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Solutions series for some non-harmonic motion equations *

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Abstract

We consider the class of nonlinear oscillators of the form

$$\frac{d^2u}{dt^2} + f(u) = \epsilon g(t)$$
$$u(0) = a_0, \quad u'(0) = 0,$$

where $g(t)$ is a $2T$-periodic function, f is a function only dependent on u, and ϵ is a small parameter. We are interested in the periodic solutions with minimal period $2T$, when the restoring term f is such that $f(u) = \omega^2 u + u^2$ and g is a trigonometric polynomial with period $2T = \frac{\pi}{\omega}$. By using method based on expanding the solution as a sine power series, we prove the existence of periodic solutions for this perturbed equation.

1 Introduction

Consider the second order differential equation

$$\begin{gathered} x'' + g(t, x, x', \epsilon) = 0 \\ u(0) = a_0, \quad u'(0) = 0\,, \end{gathered} \tag{1.1}$$

where $\epsilon > 0$ is a small parameter, g is a $2T$-periodic function in t with $g(t, x, 0, 0) = g(x)$ is independent of t.

The existence of solutions for this equation in the case where g is independent on x' and continuously differentiable has been studied by many authors. In the latter case, this proved the existence of solutions to

$$x'' + g(t, x, \epsilon) = 0.$$

For a detailed review, we refer the reader to the book by Chow and Hale [1]. Furthermore, the example given by Hartman proved non existence cases of (1). So, we cannot expect to generalize their results. On the other hand, Loud,

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Willem and others considered the case where T is independent on ϵ. Concerning the case $T = T(\epsilon)$, Fonda and Zanolin studied the system

$$u'' + f(u) = \epsilon e(t, u, \epsilon)$$

where f is semilinear with $xf(x) > 0$. Here e is a T_ϵ-periodic function such that $|e(t, x, \epsilon)| < K$, $\lim_{\epsilon\to 0} T_\epsilon = +\infty$ and $\lim_{\epsilon\to 0} \epsilon T_\epsilon = 0$. They proved the existence of periodic solutions.

In the present paper we are interested in the class of non linear oscillators of the form

$$\begin{gathered} \frac{d^2u}{dt^2} + f(u) = \epsilon g(t) \\ u(0) = a_0, \quad u'(0) = 0\,, \end{gathered} \tag{1.2}$$

where the restoring term $f \equiv f(u)$ depends on u and the function $g \equiv g(t)$ is $2T$-periodic. We look for periodic solutions with minimal period $2T$, when f and g satisfy suitable conditions.

When f is a quadratic polynomial function, we are able to solve (1.1) , in some situations and without ϵ to be small, or any restricted condition on the period T. We will prove for that the existence of a trigonometric expansions of (periodic) solutions in Sinus powers.

Note that, some times under certain conditions, (1) cannot have a periodic solution, as described in the example below. However, under other suitable conditions of controllability of the period, Equation (1) has a periodic solution.

A non existence result

According to Hartman [5, p.39], Equation (1) in general does not have a non constant periodic solution, even if $xg(t, x, x') > 0$. The following example, given by Moser, proves the non existence of a non constant periodic solution of

$$x'' + \phi(t, x, x') = 0.$$

Let

$$\phi(t, x, y) = x + x^3 + \epsilon f(t, x, y), \qquad \epsilon > 0$$

satisfying the following conditions: $\phi \in C^1(R^3)$; $f(t + 1, x, y) = f(t, x, y)$ with $f(0, 0, 0) = 0$ and $f(t, x, y) = 0$ if $xy = 0$;

$$\frac{\phi}{x} \to \infty \quad \text{as } x \to \infty$$

uniformly in $(t, y) \in R^2$;

$$\frac{\delta f}{\delta y} > 0 \quad \text{if } xy > 0, \quad \text{and} \quad \frac{\delta f}{\delta y} = 0 \quad \text{otherwise.}$$

and x, y satisfying $|x| < \epsilon$, $|y| < \epsilon$. In fact, we have $xf(t, x, y)$ and $yf(t, x, y)) > 0$ if $xy > 0$, $|x| < \epsilon$, y arbitrary and $\phi = 0$ otherwise.

We remark that $\frac{\delta f}{\delta y}$ is small. The function $V = 2x^2 + x^4 + 2x'^2$ satisfies $V' = -4\epsilon x' f(t, x, x')$, so that $V' < 0$ if $xx' > 0$, $|x| < \epsilon$ and $V' = 0$ otherwise. Thus x cannot be periodic unless $V' = 0$.

A controllably periodic perturbation

In this part, we give brief explanation of a method due to Farkas [4] inspired of the one of Poincaré. The determination of controllably periodic perturbed solution. This method appears to be efficient particularly for the perturbations of various autonomous systems. Since we know it for example a good application for perturbed Van der Pol equations type as well as Duffing equations type.

The perturbation is supposed to be 'controllably periodic', i.e., it is periodic with a period which can be chosen appropriately. Let $u_0(t)$ be the periodic solution of the (unperturbed) equation. Under very mild conditions it is proved that to each small enough amplitude of the perturbation there belongs a one parameter family of periods such that the perturbed system has a unique periodic solution with this period.

It is assumed the existence of the fundamental matrix solution of the first variational 2-dimensional system of $\dot{x} = h(x)$ and the unique periodic solution $p(t)$ corresponding to $u_0(t)$. For more details concerning applications of this method we may refer to [4].

2 Periodic perturbations of Duffing non linear equations

In this section, we will apply the methods of trigonometric series in sinus powers to resolve in an explicit way non linear oscillators of Duffing type. This approach already was used previously in an effective way, see [6] and [2]. One can also apply it for periodic perturbations of various differential equations. Our results generalize and improve those of [6] and [2]. Their also contribute in an interesting way to the general problem of the periodic perturbations roughly exposed in the introduction.

Case of finite trigonometric polynomial

Firstly, let us consider the restoring function

$$f(u) = \omega^2 u + u^2,$$

where ω is a constant and $g(t)$ a finite trigonometric polynomial with period $2T = \frac{\pi}{\omega}$, which has a finite Fourier series expansion. So, it can be written

$$g(t) = b_0 + b_2 \sin^2(\frac{\pi t}{2T}) + \cdots + b_{2p_0} \sin^{2p_0}(\frac{\pi t}{2T}) = \sum_{n \le p_0} b_{2n} \sin^{2n}(\frac{\pi t}{2T}), \quad p_0 \in \mathbb{N}.$$

Consider the following system

$$\begin{aligned} &\frac{d^2u}{dt^2} + \omega^2 u + u^2 = \epsilon g(t) \\ &u(0) = a_0, \quad u'(0) = 0\,. \end{aligned} \tag{2.1}$$

Theorem 2.1 *The T-periodic solutions of* (1.2) *may be expressed in the form*

$$u(t) = c_0 + c_2 \sin^2(\frac{\pi t}{2T}) + c_4 \sin^4(\frac{\pi t}{2T}) + c_6 \sin^6(\frac{\pi t}{2T}) + \cdots = \sum_{n\in IN} c_{2n} \sin^{2n}(\frac{\pi t}{2T})$$

The coefficients c_{2n} satisfy $|a_0| < C$, $a_0 = c_0$, $2\omega^2 c_2 = -\omega^2 c_0 - c_0^2 + \epsilon b_0$, ($\omega = \frac{\pi}{2T}$) and the recursion formula

$$(2n+1)(2n+2)c_{2n+2} = \begin{cases} (4n^2-1)c_{2n} - \frac{1}{\omega^2}\sum_{r=0}^{n} c_{2r}c_{2n-2r} + \frac{\epsilon}{\omega^2} b_{2n}, & \text{for } n \le p_0, \\ (4n^2-1)c_{2n} - \frac{1}{\omega^2}\sum_{r=0}^{n} c_{2r}c_{2n-2r}, & \text{for } n > p_0 . \end{cases} \tag{2.2}$$

Proof Suppose the solution u of equation (1.2) can be written in the form

$$u(t) = \sum_{n\in IN} c_{2n} \sin^{2n}(\omega t).$$

We shall apply the previous method in [2] to: (i) obtain recursion formula of the coefficients c_{2n}, and (ii) prove the convergence of the series expansion of the solution.

This method finds a solution to (1.2)in the form

$$u(t) = c_0 + c_1 \sin \omega t + c_2 \sin^2 \omega t + c_3 \sin^3 \omega t + \dots \tag{2.3}$$

where c_i, $i = 0, 1, 2, \dots$, are coefficients to be determined by the substitution of (2.2) in (1.2). In fact, we get $\omega = \frac{\pi}{2T}$ where T is the period of the solution. So, a trivial computation gives

$$2\omega^2 c_2 = -\omega^2 c_0 - c_0^2 + \epsilon b_0.$$

We get also

$$12 c_4 = 3 c_2 - \frac{2}{\omega^2} c_0 c_2 + \frac{\epsilon}{\omega^2} b_2$$

By identification for $n \ge 1$, we obtain the recursion formula of these coefficients

$$(2n+1)(2n+2)c_{2n+2} = (4n^2-1)c_{2n} - \frac{1}{\omega^2}\sum_{r=0}^{n} c_{2r}c_{2n-2r} + \frac{\epsilon}{\omega^2} b_{2n}, \quad \text{for } n \le p_0. \tag{2.4}$$

and

$$(2n+1)(2n+2)c_{2n+2} = (4n^2-1)c_{2n} - \frac{1}{\omega^2}\sum_{r=0}^{n} c_{2r}c_{2n-2r}, \quad \text{for } n > p_0. \tag{2.5}$$

Equation (1.2) implies $c_1 = 0$. Relations (2.3) yields $c_3 = 0$, $c_5 = 0$, ... However, the even order coefficients c_{2n} do not vanish. The solution to (1.2) can now be written as

$$u(t) = c_0 - [c_0 - \frac{1}{\omega^2} c_0^2 + \frac{\epsilon}{\omega^2} b_0] \sin^2 \omega t + [\frac{1}{4} c_2 - \frac{1}{6\omega^2} c_0 c_2 + \frac{\epsilon}{12\omega^2} b_2] \sin^2 \omega t + \dots \tag{2.6}$$

Relations of the coefficients and further induction show that c_{2i}, $i = 1, 2, \dots$ all vanish for $a_0 = 0$ and $b_{2i} = 0$. So, the trivial solution $u \equiv 0$ is included in (2.2) as a special case. The following lemma is strictly analogous of [2, Lemma 2], corresponding to the case $\epsilon = 0$. It implies that

$$\sum_{n\geq 1} |c_{2n}| < +\infty,$$

which implies the convergence of the expansion. □

Lemma 2.2 *There exist a positive constants k and α verifying $1 < \alpha < \frac{3}{2}$ such that the coefficients c_{2n} of the series expansion* (2.2) *solution of the differential equation* (1.2) *satisfy the inequality*

$$|c_{2n}| < \frac{k}{(2n)^{\alpha}}. \tag{2.7}$$

Remark Following [6], it is interesting to write the power series solution for the system (1.2) in the form

$$u(t) = v(\sin \omega t).$$

Let $\tilde{g}$ defined by $g(t) = \tilde{g}(\sin \omega t)$. Then v is a solution of the differential equation

$$\begin{gathered}(1 - x^2)\frac{d^2 v}{dx^2} - x\frac{dv}{dx} + v + \frac{1}{\omega^2}v^2 = \frac{\epsilon}{\omega^2}\tilde{g}(x) \\ v(0) = a_0, \quad \frac{dv}{dx}(0) = 0\end{gathered} \tag{2.8}$$

Recall that the latter method permits to compare approximate solutions of the non-harmonic motion of the oscillator.

A more general case

Now, consider a more general situation where the function $g(t)$ in (1.1) may be written as an infinite expansion in Sinus power

$$g(t) = b_0 + b_2 \sin^2(\frac{\pi t}{2T}) + \dots + b_{2n} \sin^{2n}(\frac{\pi t}{2T}) + \dots = \sum_{n\in IN} b_{2n} \sin^{2n}(\frac{\pi t}{2T}).$$

It is the case in particular, when the function $g(t)$ has a finite Fourier series expansion. When g has an (infinite) Fourier series expansion, $g(t)$ may be expressed as an infinite expansion in Sinus power. But we have to prove its convergence.

Now, we are interested in solving

$$\begin{gathered}\frac{d^2 u}{dt^2} + \omega^2 u + u^2 = \epsilon \sum_{n\in IN} b_{2n} \sin^{2n}(\frac{\pi t}{2T}) \\ u(0) = a_0, \quad u'(0) = 0\,.\end{gathered} \tag{2.9}$$

We prove the following theorem which extends Theorem 2.1.

Theorem 2.3 *Suppose that the coefficients of the expansions of the function $g(t)$ satisfies*

$$|b_{2n}| < \frac{1}{(2n)^\beta}, \text{ with } \beta \geq 1,$$

then the solutions of (2.8) *may be expressed in the form*

$$u(t) = a_0 + c_2 \sin^2(\frac{\pi t}{2T}) + c_4 \sin^4(\frac{\pi t}{2T}) + c_6 \sin^6(\frac{\pi t}{2T}) + \cdots = \sum_{n \in IN} c_{2n} \sin^{2n}(\frac{\pi t}{2T}).$$

The coefficients c_{2n} satisfy the conditions $|a_0| < C$, $a_0 = c_0$, $2\omega^2 c_2 = -\omega^2 c_0 - c_0^2 + \epsilon b_0$, ($\omega = \frac{\pi}{2T}$) and the recursion formula, for $n > 0$:

$$(2n+1)(2n+2)c_{2n+2} = (4n^2 - 1)c_{2n} - \frac{1}{\omega^2} \sum_{r=0}^{n} c_{2r} c_{2n-2r} + \frac{\epsilon}{\omega^2} b_{2n}. \tag{2.10}$$

Proof The proof starts in the same manner as that of Theorem 2.1. In order to establish the recursion relation between the coefficients we may proceed as previously. The difference is that one obtains a less good estimation of the c_{2n}. The crucial point is to state an analogous one of Lemma 2.4, ensuring thus the convergence of the series solution. It may be deduced from

$$\sum_{n \geq 1} |c_{2n}| < +\infty$$

For that, we prove the following lemma.

Lemma 2.4 *For any positive number α such that $1 < \alpha < 3/2$, there exists a positive constant k satisfying*

$$k < \frac{1}{\omega^2} \frac{3}{4} (\frac{3}{2} - \alpha) 4^{1-\alpha}$$

such that the coefficients c_n of the series expansion solution of the differential equation (2.8) *satisfies the inequality*

$$|c_n| < \frac{k}{n^\alpha}. \tag{2.11}$$

Proof We first notice that Lemma 2.4 gives an optimal result, because our method does not work for $\alpha = \frac{3}{2}$. The coefficients c_n of the power series solution, satisfy the recursion formula (2.9). We shall prove there exist two positive constants $k > 0$, and $\alpha > 1$, such that the following inequality holds

$$|c_n| < \frac{k}{n^\alpha}$$

for any integer $n \geq 1$. Suppose for any $n \leq p$, we get $|c_n| < \frac{k}{n^\alpha}$. In particular, it implies that

$$\sum_{0<r<p} c_r c_{p-r} < \sum_{0<r<p} \frac{k^2}{r^\alpha (r-p)^\alpha} \leq \frac{k^2}{(p-1)^{\alpha-1}}.$$

Equality (2.9) gives

$$c_{p+2}=\frac{p^2-1}{(p+1)(p+2)}c_p-\frac{1}{\omega^2(p+1)(p+2)}\sum_{r=0}^{p-1}c_rc_{p-r}+\frac{\epsilon}{\omega^2(p+1)(p+2)}b_p.$$

Thus, if we prove the inequality

$$\frac{p^2-1}{(p+1)(p+2)}\frac{k}{p^\alpha}+\frac{1}{\omega^2(p+1)(p+2)}\frac{k^2}{(p-1)^{\alpha-1}}+\frac{\epsilon|b_p|}{\omega^2(p+1)(p+2)}\le\frac{k}{(p+2)^\alpha},$$

we can conclude

$$|c_{p+2}|<\frac{k}{(p+2)^\alpha}. \tag{2.12}$$

Since the coefficients b_p satisfy the hypothesis $|b_{2n}|<\frac{1}{(2n)^\beta}$, with $\beta\ge 1$, it is not difficult to exhibit an integer p_0 depending on β such that the following inequality holds for $p\ge p_0$:

$$\frac{p^2-1}{(p+1)(p+2)}\frac{k}{p^\alpha}+\frac{1}{\omega^2(p+1)(p+2)}\frac{k^2}{(p-1)^{\alpha-1}}\le\frac{k}{(p+2)^\alpha} \tag{2.13}$$

This inequality is equivalent to

$$k<\frac{\beta}{\omega^2}pf(p)g(p) \tag{2.14}$$

where

$$f(p)=\frac{p+1}{p}(\frac{p-1}{p+2})^{\alpha-1}$$
$$g(p)=1-\frac{(p^2-\frac{\beta}{\omega^2}c_0)(p+2)^{\alpha-1}}{(p+1)p^\alpha}$$

Using MAPLE, we prove that $f(p)$ is an increasing positive function in p. Moreover, for any $p\ge 1$, $f(p)$ is bounded below as

$$f(p)\ge(\frac{3}{2})4^{1-\alpha}.$$

The function $g(p)$ is such that

$$pg(p)=p-\frac{(p^2-\frac{\beta}{\omega^2}c_0)(\frac{p+2}{p})^{\alpha-1}}{(p+1)}$$

is a strictly decreasing and bounded function. More exactly, we may calculate the lower bound

$$g(p)>\frac{(3-2\alpha)}{p}.$$

Thus, if $(3-2\alpha)=\epsilon>0$, it suffices to chose

$$k\le(\frac{3}{2})4^{1-\alpha}(3-2\alpha)$$

for inequality (17) to hold.

Remark for the case $\epsilon = 0$: Note that the choice of k depends on α value. For $\alpha = 3/2$, using MAPLE we can prove that

$$pg(p) = p - \frac{(p^2 - \frac{3}{2\omega^2}c_0)(\frac{p+2}{p})^{1/2}}{(p+1)}$$

is positive and strictly decreasing to 0. While $p^2 g(p)$ is a bounded function. Moreover, it appears that $pf(p)g(p)$ is a decreasing function which tends to 0 as p tends to infinity. Thus, our method fails since it does not permit to determine a non negative constant k.

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