Fifth Mississippi State Conference on Differential Equations and Computational Simulations,
Electronic Journal of Differential Equations, Conference 10, 2003, pp 109–113.
http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Uniform stability of multidimensional travelling waves for the nonlocal Allen-Cahn equation *

Fengxin Chen

Abstract

In this paper, we study the uniform stability of mutidimensional planar travelling waves for the nonlocal Allen-Cahn equation.

1 Introduction

The main concern of this paper is the stability of planar travelling wave solutions of the multidimensional nonlocal Allen-Cahn equation

$$u_t = J * u - u + f(u). \tag{1.1}$$

Here $J \in C^1(\mathbb{R}^n)$ is a nonnegative function with $\int_{\mathbb{R}^n} J(y)dy = 1$ and $J(0) \neq 0$; $J * u = \int_{\mathbb{R}^n} J(x-y)u(y)dy$ is the convolution of J and u; f is a smooth bistable function with three zeros, ± 1 and $a \in (-1,1)$ satisfying $f'(\pm 1) < 0$ and $f'(a) > 0$. A typical example is $f(u) = (u-a)(1-u^2)$ for some $a \in (-1,1)$.

Travelling wave solutions of the nonlocal Allen-Cahn equation in one spatial dimension have been extensively studied. It is well known that there exists a travelling wave solution of the form $u(x,t) = \phi(x - c_0 t)$ satisfying

$$c_0\phi' + J * \phi - \phi + f(\phi) = 0, \quad \phi(\pm\infty) = \pm 1, \tag{1.2}$$

where ϕ is a monotone function; If ϕ is continuous,

$$c_0 = \int_{-1}^{1} f(u)du \Big/ \int_{-\infty}^{\infty} (\phi'(z))^2 dz;$$

if $c_0 \neq 0$, the travelling wave solution is smooth and unique modulo a spatial shift; and it is uniformly and asymptotically stable (see [4], [5] and [6]). If the unique speed $c_0 = 0$, the wave may be discontinuous but monotone waves are still unique up to a spatial shift.

A planar travelling wave solutions of (1.1) is a solution of the form $\phi(\xi) = \phi(k \cdot x - ct)$ and $\phi(\pm\infty) = \pm 1$, where $k \in S^{n-1}$ is a unit vector. Without

---

* *Mathematics Subject Classifications:* 35K55, 35Q99.
Key words: nonlocal phase transition, travelling waves, continuation.

Published February 28, 2003.

loss of generality, we assume $k = (1, 0, \cdots, 0)$. Then $\phi(k \cdot x - ct) = \phi(x_1 - ct)$ satisfies (1.2) with J being replaced by $J_1(\cdot) = \int_{\mathbb{R}^{n-1}} J(\cdot, x')dx'$. Notice that $\int_{\mathbb{R}} J_1(x)dx = 1$. Therefore the existence of such planar travelling wave solutions can be derived from the one dimensional case. Therefore, in this paper, we assume that $\phi(x_1 - c_0t)$ is a planar travelling satisfying $\phi'(x) > 0$ for all $x \in \mathbb{R}$; and $\phi(\pm\infty) = \lim_{x\to\infty} \phi(\pm x) = \pm 1$, with wave speed c_0. Our main concern is the multidimensional stability for the planar travelling wave $\phi(x_1 - c_0t)$. We have the following theorem.

Theorem 1.1 (Uniform Stability) *Let $u(x, t) = \phi(x_1 - c_0t)$ be a travelling wave solution satisfying $\phi'(x) > 0$ for all $x \in \mathbb{R}$ and $\phi(\pm\infty) = \pm 1$. Then $\phi(x_1 - c_0t)$ is uniformly stable, that is, for any $\epsilon > 0$ there is $\delta(\epsilon) > 0$ such that for any $u_0 \in L^\infty(\mathbb{R}^n)$ with $\|u_0(\cdot) - \phi(\cdot)\|_{L^\infty(\mathbb{R}^n)} < \delta(\epsilon)$, one has*

$$\|u(\cdot, t; u_0) - \phi(\cdot - c_0t)\|_{L^\infty(\mathbb{R}^n)} < \epsilon$$

for all $t > 0$, where $u(\cdot, t; u_0)$ is the solution of (1.1) with initial data $u(\cdot, 0; u_0) = u_0$.

The global exponential stability in one space dimension is due to the spectral gap [2]. In the multidimensional case, however, the gap disappears due to the effects of the transverse diffusion along the planar wave front and there may exist continuous spectrum all the way up to zero. The global asymptotic stability for the multidimensional case is studied in [2] for special kernel J. For general case the asymptotic stability is still open.

2 Proof of the main Theorem

In this section, we will use super-and sub- solution method to prove the theorem. First we have the following comparison principle.

Lemma 2.1 (Comparison Principle) *Suppose R_1 is an open set in $\mathbb{R}^n$ and $R_2 = \mathbb{R}^n \setminus R_1$ is the complement of R_1. Suppose $u \in C^1([\tau, t_0], L^\infty(\mathbb{R}^n))$ and $u(x, t) \geq 0$ for almost all $x \in R_2$ and $t \in [\tau, t_0]$. Assume $u(x, t)$ satisfies*

$$u_t - K_0(x, t)u - (J * u)(x, t) \geq 0 \tag{2.1}$$

for almost all $(x, t) \in R_1 \times (\tau, t_0]$, where $K_0(x, t) \in L^\infty(\mathbb{R}^n \times [\tau, t_0])$. If $u(x, \tau) \geq 0$ for almost all $x \in \mathbb{R}^n$, then $u(x, t) \geq 0$ for almost all $x \in \mathbb{R}^n$, and $t \in [\tau, t_0]$. If, furthermore, $u \in C_{unif}(\mathbb{R}^n \times [\tau, t_0])$ and $u(x, \tau) \not\equiv 0$, then $u(x, t) > 0$ for $x \in R_1$, and $t \in (\tau, t_0]$.

Proof The proof is similar to that of one dimensional case(see[5] and [6]). We may assume $\tau = 0$. By assumption, $\operatorname{ess\,inf}_{x\in\mathbb{R}^n} u(x, t)$ is continuous. If the conclusion of the lemma is not true, then there exist constants $\epsilon > 0, T > 0$ such that $u(x,t) > -\epsilon e^{2Kt}$ for almost all $x \in \mathbb{R}^n, 0 < t < T$ and $\operatorname{ess\,inf}_{x\in\mathbb{R}} u(x,T) = -\epsilon e^{2KT}$, where

$$K = \|K_0\|_{L^\infty(\mathbb{R}^n \times [\tau,t_0])} + 1. \tag{2.2}$$

Let $z(x)$ be a smooth function such that $\min_{x\in\mathbb{R}^n} z(x) = z(0) = 1$, $\sup_{x\in\mathbb{R}^n} z(x) = z(\pm\infty) = 3$, and $|z_{x_i}(x)| \le 1$ for $i = 1, \cdots, n$. Define $w_\sigma(x,t) = -\epsilon\left(\frac{3}{4} + \sigma z(x)\right) e^{2Kt}$, for $\sigma \in [0,1]$. Since $w_1(x,t) < u(x,t)$ for almost all $x \in \mathbb{R}^n$, and $0 \le t \le T$, and $w_0(x,t) = -\frac{3}{4}\epsilon e^{2Kt}$, there is a minimum $\sigma^* \in \left(\frac{1}{8}, \frac{1}{4}\right]$ such that $w_{\sigma^*}(x,t) \le u(x,t)$ for almost all $x \in \mathbb{R}^n$, and $t \in [0,T]$. Since $w_{\sigma^*}(\pm\infty, t) \le -\frac{9}{8}\epsilon e^{2Kt} < u(x,t)$ and $u(x,t) > w_{\sigma^*}(x,t)$ for almost all $x \in R_2$, and $t \in (0,T]$, there exist $(x_n,t_n) \in R_1 \times (0,T]$ and $(\bar{x},\bar{t})$ such that $\lim_{n\to\infty}(x_n,t_n) = (\bar{x},\bar{t})$, $\lim_{n\to\infty}\{u(x_n,t_n) - w_{\sigma^*}(x_n,t_n)\} = 0$, the infimum of $u(x,t) - w_{\sigma^*}(x,t)$ on $\mathbb{R}\times[0,T]$, and $\lim_{n\to\infty}(u - w_{\sigma^*})_t(x_n,t_n) \le 0$. Therefore,

$$\begin{aligned}
0 &\ge \lim_{n\to\infty} (u - w_{\sigma^*})_t(x_n,t_n) \\
&\ge \lim_{n\to\infty} \{(J*u)(x_n,t_n) + K_0(x_n,t_n)u(x_n,t_n)\} + 2K\epsilon e^{2K\bar{t}}\big(\sigma^* z(\bar{x}) + \frac{3}{4}\big) \\
&\ge \lim_{n\to\infty} \{K_0(x_n,t_n)(u - w_{\sigma^*})(x_n,t_n) + K_0(x_n,t_n)w_{\sigma^*}(x_n,t_n) \\
&\quad + J*(u - w_{\sigma^*})(x_n,t_n) + J*w_{\sigma^*}(x_n,t_n)\} + 2K\epsilon e^{2K\bar{t}}\big(\sigma^* z(\bar{x}) + \frac{3}{4}\big) \\
&\ge \epsilon e^{2K\bar{t}}\big[\frac{7}{4}K - \frac{3}{2}\|K_0\| - \frac{3}{2}\big] > 0.
\end{aligned}$$

by the choice of K in (2.2), which is a contradiction. Therefore $u(x,t) \ge 0$ for almost all $x \in \mathbb{R}^n$ and $t \in [\tau, t_0]$.

Let $v(x,t) = e^{Kt}u(x,t)$. Then we have $v_t(x,t) \ge J * v(x,t)$ for $x \in R_1$ and $t \in (\tau, t_0]$ since $u(x,t) \ge 0$. Therefore, $v(x,t) \ge tJ*v(x,0)$. After N^{th} iteration, we have $v(x,t) \ge \frac{t^N}{N!} J*\cdots*J*u(x,0)$. If $u \in C_{unif}(\mathbb{R}^n \times [\tau,t_0])$ and $u(x,0) \not\equiv 0$, we can choose N large enough such that $J*\cdots*J*u(x,0) > 0$. Therefore, we have $v(x,t) > 0$. This completes the proof. □

Lemma 2.2 *Suppose $u_1(x,t)$ and $u_2(x,t)$ are super-solution and sub-solution of (1.1), respectively, with $u_1(x,\tau) \ge u_2(x,\tau)$, for all $x \in \mathbb{R}^n$ and for some $\tau \in \mathbb{R}^n$. Then $u_1(x,t) \ge u_2(x,t)$ for all $x \in \mathbb{R}^n$ and $t > \tau$. Moreover, if $u_1(x,\tau) \not\equiv u_2(x,\tau)$, then $u_1(x,t) > u_2(x,t)$ for all $x \in \mathbb{R}^n$ and $t > \tau$.*

Proof Let $v(x,t) = u_1(x,t) - u_2(x,t)$. Then $v(x,\tau) \ge 0$ for all $x \in \mathbb{R}^n$ and $v(x,t)$ satisfies

$$v_t - K_0(x,t)v - (J*v)(x,t) \ge 0 \tag{2.3}$$

for all $x \in \mathbb{R}^n$ and $t \ge \tau$, where

$$K_0(x,t) = \int_0^1 f_u(u_2 + \theta(u_1 - u_2))d\theta - 1. \tag{2.4}$$

The result follows from Lemma 2.2. □

We use the super- and sub-solution method employed in [6] to prove the stability in one dimensional case. To that end, we first develop the following lemma.

Lemma 2.3 *Let $\phi(x-c_0t)$ be as in Theorem 1.1 and $\beta_1 = -\frac{1}{2}\max\{f(-1), f(1)\}$. There exist $\delta_1 > 0$ and $\sigma_1 > 0$ such that, for any $\delta \in (0, \delta_1)$, $\xi_0 \in \mathbb{R}$ and $w^{\pm}(x,t)$ are super- and sub-solutions of (1.1) on $(0,\infty)$, respectively, where*

$$w^{\pm}(x,t) = \phi(x_1 + \xi_0 \pm \sigma_1\delta(1 - e^{-\beta_1 t}) - c_0t) \pm \delta e^{-\beta_1 t} \tag{2.5}$$

for $x \in \mathbb{R}^n, t \in (0,\infty)$.

Proof We prove only that $w^+(x,t)$ is a super-solution. The other can be proved similarly.

$$\begin{aligned} Lw^+ :=& w_t^+ - (J * w^+ - w^+) - f(w^+) \\ =& [\sigma_1\beta_1\phi'(\eta_+(x,t)) - \beta_1 - K_0(x,t)]\delta e^{-\beta_1 t} \end{aligned} \tag{2.6}$$

where

$$K_0(x,t) = \int_0^1 f_u(\phi(\eta_+(x,t)) + \theta\delta e^{-\beta_1 t})d\theta,$$

and $\eta_+(x,t) = x_1 + \xi_0 + \sigma_1\delta(1 - e^{-\beta_1 t}) - c_0t$. Since $\lim_{x\to\infty}\phi(\pm x) = \pm 1$, $K_0(x,t) \to f_u(\pm 1)$ uniformly in $t \in [0,\infty)$ as $\eta_+(x,t) \to \pm\infty$ and $\delta \to 0$. So, there exist $\bar{m} > 0$ and $\delta_1 > 0$ such that for $x \in \mathbb{R}^n$ with $|\eta_+(x,t)| \ge \bar{m}$ and $0 < \delta < \delta_1$,

$$K_0(x,t) < -\beta_1, \tag{2.7}$$

that is, $-\beta_1 - K_0(x,t) \ge 0$ for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$ with $|\eta_+(x,t)| \ge \bar{m}$. Therefore, $Lw^+ \ge 0$ for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$ with $|\eta_+(x,t)| \ge \bar{m}$.

For $|\eta_+(x,t))| \le \bar{m}$, choose

$$\sigma_1 = \frac{\beta_1 + K}{\beta_1\alpha(\bar{m})}, \tag{2.8}$$

where $K = \sup\{|f_u(u)| : u \in [-2,+2]\}$ and $\alpha(\bar{m}) = \min\{\phi(x) : x \in [-\bar{m},\bar{m}]\}$. We know that $\alpha(\bar{m}) > 0$ since $\phi(x) > 0$ for all $x \in \mathbb{R}$. Then, for $t \ge 0$, $x \in \mathbb{R}^n$ with $|\eta_+(x,t)| \le \bar{m}$ and any $0 < \delta \le \delta_1$, we have $Lw^+ \ge 0$.

Therefore $Lw^+ \ge 0$ for all $x \in \mathbb{R}^n$ and $t \in (0,\infty)$. That is, with the above choices of δ_1 and σ_1, the function $w^+(x,t)$ is a super-solution for (1.1). □

Proof of Theorem 1.1 For $\epsilon > 0$ given, since ϕ is uniformly continuous, there exists $k_0 > 0$ such that, for all $|k| \le k_0$,

$$|\phi(x_1 + k) - \phi(x_1)| < \frac{\epsilon}{2} \tag{2.9}$$

for all $x_1 \in \mathbb{R}$. Let β_1, σ_1 and δ_1 be as in Lemma 2.3. Choose $\delta > 0$ such that $\delta < \min\left\{\frac{\epsilon}{2}, \frac{k_0}{\sigma_1}, \delta_1\right\}$. Then by Lemma 2.2, the condition

$$\phi(x_1) - \delta < u_0(x) < \phi(x_1) + \delta$$

implies

$$\begin{aligned} &\phi(x_1 - \sigma_1\delta(1 - e^{-\beta_1 t}) - c_0 t) - \delta e^{-\beta_1 t} \\ &\le u(x,t) \\ &\le \phi(x_1 + \sigma_1\delta(1 - e^{-\beta_1 t}) - c_0 t) + \delta e^{-\beta_1 t}. \end{aligned} \tag{2.10}$$

By the choice of δ and (2.9) - (2.10), we have

$$|u(x,t) - \phi(x_0 - c_0 t)| < \epsilon$$

for all $x \in \mathbb{R}^n$ and $t > 0$. That completes the proof.

References

[1] P. W. Bates and F. Chen, Periodic travelling wave solutions of an integrodifferential model for phase transition, *Electron. J. Differential Equations* **26** (1999) 1-19.

[2] P. W. Bates and F. Chen, Spectral analysis and multidimensional stability of travelling waves for nonlocal Allen-Cahn equation, *J. Math. Anal. Appl.* **273** (2002) 45-57.

[3] P. W. Bates, F. Chen, and J. Wang, Global existence and uniqueness of solutions to a nonlocal phase-field syatem, *US-Chinese Conference on Differential Equations and Applications, P. W. Bates, S-N. Chow, K. Lu and X. Pan, Eds,International Press, Cambridge MA,*, 1997 pp 14-21.

[4] P. W. Bates, P. C. Fife, X. Ren, and X. Wang, Traveling waves in a nonlocal model of phase transitions, *Arch. Rat. Mech. Anal.* **138** (1997), 105-136.

[5] F. Chen, Almost periodic travelling waves of nonlocal evolution equations, *Nonlinear Anal.* **50** (2002) 807-838.

[6] X. Chen, Existence, uniqueness and asymptotic stability of travelling waves in nonlocal evolution equations, *Adv. Diff. Eqs.* **2** (1997), 125-160.

[7] C. D. Levermore and J. X. Xin, Multidimensional stability of travelling waves in a bistable reaction-diffusion equation. II. *Comm. Partial Differential Equations* **17** (1992), 1901–1924.

[8] J. X. Xin, Multidimensional stability of travelling waves in a bistable reaction-diffusion equation. I. *Comm. Partial Differential Equations* **17** (1992) 1889–1899.

Fengxin Chen
Division of Mathematics and Statistics
University of Texas at San Antonio
San Antonio, TX 78249, USA e-mail: feng@sphere.math.utsa.edu