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Asymptotic analysis of the Carrier-Pearson problem *

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Abstract

This paper provides a rigorous analysis of the asymptotic behavior of the solution for the boundary-value problem

$$\epsilon^2 u'' + u^2 - 1 = 0, \quad -1 < x < 1,$$
$$u(-1) = u(1) = 0$$

as the parameter ϵ approaches zero.

1 Introduction

The boundary-value problem

$$\epsilon^2 u'' + u^2 - 1 = 0, \quad -1 < x < 1, \tag{1.1}$$
$$u(-1) = u(1) = 0, \tag{1.2}$$

where $0 < \epsilon \ll 1$ is a parameter, has been studied extensively since it was introduced by Carrier and Pearson in 1968. The problem attracted their attention so much that they reintroduced the problem in 1990 [1]. It was observed that there are two boundary layers at $x = \pm 1$, and an interior layer $x_0(\epsilon) \in (-1, 1)$. There may be denumerably many solutions to the boundary-value problem, i.e., the solution may oscillate many times depending on the value of $u'(-1)$. In their book, Carrier and Pearson pointed out that the method of matched asymptotic expansions seemed to produce "spurious" solutions. This problem, in fact, is caused by the existence of transcendentally or exponentially small terms between the two boundary layers, actually, between the left boundary layer and interior layer. In order to better approximate the solution of (1.1)-(1.2), many researchers presented several ways to formally construct the asymptotic solution of (1.1)-(1.2) [2, 3, 5, 8, 10]. Some of them even considered more general boundary-value problems, the equation (1.1) with $u(-1) = a$ and $u(1) = b$ where $a, b \in (-1, 2)$. This paper uses the symmetry of the solution of (1.1)-(1.2) to rigorously analyze the asymptotic behavior of the solution. The main

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result of the paper on the uniform convergence of the solution has not been seen elsewhere. For the sake of simplicity, we only focus on the solutions with only one-spike, which have two wiggles - one is down in the left part and the other is up in the right part of the interval. Since the equation is autonomous and the solution is periodic, we can, of course, have a single spike solution that has a up wiggle in the left part and a down wiggle in the right part of the interval. For the sake of certainty, the single spike solution in this paper means the former type. Throughout the paper, the words "the single spike solution" mean the single spike solution which has a minimum first and then a maximum. The main result of the paper stated as the theorem of the paper is given at the end of section 4.

The existence and uniqueness of the single-spike solution have been proved by this author in [7]. For the need of asymptotic analysis, we will have to repeat some of the results from [7].

Preliminary Results

Lemma 1.1 *Any solution of (1.1)-(1.2) is bounded between* -1 *and 2.*

Proof Multiplying the both sides of (1.1) by u' and integrating the resulting equation, we get

$$\frac{\epsilon^2 (u')^2}{2} = u - \frac{u^3}{3} + d(\epsilon) \tag{1.3}$$

where $d(\epsilon)$ is the integrate constant depending on ϵ. Since the solution has a minimum value at $u \in (-1, 0)$ with $u - \frac{u^3}{3} + d(\epsilon) = 0$ for all ϵ, it follows that $d(\epsilon) = -u + \frac{u^3}{3}$ is bounded for all ϵ. In fact, $d(\epsilon) \in (0, 2/3)$, because the range of the function $d(\epsilon) = -u + \frac{u^3}{3}$ for $u \in (-1, 0)$ is $(0, 2/3)$. Noting that (1.3) holds for all $x \in [-1, 1]$ and that $d(\epsilon)$ only depends on ϵ, we see that $u - \frac{u^3}{3} \geq -d(\epsilon) \geq -2/3$ from which we conclude that u is also bounded above for all $x \in [-1, 1]$ and for all ϵ, and that $a, b \in (-1, 2)$. □

If $a = b = 0$, we call the boundary-value problem the homogeneous problem, otherwise, the nonhomogeneous problem.

Let x_- and x_+ be the two points where the single spike solution $u(x)$ takes extrema, i.e., $u(x_-) = \min\{u(x)\}, u(x_+) = \max\{u(x)\}$ over the interval $[-1, 1]$.

Lemma 1.2 *Any solution of the homogeneous problem* u *is symmetric with respect to the line* $x = x_-$ *on the interval* $[-1, x_0]$ *and the line* $x = x_+$ *on* $[x_0, 1]$.

See [7] for the proof. Using Lemma 1.2, we can investigate the solution over the interval $[x_-, x_+]$. Again, from the symmetry, $x_+ - x_- = 1$ and u is strictly increasing in (x_-, x_+). Since the equation is autonomous, we can transform the interval $[x_-, x_+]$ onto $[0, 1]$. Then, we only study the equation (1.1) with boundary conditions

$$u'(0) = u'(1) = 0. \tag{1.4}$$

Note that there are, in terms of the new variable, also two boundary layers at $x = 0, x = 1$ respectively, and a turning point $x = \widetilde{x}_0$ which is the transformation of the turning point x_0. For the sake of briefness, we again use x_0 to denote the turning point, and all the notations used in what follows are related to the new variables. The behavior of the solution of (1.1)-(1.4) at $x = x_0$ will provide the information about the solution at the boundary layers and interior layer for the original solution. The goal of the paper is to regorously study the solution of the boundary-value problem (1.1)-(1.4). Since $0 < \epsilon \ll 1$, we only consider the side limit, $\epsilon \to 0^+$, which is denoted by $\epsilon \to 0$ for simplicity in what follows.

Noting that $x = 0$ and $x = 1$ are the only two rest points of the solution on $[0, 1]$, we see that the solution, if it exists, must be strictly increasing in $(0, 1)$. For convenience, we rewrite (1.3) as

$$(\epsilon u')^2 = 2\big(u - \frac{u^3}{3} + c(\epsilon)\big), \tag{1.5}$$

where $c(\epsilon) = -u(0) + \frac{u^3(0)}{3}$ is determined by the boundary condition $u(0)$. Let $u(0) = \alpha \le 0$, and consider the initial value problem (1.1) with

$$u(0) = \alpha, u'(0) = 0. \tag{1.6}$$

For any given $\epsilon > 0$, the initial value problem (1.1)-(1.6) has a unique solution. If $\alpha = -1$, the initial-value problem has the constant solution $u(x) \equiv -1$. If $\alpha < -1$, then $u'' < 0$ initially, and hence, $u' < 0$, $u'' < 0$ as long as $u < -1$ which cannot provide the solution with a spike. Thus, we only consider $\alpha \in (-1, 0)$. Define $f(u) = 3u - u^3$. From (1.5),

$$u' = \frac{1}{\epsilon}\sqrt{(2/3}\sqrt{f(u) - f(\alpha)} \tag{1.7}$$

where $\alpha \in (-1, 0)$. Let $u'(x_+) = 0$ for an $x_+ > 0$. Then, $u(x_+) = \beta$ is the greatest zero point of the equation $f(u) - f(\alpha) = 0$, and α the mid-zero point of the equation, and therefore, $f(u) - f(\alpha) = (u - \mu)(u - \alpha)(\beta - u)$ with $\mu \le \alpha < 0 < \beta$. Hence,

$$\begin{aligned} f(u) - f(\alpha) &= -(u - \alpha)(u^2 + \alpha u + \alpha^2 - 3) \\ &= (\beta - u)(u^2 + \beta u + \beta^2 - 3) \end{aligned} \tag{1.8}$$

with

$$\beta = \frac{-\alpha + \sqrt{12 - 3\alpha^2}}{2} \tag{1.9}$$

It is also seen that $\beta > \sqrt{3}$ ($\alpha = 0$ implies $\beta = \sqrt{3}$ and $\mu = -\beta$) and β is a decreasing function of α because $\frac{d\beta}{d\alpha} = -\frac{1}{2} - \frac{3\alpha}{\sqrt{12-3\alpha^2}} < 0$ for $\alpha \in (-1, 0)$. We now define a function

$$I(\alpha) = \int_\alpha^\beta \frac{du}{\sqrt{f(u) - f(\alpha)}} \tag{1.10}$$

for $\alpha \in (-1, 0)$ and $\beta = \frac{-\alpha+\sqrt{12-3\alpha^2}}{2} \in (\sqrt{3}, 2)$. Notice that this function is independent of ϵ. From (1.7), the value of α required by the boundary-value problem can be determined by the equation

$$I(\alpha) = \frac{\sqrt{2}}{\sqrt{3}\epsilon}. \tag{1.11}$$

Thus, the existence and uniqueness of the one-spike solution of the boundary-value problem is equivalent to the existence and uniqueness of the value of α, the solution of (1.11) for given $\epsilon > 0$. Since the function $f(u) - f(\alpha)$ has three distinct real zero points and the two limits of the improper integral $I(\alpha)$ are regular, the improper integral $I(\alpha)$ converges and $I(\alpha)$ is well defined.

Lemma 1.3 *As* $\epsilon \to 0$, $\alpha \to -1^+$.

Proof Since

$$\begin{aligned} I(\alpha) &= \Big(\int_\alpha^0 + \int_0^\beta\Big)\frac{du}{\sqrt{f(u)-f(\alpha)}} \\ &= \int_\alpha^0 \frac{du}{\sqrt{3u-u^3-3\alpha+\alpha^3}} + \int_0^\beta \frac{du}{\sqrt{3u-u^3-3\beta+\beta^3}} \\ &\le \int_\alpha^0 \frac{du}{\sqrt{3(u-\alpha)(1-\alpha^2)}} + (\int_0^c + \int_c^\beta)\frac{du}{\sqrt{\beta u(\beta-u)}} \\ &\le \frac{2\sqrt{-\alpha}}{\sqrt{3-3\alpha^2}} + \frac{2\sqrt{c}}{\sqrt{\beta(\beta-c)}} + \frac{2\sqrt{\beta-c}}{\sqrt{\beta c}}, \end{aligned}$$

for some constant $c \in (0, \beta)$. It follows that $\sqrt{2}/(\sqrt{3}\epsilon) \le 2\sqrt{-\alpha}/\sqrt{3-3\alpha^2} + O(1)$. This proves either $\alpha \to -1$ as $\epsilon \to 0$. □

In the remainder of the paper, we use $\alpha \to -1$ to denote the side limit, $\alpha \to -1^+$.

Lemma 1.4 *The function* $I(\alpha)$ *is continuous for* $\alpha \in (-1, 0)$ *and decreasing for sufficiently small* $(\alpha + 1) > 0$.

For the proof of this lemma, see [7]. More delicate work may prove that the function $I(\alpha)$ is a monotone function in the open interval $(-1, 0)$. Because we only focus on asymptotic analysis as $\epsilon \to 0$ in the paper, we omit that proof here. As stated above, it follows from Lemma 1.4 that the single-spike solution of (1.1)-(1.2) exists and is unique.

2 Length of the interior layer

The asymptotic analysis begins in this section. From (1.7), we see that

$$\int_\alpha^x \frac{du}{\sqrt{f(u)-f(\alpha)}} = \sqrt{\frac{2}{3}}\frac{x}{\epsilon}. \tag{2.1}$$

Through out the paper, we will study the asymptotic behavior of the integral function on the left side of (2.1).

Lemma 2.1 *Let x_0 be the turning point. Then, as $\epsilon \to 0$,*

$$1 - x_0 \sim [\sqrt{2}\ln(\sqrt{3}+\sqrt{2})]\epsilon.$$

Proof It is seen from (1.9) that $\beta \to 2^-$ as $\alpha \to -1$. We will only consider the case that $(2-\beta)$ is positive and sufficiently small. From (1.7), we see that

$$\int_0^u \frac{du}{\sqrt{f(u)-f(\alpha)}} = \sqrt{\frac{2}{3}}\frac{x-x_0}{\epsilon}. \tag{2.2}$$

Define $K(\beta) = \int_0^\beta \frac{du}{\sqrt{f(u)-f(\beta)}}$, $M(\beta) = \int_0^\beta \frac{du}{\sqrt{f(u)-f(2)}}$ and $s(\beta) = K(\beta) - M(\beta)$. It follows from (2.2) that $K(\beta) = \sqrt{(2/3)}\frac{x-x_0}{\epsilon}$. Since

$$\begin{aligned} K(\beta) &= \Big(\int_0^{\sqrt{3}} + \int_{\sqrt{3}}^\beta\Big)\frac{du}{\sqrt{f(u)-f(\beta)}} \\ &= \int_0^{\sqrt{3}} \frac{du}{\sqrt{f(u)-f(\beta)}} + 2\int_{\sqrt{3}}^\beta \frac{d(\sqrt{f(u)-f(\beta)})}{f'(u)} \\ &= \int_0^{\sqrt{3}} \frac{du}{\sqrt{f(u)-f(\beta)}} - \frac{2\sqrt{f(\sqrt{3})-f(\beta)}}{f'(\sqrt{3})} \\ &\quad + \int_{\sqrt{3}}^\beta 2\sqrt{f(u)-f(\beta)}\frac{f''(u)}{[f'(u)]^2}du. \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} M(\beta) &= \Big(\int_0^{\sqrt{3}} + \int_{\sqrt{3}}^\beta\Big)\frac{du}{\sqrt{f(u)-f(2)}} \\ &= \int_0^{\sqrt{3}} \frac{du}{\sqrt{f(u)-f(2)}} + 2\int_{\sqrt{3}}^\beta \frac{d(\sqrt{f(u)-f(2)})}{f'(u)} \\ &= \int_0^{\sqrt{3}} \frac{du}{\sqrt{f(u)-f(2)}} + \frac{2\sqrt{f(\beta)-f(2)}}{f'(\beta)} - \frac{2\sqrt{2}}{f'(\sqrt{3})} \\ &\quad +2\int_{\sqrt{3}}^\beta \sqrt{f(u)-f(2)}\frac{f''(u)}{[f'(u)]^2}\,du, \end{aligned} \tag{2.4}$$

it follows that

$$\begin{aligned} s(\beta) &= \int_0^{\sqrt{3}} \Big[\frac{1}{\sqrt{f(u)-f(2)}} - \frac{1}{\sqrt{f(u)-f(\beta)}}\Big] du \\ &\quad -\frac{2\sqrt{-f(\beta)}}{f'(\sqrt{3})} - \Big[\frac{2\sqrt{f(\beta)-f(2)}}{f'(\beta)} - \frac{2\sqrt{2}}{f'(\sqrt{3})}\Big] \\ &\quad +2\int_{\sqrt{3}}^{\beta} \Big[\sqrt{f(u)-f(\beta)} - \sqrt{f(u)-f(2)}\Big] \frac{f''(u)}{[f'(u)]^2}\, du. \end{aligned} \tag{2.5}$$

The last integral in (2.5) can be written as

$$p(\beta) = \int_{\sqrt{3}}^{\beta} \frac{(2-\beta)(1+\beta)^2}{\big[\sqrt{f(u)-f(\beta)} + \sqrt{f(u)-f(2)}\big]} \frac{(-6u)}{9(1-u^2)^2}\, du. \tag{2.6}$$

It then follows that $p(\beta) = O(\sqrt{2-\beta})$ and $s(\beta) \to 0$ as $\beta \to 2$. From $M(\beta) = -\frac{1}{\sqrt{3}}\big[\ln\frac{\sqrt{3}+\sqrt{2-\beta}}{\sqrt{3}-\sqrt{2-\beta}} - \ln\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}\big]$, we see that $\lim_{\beta\to 2} K(\beta) = \lim_{\beta\to 2} M(\beta) = \frac{2}{\sqrt{3}}\ln(\sqrt{3}+\sqrt{2})$. Since, as $\beta \to 2$,

$$\frac{\sqrt{2}(1-x_0)}{\sqrt{3}\epsilon} = K(\beta) \to \frac{2}{\sqrt{3}}\ln(\sqrt{3}+\sqrt{2}),$$

or

$$\lim_{\epsilon\to 0}\Big\{\frac{\sqrt{2}(1-x_0)}{\sqrt{3}\epsilon} - \frac{2}{\sqrt{3}}\ln(\sqrt{3}+\sqrt{2})\Big\} = 0,$$

it turns out that, as $\epsilon \to 0$,

$$1 - x_0 \sim \epsilon\sqrt{2}\ln(\sqrt{3}+\sqrt{2}),$$

which is the conclusion of Lemma 2.1. □

We are now ready to study the asymptotic behavior of the solution at the boundary layers.

3 Asymptotic analysis for $x \le x_0$

Let $x < x_0$, and $u = u(x)$. Then, from (1.7),

$$\int_u^0 \frac{du}{\sqrt{f(u)-f(\alpha)}} = \sqrt{\frac{2}{3}}\frac{x_0 - x}{\epsilon}. \tag{3.1}$$

Define $J(u,\alpha) = \int_u^0 \frac{du}{\sqrt{f(u)-f(\alpha)}}$ and $L(u) = \int_u^0 \frac{du}{\sqrt{f(u)-f(-1)}}$ for $u \in [\alpha, 0]$. Note that $L(u)$ can be expressed explicitly, i.e.,

$$L(u) = -\frac{1}{\sqrt{3}}\Big[2\ln(\sqrt{3}+\sqrt{2}) - \ln\frac{\sqrt{3}+\sqrt{2-u}}{\sqrt{3}-\sqrt{2-u}}\Big]. \tag{3.2}$$

On the other hand, by integration by parts,

$$L(u)=\frac{2}{3}\Big[\sqrt{2}-\frac{\sqrt{f(u)+2}}{1-u^2}+2\int_u^0\frac{-u\sqrt{f(u)+2}du}{(1-u^2)^2}\Big] \tag{3.3}$$

and

$$J(u,\alpha)=\frac{2}{3}\Big[\sqrt{-f(\alpha)}-\frac{\sqrt{f(u)-f(\alpha)}}{1-u^2}+2\int_u^0\frac{-u\sqrt{f(u)-f(\alpha)}du}{(1-u^2)^2}\Big]. \tag{3.4}$$

Therefore,

$$\begin{aligned} J(u,\alpha)-L(u) &= \frac{2}{3}\Big[\sqrt{-f(\alpha)}-\frac{\sqrt{f(u)-f(\alpha)}}{1-u^2}-\sqrt{2}+\frac{\sqrt{f(u)+2}}{1-u^2}\Big] \\ &\quad +\frac{4}{3}\int_u^0\frac{-u[\sqrt{f(u)-f(\alpha)}-\sqrt{f(u)+2}]}{(1-u^2)^2}du. \end{aligned} \tag{3.5}$$

Noting that

$$\begin{aligned} \frac{\sqrt{f(u)-f(\alpha)}-\sqrt{f(u)+2}}{1-u^2} &= \frac{(\alpha+1)^2(2-\alpha)}{\big[\sqrt{f(u)-f(\alpha)}+\sqrt{f(u)+2}\big](1-u^2)} \\ &\le \frac{(\alpha+1)^2(2-\alpha)}{(1+u)^2(1-u)\sqrt{2-u}} \end{aligned} \tag{3.6}$$

and

$$\sqrt{-f(\alpha)}-\sqrt{2}=\frac{(2-\alpha)(\alpha+1)^2}{\sqrt{\alpha^3-3\alpha}-2}, \tag{3.7}$$

we see that

$$\int_u^0\frac{-u\big[\sqrt{f(u)-f(\alpha)}-\sqrt{f(u)+2}\big]du}{(1-u^2)^2}\le\int_u^0\frac{-(\alpha+1)^2(2-\alpha)u\,du}{(1+u)^3(1-u)^2\sqrt{2-u}} \tag{3.8}$$

and hence, for all $u(x)\in[\alpha,0]$, equivalently for $x\in[0,x_0]$

$$0\le J(u,\alpha)-L(u)\le\sqrt{3}|u|\big(\frac{1+\alpha}{1+u}\big)^2 \tag{3.9}$$

which is either $o(1)$ or $O(1)$ as $\alpha\to-1$. From (3.9),

$$0\le\frac{J(u,\alpha)}{L(u)}-1\le\frac{\sqrt{3}|u|\big(\frac{1+\alpha}{1+u}\big)^2}{L(u)}. \tag{3.10}$$

Our next goal is to prove $\frac{J(u,\alpha)}{L(u)}\to1$ uniformly on $[\alpha,0]$ as $\alpha\to-1$. To study the convergence, we denote $S(u)=(1+u)^2L(u)$. A differentiation of $S(u)$ with respect to u produces

$$S'(u)=2(1+u)\int_u^0\frac{du}{(u+1)\sqrt{2-u}}-\frac{1+u}{\sqrt{2-u}} \tag{3.11}$$

which implies that $S'(u) > 0$ for $u \in [\alpha, u_1]$ with some constant $u_1 \in (\alpha, 0)$. Noting that $S(u)$ is independent of α, we see that u_1 is independent of α. Thus, $S'(u) > 0$ for $u \in [\alpha, u_1]$, and hence, $S(u) \geq S(\alpha)$. It follows that

$$0 \leq \frac{\breve{J}(u,\alpha)}{L(u)} - 1 \leq \frac{\sqrt{3}|u|}{L(\alpha)} \to 0 \tag{3.12}$$

uniformly for $u \in [\alpha, u_1]$ because $L(\alpha) \sim \frac{1}{\sqrt{3}} \ln \frac{\sqrt{3}+\sqrt{2-\alpha}}{\sqrt{3}-\sqrt{2-\alpha}}$. For any fixed $u_0 \in (u_1, 0)$, the convergence of $\frac{J(u,\alpha)}{L(u)} \to 1$ is uniform on $[u_1, u_0]$. This can be obtained by the uniform convergence of $J(u,\alpha) \to L(u)$ on the interval because $f(u) - f(\alpha)$ uniformly converges to $f(u) - f(-1)$ for $u \in [u_1, u_0]$ as $\alpha \to -1$. Next, we will prove that the convergence is also uniform on $[u_0, 0]$ by choosing a u_0 independent of α. First, we see that $L(u) \sim -\frac{u}{\sqrt{2}}$ because $-\frac{u}{\sqrt{2}}$ is the linear part of $L(u)$ for sufficiently small $|u|$. We then consider the function $Q(u,\alpha) = \frac{J(u,\alpha)}{L(u)}$ for $u < 0$ where $|u|$ is sufficiently small, noting that $Q(0,\alpha)$ is defined by an application of L'Hospital's rule. From

$$\frac{\partial Q}{\partial \alpha} = \frac{(f(u) - f(\alpha))^{-3/2} f'(\alpha)}{L(u)}, \tag{3.13}$$

we find $\frac{\partial Q}{\partial \alpha} > 0$ for $u < 0$ and for $|\alpha| < 1$. This shows that $Q(u,\alpha) < Q(0,\alpha) = \frac{\sqrt{2}}{\sqrt{-3\alpha+\alpha^3}}$. Thus, if we choose $u_0 < 0$ with sufficiently small $|u_0|$, then for $u \in [u_0, 0]$, $Q(u,\alpha) - 1 \leq \frac{\sqrt{2}}{\sqrt{-3\alpha+\alpha^3}} - 1 \to 0$ uniformly. This completes the proof that $J(u,\alpha) \sim L(u)$ uniformly on $u \in [\alpha, 0]$ as $\alpha \to -1$, and hence, as $\epsilon \to 0$,

$$-2\ln(\sqrt{3}+\sqrt{2}) + \ln \frac{\sqrt{3}+\sqrt{2-u}}{\sqrt{3}-\sqrt{2-u}} \sim \sqrt{\frac{2}{3}} \frac{x_0 - x}{\epsilon}, \tag{3.14}$$

or

$$\ln \frac{\sqrt{3}+\sqrt{2-u}}{\sqrt{3}-\sqrt{2-u}} \sim \sqrt{\frac{2}{3}} \frac{x_0 - x}{\epsilon} + 2\ln(\sqrt{3}+\sqrt{2}). \tag{3.15}$$

This implies that for the compact interval $[0, x_0]$,

$$u(x) \sim 2 - 3\Big[\frac{(5+2\sqrt{6})e^{\frac{\sqrt{2}(x_0-x)}{\epsilon}} - 1}{(5+2\sqrt{6})e^{\frac{\sqrt{2}(x_0-x)}{\epsilon}} + 1}\Big]^2, \tag{3.16}$$

or

$$u \sim -1 + 3\,\mathrm{sech}^2\left(\frac{x_0 - x}{\epsilon\sqrt{2}} + 2\ln(\sqrt{3}+\sqrt{2})\right) \tag{3.17}$$

uniformly as $\epsilon \to 0$ for $x \in [0, x_0]$. Note that as $\epsilon \to 0$, $x_0 = x_0(\epsilon) \to 1$, and therefore, the compact interval $[0, x_0]$ is a moving interval. Here we adopt the definition of uniform convergence defined on the moving interval as stated in [4] and applied for rigorous proofs of asymptotic solutions by this author in [6].

4 Asymptotic analysis for $x \geq x_0$

For $u \in [0, \beta]$, we define

$$K(u,\beta) = \int_0^u \frac{du}{\sqrt{f(u) - f(\beta)}} \,. \tag{4.1}$$

From (2.2), $K(u,\beta) = \sqrt{(2/3)}\frac{x-x_0}{\epsilon}$. An argument similar to the previous section can be carried out for $u \in [0, u_2]$, where $u_2 > 0$ is sufficiently small and independent of α, to prove the uniform convergence of $K(u,\alpha) \sim M(u)$ where the function

$$\begin{aligned} M(u) &= \int_0^u \frac{du}{\sqrt{f(u) - f(2)}} \\ &= \frac{1}{\sqrt{3}} \ln \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} - \frac{1}{\sqrt{3}} \ln \frac{\sqrt{3}+\sqrt{2-u}}{\sqrt{3}-\sqrt{2-u}}. \end{aligned} \tag{4.2}$$

Then for $u \in [u_2, 2-\delta]$, where δ is a sufficiently small positive number, the convergence $\int_0^u \frac{du}{\sqrt{f(u)-f(\alpha)}} \to M(u)$ is also uniform because of the uniform convergence of $f(u) - f(\beta)$ to $f(u) - f(2)$. This means that $K(u,\beta) \sim M(u)$ holds for $u \in [u_2, 2-\delta]$. Finally, we prove that the uniform convergence holds on the interval $[2-\delta, \beta]$. Following the computation in the proof of lemma 2.1, by integration by parts, we get

$$\begin{aligned} K(u,\beta) &= \int_0^{\sqrt{3}} \frac{du}{\sqrt{f(u)-f(\beta)}} + \frac{2\sqrt{f(u)-f(\beta)}}{f'(u)} - \frac{2\sqrt{f(\sqrt{3})-f(\beta)}}{f'(\sqrt{3})} \\ &\quad + \int_{\sqrt{3}}^u 2\sqrt{f(u)-f(\beta)} \frac{f''(u)}{[f'(u)]^2}\, du. \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} M(\beta) &= \int_0^{\sqrt{3}} \frac{du}{\sqrt{f(u)-f(2)}} + \frac{2\sqrt{f(u)+2}}{f'(u)} - \frac{2\sqrt{2}}{f'(\sqrt{3})} \\ &\quad +2\int_{\sqrt{3}}^u \sqrt{f(u)-f(2)} \frac{f''(u)}{[f'(u)]^2}\, du. \end{aligned} \tag{4.4}$$

Let $s(u) = K(u,\beta) - M(u)$. Then

$$\begin{aligned} s(u) &= \int_0^{\sqrt{3}} \Big[\frac{1}{\sqrt{f(u)-f(2)}} - \frac{1}{\sqrt{f(u)-f(\beta)}}\Big] du \\ &\quad + \frac{2\sqrt{f(u)-f(\beta)}}{f'(u)} - \frac{2\sqrt{-f(\beta)}}{f'(\sqrt{3})} - \Big[\frac{2\sqrt{f(u)-f(2)}}{f'(u)} - \frac{2\sqrt{2}}{f'(\sqrt{3})}\Big] \\ &\quad +2\int_{\sqrt{3}}^u \Big[\sqrt{f(u)-f(\beta)} - \sqrt{f(u)-f(2)}\Big] \frac{f''(u)}{[f'(u)]^2} du. \end{aligned} \tag{4.5}$$

The last integral in (4.5) can be written as

$$p(u)=\int_{\sqrt{3}}^{u}\frac{(2-\beta)(1+\beta)^2}{\Big[\sqrt{f(u)-f(\beta)}+\sqrt{f(u)-f(2)}\Big]}\frac{-6u}{9(1-u^2)^2}\,du. \tag{4.6}$$

It is observed that $p(u)=O(\sqrt{2-\beta})$. It follows that $s(u)\to 0$ uniformly for $u\in[2-\delta,b]$, as $\beta\to 2$. Therefore, $K(u,\alpha)\sim M(u)$ uniformly for $u\in[2-\delta,\beta]$. We now can conclude that

$$\frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}-\frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+\sqrt{2-u}}{\sqrt{3}-\sqrt{2-u}}\sim\sqrt{\frac{2}{3}}\frac{x-x_0}{\epsilon} \tag{4.7}$$

which implies that

$$u\sim -1+3\,\mathrm{sech}^2\big(\frac{x_0-x}{\epsilon\sqrt{2}}+2\ln(\sqrt{3}+\sqrt{2})\big) \tag{4.8}$$

uniformly for $x\in[x_0,1]$

Summing up, we have proved that, as $\epsilon\to 0$, the asymptotic formula

$$\int_{\alpha}^{u}\frac{du}{\sqrt{f(u)-f(\alpha)}}\sim\int_{\alpha}^{u}\frac{du}{\sqrt{f(u)-f(-1)}} \tag{4.9}$$

uniformly holds for all $u\in[\alpha,\beta]$ and the asymptotic expression (4.8) holds uniformly for $x\in[0,1]$. One can then easily get the asymptotic solution for the homogeneous problem. Set $\Delta=1-x_0$. Applying the symmetry, by Lemma 1.2, to expand the solution, we can obtain the single spike solution defined on $[-1+\Delta,1-\Delta]$. Then, using the translation $x=y+\Delta$, one can get the solution of (1.1)-(1.2) with the single spike. The interval $[-1+\Delta,1-\Delta]$ is translated to $[-1,1]$, and $[0,1]$ to $[-\Delta,1-\Delta]$. The uniform convergence proved above implies that the single spike solution $u(x,\epsilon)$ of (1.1)-(1.2) satisfies

$$u(x,\epsilon)\sim -1+3\,\mathrm{sech}^2\big(\frac{1-2\Delta-x}{\epsilon\sqrt{2}}+2\ln(\sqrt{3}+\sqrt{2})\big) \tag{4.10}$$

uniformly on $[-\Delta,1-\Delta]$.

For $x\in[-1,-\Delta]$, we use the symmetry about $x=-\Delta$, i.e., $u(x)=u(-x-2\Delta)$, replacing x by $-x-2\Delta$ in (4.10) to get

$$u\sim -1+3\,\mathrm{sech}^2\big(\frac{1+x}{\epsilon\sqrt{2}}+2\ln(\sqrt{3}+\sqrt{2})\big). \tag{4.11}$$

For $x\in[1-\Delta,1]$, we use the symmetry about $x=1-\Delta$, $u(x)=u(1-\Delta-(x-1+\Delta))=u(2-2\Delta-x)$, replacing $x=2-2\Delta-x$ in (4.10), to get

$$u\sim -1+3\,\mathrm{sech}^2\big(\frac{x-1}{\epsilon\sqrt{2}}+2\ln(\sqrt{3}+\sqrt{2})\big), \tag{4.12}$$

Summing up, we have proved the main result of this paper:

Theorem 4.1 *The single spike solution $u(x,\epsilon)$ of (1.1)-(1.2) satisfies*

$$u(x,\epsilon) \sim -1 + 3\,\mathrm{sech}^2\big(\frac{1-2\Delta-x}{\epsilon\sqrt{2}} + 2\ln(\sqrt{3}+\sqrt{2})\big)$$

uniformly on $[-\Delta, 1-\Delta]$,

$$u \sim -1 + 3\,\mathrm{sech}^2\big(\frac{x-1}{\epsilon\sqrt{2}} + 2\ln(\sqrt{3}+\sqrt{2})\big),$$

uniformly on $[1-\Delta, 1]$, *and*

$$u \sim -1 + 3\,\mathrm{sech}^2\big(\frac{1+x}{\epsilon\sqrt{2}} + 2\ln(\sqrt{3}+\sqrt{2})\big)$$

uniformly on $[-1, -\Delta]$, *where* $\Delta \sim \epsilon\sqrt{2}\ln(\sqrt{3}+\sqrt{2})$.

Applying this theorem, one can easily get the asymptotic solutions for the entire interval [-1,1]. This result supports the formal asymptotic work of [5] and [8].

5 Nonhomogeneous problem

For the nonhomogeneous problem, equation (1.1) with

$$u(-1) = a, \quad u(1) = b \tag{5.1}$$

where not both a and b are zero, and $-1 < a, b < 2$, we assume $u'(x_-) = u'(x^+) = 0$ for $-1 < x_- < x^+$. We may also assume that $x_- = 0$. Set $u(x_-) = \alpha, u(x^+) = \beta$. Then we use the symmetry around the rest points to get

$$\Big(2\int_\alpha^a + \int_a^b + 2\int_b^\beta\Big)\frac{du}{\sqrt{f(u)-f(\alpha)}} = \frac{2\sqrt{2}}{\sqrt{3}\epsilon}. \tag{5.2}$$

Similar to proving Lemma 1.1, we can show that α must approach -1 as $\epsilon \to 0$, because $\int_a^b \frac{du}{\sqrt{f(u)-f(\alpha)}} = O(1)$. It then follows that $u \to -1$ on any compact subset of (-1,1). Set

$$I(\alpha) = \int_\alpha^\beta \frac{du}{\sqrt{f(u)-f(\alpha)}} - \int_a^b \frac{du}{\sqrt{f(u)-f(\alpha)}}. \tag{5.3}$$

Then, from (5.2), $I(\alpha) = \frac{\sqrt{2}}{\sqrt{3}\epsilon}$. Note that $\int_a^b \frac{du}{\sqrt{f(u)-f(\alpha)}} \to \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+\sqrt{2-a}}{\sqrt{3}-\sqrt{2-a}} - \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+\sqrt{2-b}}{\sqrt{3}-\sqrt{2-b}}$. Denote

$$\frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+\sqrt{2-a}}{\sqrt{3}-\sqrt{2-a}} - \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+\sqrt{2-b}}{\sqrt{3}-\sqrt{2-b}} = \frac{g(a,b)}{\sqrt{3}}. \tag{5.4}$$

Without repeating the arguments similar to above, we can get the following results:

$$\int_\alpha^\beta \frac{du}{\sqrt{f(u)-f(\alpha)}} = \frac{\sqrt{2}}{\sqrt{3}\epsilon} - \int_a^b \frac{du}{2\sqrt{f(u)-f(\alpha)}}, \tag{5.5}$$

or

$$\ln \frac{\sqrt{3}+\sqrt{2-a}}{\sqrt{3}-\sqrt{2-a}} \sim \frac{\sqrt{2}}{\epsilon} - \frac{g(a,b)}{2}. \tag{5.6}$$

It turns out that

$$\sqrt{2-\alpha} \sim \sqrt{3}\frac{e^{\frac{\sqrt{2}}{\epsilon}-\frac{g(a,b)}{2}}-1}{e^{\frac{\sqrt{2}}{\epsilon}-\frac{g(a,b)}{2}}+1}, \tag{5.7}$$

and hence

$$\alpha \sim -1 + 3\,\mathrm{sech}^2\Big(\frac{\sqrt{2}}{2\epsilon} - \frac{g(a,b)}{4}\Big). \tag{5.8}$$

Therefore, the asymptotic behavior of the solution is the same as that of the homogeneous problem. As to the asymptotic solution at the interior x_0, the result is similar. Assume that $x_1 < x_2$ with $u(x_1) = a$ and $u(x_2) = b$. Without loss of generality, we may also assume $a < b$. Thus,

$$\int_a^b \frac{du}{\sqrt{f(u)-f(\alpha)}} = \frac{\sqrt{2}(x_2-x_1)}{\epsilon\sqrt{3}}. \tag{5.9}$$

It then follows that

$$x_2 - x_1 \sim \frac{g(a,b)\epsilon}{\sqrt{2}}. \tag{5.10}$$

To find x_0, we consider three possibilities: (1) $a < b \le 0$, (2) $a < 0 < b$, and (3) $0 \le a < b$. For the sake of briefness we only give the analysis for the case (1) and omit analysis for the other two cases.

If $a < b \le 0$, then $x_- < x_1 < x_2 \le x_0$ and

$$\Big(\int_a^b + \int_b^0\Big)\frac{du}{\sqrt{f(u)-f(\alpha)}} = \frac{\sqrt{2}(x_2-x_1)}{\epsilon\sqrt{3}} + \frac{\sqrt{2}(x_0-x_2)}{\epsilon\sqrt{3}} \tag{5.11}$$

and

$$2\int_b^\beta \frac{du}{\sqrt{f(u)-f(\alpha)}} = \frac{\sqrt{2}(1-x_2)}{\epsilon\sqrt{3}}. \tag{5.12}$$

Since $\alpha \to -1$ implies $\beta \to 2$ and

$$\begin{aligned}\int_b^\beta \frac{du}{\sqrt{f(u)-f(\alpha)}} &\sim \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+\sqrt{2-b}}{\sqrt{3}-\sqrt{2-b}} - \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+\sqrt{2-\beta}}{\sqrt{3}-\sqrt{2-\beta}}\\ &\sim \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+\sqrt{2-b}}{\sqrt{3}-\sqrt{2-b}},\end{aligned} \tag{5.13}$$

It turns out that

$$\frac{\sqrt{2}(1-x_2)}{\epsilon\sqrt{3}} \sim \frac{2}{\sqrt{3}}\ln\frac{\sqrt{3}+\sqrt{2-b}}{\sqrt{3}-\sqrt{2-b}}, \tag{5.14}$$

and hence,

$$x_2 \sim 1 - \epsilon\sqrt{2}\ln\frac{\sqrt{3}+\sqrt{2-b}}{\sqrt{3}-\sqrt{2-b}}. \tag{5.15}$$

Also,

$$\int_b^0 \frac{du}{\sqrt{f(u)-f(\alpha)}} = \frac{\sqrt{2}(x_0-x_2)}{\epsilon\sqrt{3}} \sim \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+\sqrt{2-b}}{\sqrt{3}-\sqrt{2-b}}, \tag{5.16}$$

from which,

$$x_0 \sim 1 - \epsilon\frac{\sqrt{2}}{2}\ln\frac{\sqrt{3}+\sqrt{2-b}}{\sqrt{3}-\sqrt{2-b}}. \tag{5.17}$$

Consequently,

$$\begin{aligned} x_1 &\sim 1 - \epsilon\sqrt{2}\ln\frac{\sqrt{3}+\sqrt{2-b}}{\sqrt{3}-\sqrt{2-b}} - \frac{g(a,b)\epsilon}{\sqrt{2}} \\ &\sim 1 - \epsilon\sqrt{2}\ln\frac{\sqrt{3}+\sqrt{2-b}}{\sqrt{3}-\sqrt{2-b}} - \frac{\epsilon}{\sqrt{2}}\Big(\ln\frac{\sqrt{3}+\sqrt{2-a}}{\sqrt{3}-\sqrt{2-a}} - \ln\frac{\sqrt{3}+\sqrt{2-b}}{\sqrt{3}-\sqrt{2-b}}\Big) \\ &= 1 - \frac{\epsilon}{\sqrt{2}}\Big(\ln\frac{\sqrt{3}+\sqrt{2-a}}{\sqrt{3}-\sqrt{2-a}} + \ln\frac{\sqrt{3}+\sqrt{2-b}}{\sqrt{3}-\sqrt{2-b}}\Big). \end{aligned} \tag{5.18}$$

It follows that

$$\int_\alpha^a \frac{du}{\sqrt{f(u)-f(\alpha)}} = \frac{\sqrt{2}(x_-+1)}{\epsilon\sqrt{3}} \sim \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+\sqrt{2-\alpha}}{\sqrt{3}-\sqrt{2-\alpha}} - \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}+\sqrt{2-a}}{\sqrt{3}-\sqrt{2-a}}$$

and

$$x_- \sim -1 + \frac{\epsilon}{\sqrt{2}}\Big[\ln\frac{\sqrt{3}+\sqrt{2-\alpha}}{\sqrt{3}-\sqrt{2-\alpha}} - \ln\frac{\sqrt{3}+\sqrt{2-a}}{\sqrt{3}-\sqrt{2-a}}\Big] \sim -\epsilon\ln\frac{\sqrt{3}+\sqrt{2-b}}{\sqrt{3}-\sqrt{2-b}}.$$

From this, we see that $x_- \to 0$ and $x_+ \to 1$. Thus, the asymptotic behavior found above for the homogeneous case still holds for the nonhomogeneous problem.

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