by this condition at infinity and then by the choice of the unbounded regular open set Ω.

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In this work, the unbounded set is taken in the case $n = 1$ as an interval of $\mathbb{R}$ and as the exterior of a ball of $\mathbb{R}^n$ for $n \geq 2$. When $n = 2$, we deal with the heat equation with homogeneous boundary conditions and for $n \geq 3$, the heat equation with inhomogeneous boundary conditions is studied for an exterior domain.

The organization of this work is as follows: In Section 2, we recall some preliminaries results lying between the regularity property for unbounded sets and the well-posedness of the Dirichlet problem (1.2). We also recall some existence results for Cauchy problems with resolvent positive operators. We present in Section 3 our main result, the method of proof consists on reformulating (1.1) as a Cauchy problem with the Poisson operator. Section 4 is devoted to the study of the well-posedness of this Cauchy problem, we first show that the Poisson operator has a positive resolvent in $X \times C(\Gamma)$. Using results of Section 2, we then show the well-posedness of (1.1).

2. Preliminaries

The Dirichlet Problem. Let Ω be an open set of $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$.

Definition 2.1 ([5]). (a) Let $z \in \Gamma$. we say that z is a regular boundary point of Ω if there exists $r > 0$, and $w \in C(\overline{\Omega \cap B(z,r)})$ such that

$$\begin{gathered}\Delta w \leq 0, \quad \text{in } \mathcal{D}(\Omega \cap B(z,r))' \\ w(x) > 0, \quad x \in (\Omega \cap B(z,r))\backslash\{z\} \\ w(z) = 0\,.\end{gathered}$$

Then the function w is called a barrier.
(b) We say that Ω is regular if all boundary points are regular.

This regularity property is related to the Dirichlet problem (1.2) as follows.

Theorem 2.2 ([5]). *Let Ω be an unbounded set, not dense in $\mathbb{R}^n (n \geq 2)$ with boundary Γ. Then the following two assertions are equivalent:*
(i) For every continuous ϕ with compact support in Γ, there exists a classical solution of (1.2) *satisfying the following null condition at infinity*

(NC) *There exists h harmonic on Ω such that $h \in C(\overline{\Omega})$, with $h(x) > 0$ for $|x|$ large so that $\lim_{|x|\to+\infty} \frac{u(x)}{h(x)} = 0$.*

(ii) All boundary points of Ω are regular.

Example 2.3 ([5]). Let $n \in \mathbb{N}^*$.
(a) Case of an interval of $\mathbb{R}$. Let $\Omega_1 =]1, \infty[$, then Ω_1 is regular and for all $\phi \in \mathbb{R}$ and all $c \in \mathbb{R}$, there exists a unique classical solution of (1.2) satisfying the condition at infinity:

$$\lim_{x\to+\infty} \frac{u(x)}{x} = c.$$

(b) Case of the exterior of a ball of $\mathbb{R}^n$, $n \geq 2$. Let $\Omega_n = \mathbb{R}^n \setminus \overline{B(0,1)}$, then Ω_n is regular and given u a bounded classical solution of (1.2), then $c = \lim_{|x|\to\infty} u(x)$ exists and

$$u(x) = (1 - \frac{1}{|x|^{n-2}})c + \frac{1}{\sigma_n} \int_{\partial B} \frac{|x|^2 - 1}{|t-x|^n} \phi(t) d\gamma(t)$$

is a classical solution of (1.2), with σ_n being the total surface area of the unit sphere in $\mathbb{R}^n$. Conversely, the function u given by the last formula is a classical solution

of (1.2). Moreover, one has:
If $n = 2$, then for all $\phi \in C(\partial B)$, there exists a unique classical solution of (1.2) satisfying the condition at infinity:

$$u \text{ is bounded on } \Omega.$$

This solution will have a limit at infinity which is imposed by the giving ϕ:

$$\lim_{|x|\to\infty} u(x) = \frac{1}{2\pi}\int_{\partial B} \phi(t)d\gamma(t). \tag{2.1}$$

If $n \geq 3$, then for all $\phi \in C(\partial B)$ and all $c \in \mathbb{R}$, there exists a unique classical solution of (1.2) satisfying the condition at infinity:

$$\lim_{|x|\to\infty} u(x) = c.$$

Note that a solution u of (1.2) satisfying the null condition at infinity (**NC**) does not necessarily satisfy:

$$\lim_{|x|\to\infty} u(x) = 0.$$

This remains true for the exterior of a compact set of $\mathbb{R}^n$, $n \geq 3$.

Proposition 2.4 ([5])**.** *Let K be a compact set of $\mathbb{R}^n$, $n \geq 3$ with boundary Γ. If $\Omega = \mathbb{R}^n \setminus K$ is regular, then for all $\phi \in C(\Gamma)$, there exists a unique classical solution of (1.2) satisfying the condition at infinity:*

$$\lim_{|x|\to\infty} u(x) = 0.$$

Cauchy Problem. Let X be a Banach space and consider the inhomogeneous Cauchy Problem:

$$\begin{gathered} u'(t) = Au(t) + f(t), \quad t \in [0,\tau] \\ u(0) = u_0, \end{gathered} \tag{2.2}$$

where $u_0 \in X$ and $f \in C([0,\tau];X)$.

Definition 2.5. A mild solution of (ACP_f) is a function $u \in C([0,\tau];X)$ such that $\int_0^t u(s)ds \in D(A)$ and for all $t \in [0,\tau]$,

$$u(t) = u_0 + A\int_0^t u(s)ds + \int_0^t f(s)ds\,.$$

We recall now some results on resolvent positive operators and Cauchy problems, we refer to [2, Chapter 3], for more details.

Theorem 2.6 ([2])**.** *Let A be a resolvent positive operator on a Banach lattice X, that means, there exists $w \in \mathbb{R}$ such that $(w,\infty) \subset \rho(A)$ and $R(\lambda, A) \geq 0$ for all $\lambda > w$.*
(i) Let $u_0 \in D(A)$, $f_0 \in X$ such that $Au_0 + f_0 \in \overline{D(A)}$. Let $f(t) = f_0 + \int_0^t f'(s)ds$ where $f' \in L^1((0,\tau);X)$. Then (ACP_f) has a unique mild solution.
(ii) Let $f \in C([0,\tau];X_+)$, $u_0 \in X_+$ and let u be a mild solution of (ACP_f). Then $u(t) \geq 0$ for all $t \in [0,\tau]$.

Define now the Gaussian semigroup $(G(t))_{t\geq 0}$ on the space $C_0(\mathbb{R}^n)$ of all continuous functions vanishing at infinity by:

$$G(t)f(x) = (4\pi t)^{-n/2}\int_{\mathbb{R}^n} f(x-y)e^{-|y|^2/(4t)}dy, \quad t > 0, x \in \mathbb{R}^n, f \in C_0(\mathbb{R}^n).$$

Theorem 2.7 ([2]). *The family $(G(t))_{t\geq 0}$ defines a bounded holomorphic C_0-semigroup of angle $\frac{\pi}{2}$ on $C_0(\mathbb{R}^n)$. Its generator is the Laplacian Δ_G on $C_0(\mathbb{R}^n)$ with maximal domain; i.e.,*

$$D(\Delta_G) = \{f \in C_0(\mathbb{R}^n),\ \Delta f \in C_0(\mathbb{R}^n)\},$$
$$\Delta_G f = \Delta f,$$

here one identifies $C_0(\mathbb{R}^n)$ with a subspace of $\mathcal{D}(\mathbb{R}^n)'$.

Proposition 2.8 ([2]). *Let A be the generator of a bounded C_0-group $(U(t))_{t\in\mathbb{R}}$ on X. Then A^2 generates a bounded holomorphic C_0-semigroup $(T(t))_{t\geq 0}$ of angle $\frac{\pi}{2}$ on X. Moreover, for $t > 0$,*

$$T(t) = (4\pi t)^{-1/2} \int_{\mathbb{R}} e^{-|y|^2/(4t)} U(y)\, dy.$$

3. Main result

We consider the problem (1.1) with Ω presenting the cases in Example 2.3 and Proposition 2.4. Since in the case $n = 2$, the condition at infinity (2.1) is imposed by the boundary function, we restrict our study of (1.1) for $n = 2$ to the case where $\varphi = 0$.

Theorem 3.1. *Let $n \in \mathbb{N}$.* ***Case*** *$n = 1$: Let $\Omega_1 =]1, +\infty[$ with boundary $\Gamma_1 = \{1\}$ and denote by $\big(C_\infty(\overline{\Omega}_1); \|.\|_{C_\infty(\overline{\Omega}_1)}\big)$ the Banach space*

$$C_\infty(\overline{\Omega}_1) := \big\{u \in C([1, +\infty[),\ \lim_{x\to+\infty} \frac{u(x)}{x} \text{ exists}\big\}$$

with the norm $\|u\|_{C_\infty(\overline{\Omega}_1)} = \max_{x\in[1,\infty[} |u(x)/x|$.

Then for all $u_0 \in C_\infty(\overline{\Omega}_1)$ and all $\varphi \in C([0, \tau])$ such that $u_0(1) = \varphi(0)$, there exists a unique mild solution $u \in C([0, \tau]; C_\infty(\overline{\Omega}_1))$ of the problem

$$\begin{gathered} u_t(t, x) = u''(t, x), \quad t \in [0, \tau],\ x \in]1, +\infty[\\ u(t, 1) = \varphi(t), \quad t \in [0, \tau] \\ u(0, x) = u_0(x). \end{gathered} \tag{3.1}$$

Case *$n = 2$: Let $\Omega_2 = \mathbb{R}^2 \setminus B(0, 1)$ with boundary $\Gamma_2 = \partial B$ and set*

$$C_\infty(\overline{\Omega}_2) := \{u \in C(\overline{\Omega}_2),\ u\big|_{\Gamma_2} = 0 \text{ and } \lim_{|x|\to+\infty} u(x) = 0\}$$

with the supremum norm$\|u\|_{C_\infty(\overline{\Omega}_2)} = \max_{x\in\Omega_2} |u(x)|$. Then for all $u_0 \in C_\infty(\overline{\Omega}_2)$, there exists a unique mild solution $u \in C([0, \tau]; C_\infty(\overline{\Omega}_2))$ of the problem

$$\begin{gathered} u'(t) = \Delta u(t), \quad t \in [0, \tau] \\ u\big|_{\Gamma_2} = 0, \\ u(0) = u_0. \end{gathered} \tag{3.2}$$

Case *$n \geq 3$: Let $\Omega_n = \mathbb{R}^n \setminus B(0, 1)$ or more generally $\Omega_n = \mathbb{R}^n \setminus K$ with K being a compact set of $\mathbb{R}^n$ with boundary Γ_n, and set*

$$C_\infty(\overline{\Omega}_n) := \{u \in C(\overline{\Omega}_n),\ \lim_{|x|\to+\infty} u(x) = 0\}$$

with the supremum norm. If Ω_n is regular, then for all $u_0 \in C_\infty(\overline{\Omega}_n)$ and all $\varphi \in C([0,\tau]; C(\Gamma_n))$ such that $u_0\big|_{\Gamma_n} = \varphi(0)$, there exists a unique mild solution $u \in C([0,\tau]; C_\infty(\overline{\Omega}_n))$ of the problem

$$\begin{aligned} u'(t) &= \Delta u(t), \quad t \in [0,\tau] \\ u(t)\big|_{\Gamma_n} &= \varphi(t), \quad t \in [0,\tau] \\ u(0) &= u_0. \end{aligned} \tag{3.3}$$

Let Ω_n, $n \geq 1$ be defined as in Theorem 3.1 and define the operator $\Delta^n_{\max}$ on $C_\infty(\overline{\Omega}_n)$ as follows

$$\begin{gathered} D(\Delta^n_{\max}) = \{u \in C_\infty(\overline{\Omega}_n), \Delta u \in C_\infty(\overline{\Omega}_n)\} \\ \Delta^n_{\max} u = \Delta u \quad \text{in } \mathcal{D}(\Omega_n)'. \end{gathered}$$

We mean by mild solution of (3.3) a function $u \in C([0,\tau]; C_\infty(\overline{\Omega}_n))$ such that $\int_0^t u(s)ds \in D(\Delta^n_{\max})$ and for all $t \in [0,\tau]$,

$$\begin{gathered} u(t) = u_0 + \Delta \int_0^t u(s)ds \quad \text{in } \mathcal{D}(\Omega_n)' \\ u(t)\big|_{\Gamma_n} = \varphi(t). \end{gathered}$$

To prove Theorem 3.1, we will reformulate the problem (3.3) as an inhomogeneous Cauchy problem with resolvent positive operator.

4. Inhomogeneous Cauchy Problem

Define for $n \geq 1$ the Poisson operators A_n with domain $D(A_n) = D(\Delta^n_{\max}) \times \{0\}$ by

$$\begin{gathered} A_1(u,0) = (\Delta u, -u(1)), \\ A_2(u,0) = (\Delta u, 0), \\ A_n(u,0) = (\Delta u, -u\big|_{\Gamma_n}), \quad n \geq 3, \end{gathered}$$

and consider the Cauchy problem

$$\begin{aligned} U'(t) &= A_n U(t) + \Phi_n(t), \quad t \in [0,\tau] \\ U(0) &= U_0, \end{aligned} \tag{4.1}$$

where $U_0 = (u_0, 0)$, $u_0 \in C_\infty(\overline{\Omega}_n)$ is the initial data , $\Phi_2 = (0,0)$ and for $n \neq 2$, $\Phi_n(t) = (0, \varphi(t)), \varphi \in C([0,\tau]; C(\Gamma_n))$ is the boundary condition.

Proposition 4.1. *Let $n \geq 1$ and $U \in C([0,\tau]; C_\infty(\overline{\Omega}_n) \times C(\Gamma_n))$. Then U is a mild solution of* (4.1) *if and only if $U(t) = (u(t), 0)$ where $u \in C([0,\tau]; C_\infty(\overline{\Omega}_n))$ is the mild solution of* (1.1).

The proof is immediate from the definition of A_n and the fact that $\overline{D(A_n)} = C_\infty(\overline{\Omega}_n) \times \{0\}$.

To show the well-posedness of (4.1), we first prove that A_n is a resolvent positive operator.

Theorem 4.2. *Let $\lambda > 0$, if $n \neq 2$ then for all $(f, \phi) \in C_\infty(\overline{\Omega}_n) \times C(\Gamma_n)$ there exists a unique function $u \in D(\Delta^n_{\max})$ such that*

$$\begin{gathered}(\lambda - \Delta)u = f \quad \text{in } \mathcal{D}(\Omega_n)' \\ u\big|_{\Gamma_n} = \phi.\end{gathered} \tag{4.2}$$

Moreover, if $f \leq 0$, $\phi \leq 0$, then $u \leq 0$. If $n = 2$, then for all $f \in C_\infty(\overline{\Omega}_2)$, there exists a unique function $u \in D(\Delta^2_{\max})$ such that

$$\begin{gathered}(\lambda - \Delta)u = f \qquad \text{in } \mathcal{D}(\Omega_2)' \\ u\big|_{\Gamma_2} = 0.\end{gathered} \tag{4.3}$$

Moreover, if $f \leq 0$, then $u \leq 0$.

Proof. (1) **Existence.** (a) Case $n = 1$:
Set $C_\infty(\mathbb{R}) = \{f \in C(\mathbb{R}),\ \lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to +\infty} \frac{f(x)}{x}$ exists$\}$ and define on $C_\infty(\mathbb{R})$ the translation group

$$T(t)f(x) = f(x - t), \quad t \in \mathbb{R},\ x \in \mathbb{R}.$$

Then $(T(t))_{t \in \mathbb{R}}$ is a C_0-group with generator A_T defined by

$$\begin{gathered}D(A_T) = \{f \in C_\infty(\mathbb{R}), f' \in C_\infty(\mathbb{R})\} \\ A_T f = f'.\end{gathered}$$

It follows from Proposition 2.8 that A_T^2 generates a C_0-semigroup $(G(t))_{t \geq 0}$ which is the Gaussian semigroup:

$$G(t)f(x) = (4\pi t)^{-1/2} \int_{\mathbb{R}} e^{-|y|^2/(4t)} f(x - y)dy.$$

Moreover, $G(t)C_\infty(\overline{\Omega}_1) \subset C_\infty(\overline{\Omega}_1)$ for all $t \geq 0$. Let $\lambda > 0$ and $(f, \phi) \in C_\infty(\overline{\Omega}_1) \times \mathbb{R}$, and take

$$\begin{gathered}v_0(x) = \int_0^{+\infty} e^{-\lambda t} G(t)f(x)dt, \\ v(x) = (\phi - v_0(1))e^{-\sqrt{\lambda}(x-1)},\end{gathered}$$

Then $u = v + v_0$ is a solution of (4.2).
(b) Case $n \geq 2$: Let $f \in C_\infty(\overline{\Omega}_n)$. Then f can be extended to $C_0(\mathbb{R}^n)$. Since the Gaussian semigroup generates an holomorphic C_0-semigroup on $C_0(\mathbb{R}^n)$, we get that

$$v_0(x) = \int_0^{+\infty} e^{-\lambda t} G(t)f(x)dt,$$

is a solution of

$$(\lambda - \Delta)v = f \text{ for all } \lambda > 0. \tag{4.4}$$

Moreover, if $f \in C_\infty(\overline{\Omega}_n)$, then $v_0 \in C_\infty(\overline{\Omega}_n)$.
If $n = 2$, then v_0 is a solution of $(\lambda - A_2)(u_0, 0) = (f, 0)$.
If $n \geq 3$, it remains to show that there exists a solution of

$$\begin{gathered}(\lambda - \Delta)v = 0, \quad \text{in } \mathcal{D}(\Omega_n)' \\ v\big|_{\Gamma_n} = \phi - v_0\big|_{\Gamma_n} =: \psi.\end{gathered} \tag{4.5}$$

Let $\Omega_{nk} = \Omega_n \cap B(0, R_k)$ where $(R_k)_{k\geq 1}$ is an increasing sequence of positif reals such that $R_k \to \infty$ as $k \to \infty$ and consider the following problem on $C(\overline{\Omega}_{nk})$.

$$\begin{gathered}(\lambda - \Delta)v_k = 0 \quad \text{in } \mathcal{D}(\Omega_{nk})' \\ v_k\big|_{\Gamma_k} = 0 \quad \text{on } \Gamma_k = \partial B(0, R_k) \\ v_k\big|_{\Gamma_n} = \psi.\end{gathered} \tag{4.6}$$

Since Ω_n is regular, Ω_{nk} is regular and it follows from [10], [13] that (4.6) has a solution$v_k \in C(\overline{\Omega}_{nk})$. Our aim now is to show that the sequence $(v_k)_{k\geq 1}$ converges to the solution of (4.5), for that, we use the following maximum principle due to [2].

Theorem 4.3 (Maximum Principle for distributional solutions)**.** *Let Ω_0 be a bounded open set of $\mathbb{R}^n$ with boundary Γ. Let $M \geq 0$, $\lambda \geq 0$, $u \in C(\overline{\Omega}_0)$ such that*
(i) $\lambda u - \Delta u \leq 0$, in $\mathcal{D}(\Omega_0)'$
(ii) $u\big|_{\Gamma} \leq M$,
Then $u \leq M$ on $\overline{\Omega}_0$.

Without loss of generality, we can assume that $\psi \geq 0$.
Claim 1: $(v_k)_{k\geq 1}$ is an increasing bounded sequence. Indeed, by applying the Maximum principle in Ω_{nk} to v_k and $v_k - v_{k+1}$ respectively, we obtain:

$$0 \leq v_k \leq \|\psi\|,$$

and

$$\begin{gathered}(\lambda - \Delta)(v_k - v_{k+1}) = 0, \ \text{ in } \mathcal{D}(\Omega_{nk})' \\ (v_k - v_{k+1})\big|_{\Gamma_k} = -v_{k+1} \leq 0, \\ (v_k - v_{k+1})\big|_{\Gamma_n} = 0.\end{gathered}$$

Hence $v_k \leq v_{k+1}$ in Ω_{nk}.
Claim 2: Let $v = \lim_{k\to\infty} v_k$, then $v \in C_\infty(\overline{\Omega}_n)$. Indeed, denote by w_k the solution of the problem

$$\begin{gathered}\Delta w_k = 0, \quad \text{in } \mathcal{D}(\Omega_{nk})' \\ w_k\big|_{\Gamma_k} = 0, \\ w_k\big|_{\Gamma_n} = \psi.\end{gathered}$$

Then $w_k \geq 0$. Define the Poisson operator B_k on $C(\overline{\Omega}_{nk}) \times C(\Gamma_n \cup \Gamma_k)$ by

$$\begin{gathered}D(B_k) = \{w \in C(\overline{\Omega}_{nk}), \Delta w \in C(\overline{\Omega}_{nk})\} \times \{0\}, \\ B_k(w, 0) = (\Delta w, -(w\big|_{\Gamma_n}, w\big|_{\Gamma_k})).\end{gathered}$$

Since Ω_{nk} is regular, we deduce from [2, Chapter 6], that B_k is a resolvent positive operator and then

$$(w_k, 0) = R(\lambda, B_k)(\lambda w_k, (\psi, 0)) \geq R(\lambda, B_k)(0, (\psi, 0)) = (v_k, 0). \tag{4.7}$$

On the other hand, it follows from Proposition 2.4 that for all $\Phi \in C(\Gamma_n)$, the Dirichlet problem (1.2)(with $\phi = \Phi$) has a unique solution w satisfying the condition at infinity $\lim_{|x|\to\infty} w(x) = 0$. Moreover, if $\Phi \leq 0$, then $w \leq 0$. Indeed, let $\varepsilon > 0$, since $w \in C^1(\Omega_n)$, there exists $\Omega_0 \subset\subset \Omega_n$ such that $\operatorname{supp}(w-\varepsilon)^+ \subset \Omega_0$, Thus $(w-\varepsilon)^+ \in H_0^1(\Omega_0)$ and $\int_{\{w>\varepsilon\}} |\nabla w|^2 = 0$. Hence,

$$w \le \varepsilon.$$

Denote by w_0 the solution of (1.2) (with $\phi = \psi$) vanishing to zero at infinity, then

$$\begin{gathered}\Delta(w_k - w_0) = 0, \quad \text{in } \mathcal{D}(\Omega_{nk})' \\ (w_k - w_0)\big|_{\Gamma_k} = -w_{0|\Gamma_k} \le 0, \\ (w_k - w_0)\big|_{\Gamma_n} = 0.\end{gathered}$$

Theorem 4.3 and (4.7) imply that $0 \le v_k \le w_k \le w_0$. Hence

$$\lim_{|x|\to\infty} v(x) = \lim_{|x|\to\infty} w_0(x) = 0.$$

Finally, $u = v_0 + v$ is a solution of (4.2).
(2)**Positivity and Uniqueness.** Let $(f,\phi) \in C_\infty(\overline{\Omega}_n) \times C(\Gamma_n)$ such that $f \le 0$, $\phi \le 0$ and u a solution of (4.2).
Case $n = 1$: Since in that case $u \in C^2(\Omega_1) \cap C(\overline{\Omega}_1)$, we apply the Phragmèn-Lindelöf principle to deduce that $u \le 0$ whenever $f \le 0$, $\phi \le 0$. (See [12, Chapter 2]). By applying this maximum principle to u and $-u$ respectively when $f = 0$, we get uniqueness.
Case $n \ge 2$: Since $u \in D(\Delta^n_{\max})$, we get $u \in C^1(\Omega_n)$. Let $\Omega_0 \subset\subset \Omega_n$ such that $\operatorname{supp}(u-\varepsilon)^+ \subset \Omega_0$, $\varepsilon > 0$. then $(u-\varepsilon)^+ \in H_0^1(\Omega_0)$ and

$$\begin{aligned}\int f(u-\varepsilon)^+ &= \lambda \int u(u-\varepsilon)^+ + \int \nabla u \nabla (u-\varepsilon)^+ \\ &= \lambda \int (u-\varepsilon)(u-\varepsilon)^+ + \varepsilon\lambda \int (u-\varepsilon)^+ + \int_{\{u>\varepsilon\}} |\nabla u|^2 \\ &\le 0.\end{aligned}$$

Hence $u \le \varepsilon$. □

We are now in position to show the well-posedness of the Cauchy problem (4.1). If $n \ne 2$, let $\varphi \in W^{1,1}((0,\tau); C(\Gamma_n))$ and $U_0 = (u_0, 0) \in D(A_n) = D(\Delta^n_{\max}) \times \{0\}$, then

$$A_n U_0 + \Phi_n(0) = (\Delta u_0, -u_{0|\Gamma_n} + \varphi(0)).$$

Hence $A_n U_0 + \Phi_n(0) \in \overline{D(A_n)} = C_\infty(\overline{\Omega}_n) \times \{0\}$ if and only if

$$u_0\big|_{\Gamma_n} = \varphi(0). \tag{4.8}$$

Assumption (4.8) becomes trivial in the case $n = 2$ since we have assumed $\varphi = 0$. On the other hand, it follows from Theorem 4.2 that A_n is a resolvent positive operator. Hence, by applying Theorems 2.6 we obtain the following result.

Proposition 4.4. *Let $n \in \mathbb{N}$.*
***Case** $n = 1$: Let $\Omega_1 =]1, +\infty[$. Then for all $u_0 \in D(\Delta^1_{\max})$ and all $\varphi \in W^{1,1}((0,\tau))$ such that $u_0(1) = \varphi(0)$, there exists a unique mild solution of (4.1) with $n = 1$.*
***Case** $n = 2$: Let $\Omega_2 = \mathbb{R}^2 \setminus B(0,1)$. Then for all $u_0 \in D(\Delta^2_{\max})$, there exists a unique mild solution of (4.1) with $n = 2$.*
***Case** $n \ge 3$: Let $\Omega_n = \mathbb{R}^n \setminus K$ with boundary Γ_n, K being a compact set of $\mathbb{R}^n$. If Ω_n is regular, then for all $u_0 \in D(\Delta^n_{\max})$ and all $\varphi \in W^{1,1}((0,\tau); C(\Gamma_n))$ such that $u_0\big|_{\Gamma_n} = \varphi(0)$, there exists a unique mild solution of (4.1).*

The proof of Theorem 3.1 will be complete by combining Theorem 4.1 and the following result.

Proposition 4.5. *(i) Let* $u_0 \in C_\infty(\overline{\Omega}_1)$ *and* $\varphi \in C([0,\tau])$ *such that* $u_0(1) = \varphi(0)$*, then there exists a unique mild solution of* (4.1) *with* $n = 1$.
(ii) Let $u_0 \in C_\infty(\overline{\Omega}_2)$*, then there exists a unique mild solution of* (4.1) *with* $n = 2$.
(iii) Assume that $\Omega_n = \mathbb{R}^n \backslash K$ *is regular and let* $u_0 \in C_\infty(\overline{\Omega}_n)$ *and* $\varphi \in C([0,\tau]; C(\Gamma_n))$ *such that* $u_0\big|_{\Gamma n} = \varphi(0)$*, then there exists a unique mild solution of* (4.1).

Proof. Choose $u_{0_k} \in D(\Delta^n_{\max})$ such that $u_{0_k} \to u_0$ as $k \to \infty$ in $C_\infty(\overline{\Omega}_n)$. Choose $\varphi_k \in W^{1,1}((0,\tau); C(\Gamma_n))$ such that $\varphi_k(0) = u_{0_k}\big|_{\Gamma_n}$ and $\varphi_k \to \varphi$as $k \to \infty$ in $C([0,\tau]; C(\Gamma_n))$. By applying Proposition 4.4 and Theorem 4.1, we deduce that there exists a unique mild solution $u_k \in C_\infty(\overline{\Omega}_n)$ of $P_\tau(u_{0_k}, \varphi_k)$. We can show that

$$\|u_k\|_{C([0,\tau]; C_\infty(\overline{\Omega}_n))} \le \max\{\|\varphi_k\|_{C([0,\tau]; C(\Gamma_n))}, \|u_{0k}\|_{C_\infty(\overline{\Omega}_n)}\}.$$

where

$$\|\varphi_k\|_{C([0,\tau]; C(\Gamma_n))} = \sup_{0 \le t \le \tau} \|\varphi_k(t)\|_{C(\Gamma_n)}$$
$$\|u_k\|_{E_\infty(\overline{\Omega}_n)} = \sup_{0 \le t \le \tau} \|u_k(t)\|_{E_\infty(\overline{\Omega}_n)}.$$

Hence $(u_k)_{k \ge 1}$ is a Cauchy sequence in $C([0,\tau]; C_\infty(\overline{\Omega}_n))$. Let $u = \lim_{k\to\infty} u_k$, then $\int_0^t u(s)ds = \lim_{k\to\infty} \int_0^t u_k(s)ds \in D(\Delta^n_{\max})$ and

$$u(t) = u_0 + \Delta \int_0^t u(s)ds \quad \text{in } \mathcal{D}(\Omega_n)'$$
$$u(t)\big|_{\Gamma_n} = \lim_{k\to\infty} \varphi_k(t) = \varphi(t).$$

for all $t \in [0,\tau]$. ☐

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