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Monotone method for nonlinear nonlocal hyperbolic problems *

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Abstract

We present recent results concerning the application of the monotone method for studying existence and uniqueness of solutions to general first-order nonlinear nonlocal hyperbolic problems. The limitations of comparison principles for such nonlocal problems are discussed. To overcome these limitations, we introduce new definitions for upper and lower solutions.

1 Introduction

This paper is concerned with the first-order hyperbolic initial-boundary value problem

$$\begin{gathered} u_t + (g(x,t)u)_x = F(x,t,u,\phi(u)(x,t)) \quad \text{in } D_T, \\ g(a,t)u(a,t) = \int_a^b \beta(y,t)u(y,t)dy \quad \text{on } (0,T), \\ u(x,0) = u_0(x) \quad \text{in } [a,b]\,. \end{gathered} \tag{1.1}$$

Here $D_T = (a,b) \times (0,T)$ for some $T > 0$, $0 \le a < b \le \infty$, and ϕ is a function of u. Problem (1.1) often arises in applications. For example, the well-known size-structured model, where $F(x,t,u,\phi) = -m\,(x,t,\phi)\,u$ and $\phi(u)(t) = \int_a^b d(y)u(y,t)dy$, fits under the class of problems given in (1.1). For the size-structured model g, m and β denote the individuals growth, mortality and reproduction rates, respectively, and ϕ denotes a population weight.

There are three common methods used in the literature to prove the existence and uniqueness of solutions to certain cases of (1.1). One is the semigroup of operators theoretical approach. This approach is very elegant but has been applied only to special cases where the parameters are time-independent, i.e., $g = g(x)$ and $\beta = \beta(x)$. The idea is to write the PDE as an abstract evolution equation of the form $du/dt = \mathcal{A}u + \mathcal{F}(u)$, $u(0) = u_0$ and show that $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of operators, and hence establish the existence and uniqueness of solutions under some regularity conditions on the

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mapping $\mathcal{F}(u)$ (see, e.g.,[1, 9, 11]). The second approach is based on the classical characteristics method [12, 14]. This approach is, in general, feasible and relatively easy to apply provided the function $F(x,t,u,\phi)$ is linear in u. In such a case one can obtain an implicit representation of the solution and use this representation to transform the PDE into an equivalent system of nonlinear integral equations. Then the contraction mapping principle is applied to establish the result. The third approach to study the well-posedness of solutions is via the finite difference approximation technique used for the classical conservation laws (see, e.g., [13, 15]). The crucial step in this technique is to show that the developed finite difference approximation has a bounded total variation. Then, through the compact imbedding of the space of functions of bounded variation in $L^1(a,b)$ one can extract a convergent subsequence and show that the limit is indeed a solution [2, 8, 10]. Application of such a technique results not only in the existence of solutions but also in a numerical scheme that can be used to investigate the solution quantitatively.

The goal of this paper is to present recent results on the employment of the monotone method for investigating the existence and uniqueness of solutions of (1.1). The idea behind such a method is to replace the actual solution in all the nonlinear and nonlocal terms with some previous guess for the solution, then solve the resulting linear model to obtain a new guess for the solution. Iteration of such a procedure yields the solution of the original problem upon passage to the limit. A novelty of such a technique when applied to (1.1) is that an explicit solution representation for each of these iterates is obtained, and hence an efficient numerical scheme can be developed (see [4, 5]). The key step is a comparison principle between consecutive guesses.

To carry out our program, let $C_B(\Omega)$ denote the space of continuous and uniformly bounded functions on Ω. The following assumptions will be imposed on our parameters throughout the paper:

(A1) $g \in C_B^1(D_T)$, $g > 0$ on $[a,b)\times[0,T]$. In addition, if $b < \infty$ then $g(b,t) = 0$, $t \in [0,T]$. Otherwise, $\lim_{x\to\infty} g(x,t) = 0$ for $t \in [0,T]$.

(A2) $\beta \in C_B(D_T)$ is a nonnegative function.

(A3) $F \in C_B^1(D_T \times \mathbb{R} \times \mathbb{R})$.

(A4) $u_0 \in C_B^1(a,b)$ is nonnegative and satisfies the compatibility condition
$$g(a,0)u_0(a) = \int_a^b \beta(y,0)u_0(y)dy.$$

It is worth noting that (A4) can be considerably relaxed for certain cases (see, e.g., [4, 6, 7]).

The paper is organized as follows. In Section 2, we present a comparison principle and show that this principle holds for the case $F_\phi(x,t,u,\phi) \geq 0$. In Section 3, we discuss the case $F_\phi(x,t,u,\phi) \leq 0$. Section 4 is devoted to the case where F_ϕ has no sign restriction. An unbounded domain (i.e., $b = \infty$) is considered in Section 5.

2 The case $F_\phi(x,t,u,\phi) \ge 0$

In this section we assume that $g = g(x)$, $\beta = \beta(x)$, $b < \infty$, $\phi(u)(t) = \int_a^b u(y,t)dy, F_\phi \ge 0$ and $F_u + M \ge 0$ for some positive constant M. We begin with the definition of upper and lower solutions of problem (1.1).

Definition 2.1 A function $u(x,t)$ is called an upper (a lower) solution of (1.1) on D_T if all the following hold.

(i) $u \in C(D_T) \cap L^\infty(D_T)$.

(ii) $u(x,0) \ge (\le)\ u_0(x)$ in $[a,b]$.

(iii) For every $t \in (0,T)$ and every nonnegative $\xi(x,t) \in C^1(\overline{D_T})$,

$$\begin{aligned}
&\int_a^b u(x,t)\xi(x,t)dx \\
&\ge (\le) \int_a^b u(x,0)\xi(x,0)dx + \int_0^t \xi(a,\tau)\int_a^b \beta(x)u(x,\tau)\,dx\,d\tau \\
&\quad + \int_0^t\int_a^b u(x,\tau)[\xi_\tau(x,\tau) + g(x)\xi_x(x,\tau)]\,dx\,d\tau \\
&\quad + \int_0^t\int_a^b \xi(x,\tau)F(x,\tau,u,\phi(u)(\tau))\,dx\,d\tau.
\end{aligned}$$

The following comparison principle is established in [3]. To our knowledge, this result is the only comparison result for problem (1.1) available in the literature.

Theorem 2.2 *Let u and v be an upper solution and a lower solution of (1.1), respectively. Then $u \ge v$ in $\overline{D_T}$.*

Next we construct the following monotone approximation. Let $\underline{u}^0$ and $\overline{u}^0$ be a lower solution and an upper solution of (1.1), respectively. For $k = 1, 2, \dots$ let $\underline{u}^k$ and $\overline{u}^k$ satisfy the uncoupled systems

$$\begin{gathered}
(\underline{u}^k)_t + (g\underline{u}^k)_x = F(x,t,\underline{u}^{k-1},\phi(\underline{u}^{k-1})) - M(\underline{u}^k - \underline{u}^{k-1}) \quad \text{in } D_T, \\
g(a)\underline{u}^k(a,t) = \int_a^b \beta(y)\underline{u}^k(y,t)dy \quad \text{on } (0,T), \\
\underline{u}^k(x,0) = u_0(x) \quad \text{in } [a,b]
\end{gathered}$$

and

$$\begin{gathered}
(\overline{u}^k)_t + (g\overline{u}^k)_x = F(x,t,\overline{u}^{k-1},\phi(\overline{u}^{k-1})) - M(\overline{u}^k - \overline{u}^{k-1}) \quad \text{in } D_T, \\
g(a)\overline{u}^k(a,t) = \int_a^b \beta(y)\overline{u}^k(y,t)dy \quad \text{on } (0,T), \\
\overline{u}^k(x,0) = u_0(x) \quad \text{in } [a,b].
\end{gathered}$$

The functions $\underline{u}^k$ and $\overline{u}^k$ exist since they satisfy linear equations. Furthermore, it can be shown that (see [3])

$$\underline{u}^0 \le \underline{u}^1 \le \cdots \le \underline{u}^k \le \overline{u}^k \le \cdots \le \overline{u}^k \le \overline{u}^0 \quad \text{in } \overline{D_T}.$$

The following convergence result was established in [3].

Theorem 2.3 *Let $\underline{u}^0$ and $\overline{u}^0$ be a lower solution and an upper solution of (1.1), respectively, and they are continuously differentiable in t. Then the monotone sequences defined above converge in $L^2(a,b)$ to the unique solution $u(x,t)$ uniformly on $0 \le t \le T$. Moreover, the order of convergence is linear.*

In [3] it was shown via a counter example that the restriction $F_\phi \ge 0$ is necessary for establishing a comparison between upper and lower solutions. To overcome this obstacle, in the next section we define a new pair of upper and lower solutions and use this definition to establish a comparison principle.

3 The case $\mathbf{F}_\phi(x,t,u,\phi) \le 0$

In this section we restrict our attention to the case $F(x,t,u,\phi) = -m(x,t,\phi)u$. We assume that $b < \infty$, $\phi(u)(t) = \int_a^b u(y,t)dy$, and $m_\phi \ge 0$. We introduce the following definition of upper and lower solutions.

Definition 3.1 A pair of functions $u(x,t)$ and $v(x,t)$ are called an upper solution and a lower solution of (1.1) on D_T, respectively, if all the following hold.

(i) $u, v \in L^\infty(D_T)$.

(ii) $u(x,0) \ge u_0(x) \ge v(x,0)$ in $[a,b]$.

(iii) For every $t \in (0,T)$ and every nonnegative $\xi(x,t) \in C^1(\overline{D_T})$,

$$\begin{aligned}
&\int_a^b u(x,t)\xi(x,t)dx \\
&\ge \int_a^b u(x,0)\xi(x,0)dx + \int_0^t \xi(a,\tau)\int_a^b \beta(x,\tau)u(x,\tau)\,dx\,d\tau \\
&\quad + \int_0^t\int_a^b [\xi_\tau(x,\tau) + g(x,\tau)\xi_x(x,\tau)]u(x,\tau)\,dx\,d\tau \\
&\quad - \int_0^t\int_a^b \xi(x,\tau)m(x,\tau,\phi(v)(\tau))u(x,\tau)\,dx\,d\tau
\end{aligned} \tag{3.1}$$

and

$$
\begin{aligned}
&\int_a^b v(x,t)\xi(x,t)dx \\
&\leq \int_a^b v(x,0)\xi(x,0)dx + \int_0^t \xi(a,\tau)\int_a^b \beta(x,\tau)v(x,\tau)\,dx\,d\tau \\
&\quad + \int_0^t \int_a^b [\xi_\tau(x,\tau) + g(x,\tau)\xi_x(x,\tau)]v(x,\tau)\,dx\,d\tau \\
&\quad - \int_0^t \int_a^b \xi(x,\tau)m(x,\tau,\phi(u)(\tau))v(x,\tau)\,dx\,d\tau.
\end{aligned}
\tag{3.2}
$$

A function $u(x,t)$ is called a solution of (1.1) on D_T if u satisfies (3.1) with "$\geq$" replaced by "$=$" and $\phi(v)(\tau)$ by $\phi(u)(\tau)$. Based on this definition, the following comparison result was established in [4].

Theorem 3.2 *Let u and v be a nonnegative upper solution and a nonnegative lower solution of (1.1), respectively. Then $u \geq v$ a.e. in D_T.*

As a consequence, the following uniqueness result can be proved (see [4]).

Theorem 3.3 *Let $u(x,t)$ be a nonnegative solution of (1.1) with $\phi(u)(t) \in C([0,T])$. Then u is unique.*

We now construct monotone sequences of upper and lower solutions. To this end, let $\underline{u}^0(x,t)$ and $\overline{u}^0(x,t)$ be a nonnegative lower solution and a nonnegative upper solution of (1.1), respectively. We then define two sequences $\{\underline{u}^k\}_{k=0}^\infty$ and $\{\overline{u}^k\}_{k=0}^\infty$ as follows: For $k = 1, 2, \dots$

$$
\begin{gathered}
\underline{u}_t^k + (g(x,t)\underline{u}^k)_x = -m(x,t,\phi(\overline{u}^{k-1}))\underline{u}^k \quad \text{in } D_T, \\
g(a,t)\underline{u}^k(a,t) = \underline{B}^{k-1}(t) \quad \text{on } (0,T), \\
\underline{u}^k(x,0) = u_0(x) \quad \text{in } [a,b],
\end{gathered}
$$

where $\underline{B}^{k-1}(t) \equiv \int_a^b \beta(y,t)\underline{u}^{k-1}(y,t)dy$, and

$$
\begin{gathered}
\overline{u}_t^k + (g(x,t)\overline{u}^k)_x = -m(x,t,\phi(\underline{u}^{k-1}))\overline{u}^k \quad \text{in } D_T, \\
g(a,t)\overline{u}^k(a,t) = \overline{B}^{k-1}(t) \quad \text{on } (0,T), \\
\overline{u}^k(x,0) = u_0(x) \quad \text{in } [a,b],
\end{gathered}
$$

where $\overline{B}^{k-1}(t) \equiv \int_a^b \beta(y,t)\overline{u}^{k-1}(y,t)dy$.

Since $\underline{B}^{k-1}$ and $\overline{B}^{k-1}$ are given functions, the existence of solutions $\underline{u}^k$ and $\overline{u}^k$ easily follows. Furthermore, we can show that these sequences satisfy

$$
\underline{u}^0 \leq \underline{u}^1 \leq \dots \leq \underline{u}^k \leq \overline{u}^k \leq \dots \leq \overline{u}^1 \leq \overline{u}^0 \quad \text{a.e. in } D_T.
$$

Upon establishing the monotonicity of our sequences, we can prove the following convergence result (see [4]).

Theorem 3.4 *Suppose that* $\underline{u}^0(x,t)$ *and* $\overline{u}^0(x,t)$ *are a nonnegative lower solution and a nonnegative upper solution of (1.1), respectively. Then, the sequences* $\{\underline{u}^k\}_{k=0}^{\infty}$ *and* $\{\overline{u}^k\}_{k=0}^{\infty}$ *converge uniformly to the unique solution* $u(x,t)$ *of problem (1.1) on* D_T. *Moreover, the order of convergence is linear.*

Remark 3.5 In [4] this monotone method was used to numerically solve (1.1). The resutls in that paper indicate that such a scheme converges rapidly to the solution.

4 No restriction on the sign of $\mathbf{F}_\phi(x,t,u,\phi)$

In this section we assume that $b < \infty$, $\phi(u)(t) = \int_a^b d(y)u(y,t)dy$, and that $F(x,t,u,\phi) = -m(x,t,\phi)u$ with $M + m_\phi \geq 0$ for some positive constant M. Consider the following new definition of upper and lower solutions:

Definition 4.1 A pair of functions $u(x,t)$ and $v(x,t)$ are called an upper solution and a lower solution of (1.1) on D_T, respectively, if all the following hold.

(i) $u, v \in L^\infty(D_T)$.

(ii) $u(x,0) \geq u_0(x) \geq v(x,0)$ a.e. in (a,b).

(iii) For every $t \in (0,T)$ and every nonnegative $\xi(x,t) \in C^1(\overline{D_T})$,

$$\begin{aligned}
&\int_a^b u(x,t)\xi(x,t)dx \\
&\geq \int_a^b u(x,0)\xi(x,0)dx + \int_0^t \xi(a,\tau)\int_a^b \beta(x,\tau)u(x,\tau)dx\,d\tau \\
&\quad + \int_0^t\int_a^b [\xi_\tau(x,\tau) + g(x,\tau)\xi_x(x,\tau)]u(x,\tau)\,dx\,d\tau \\
&\quad - \int_0^t\int_a^b \xi(x,\tau)\,[m(x,\tau,\phi(v)(\tau)) + M\phi(v)(\tau) - M\phi(u)(\tau)]\,u(x,\tau)\,dx\,d\tau
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
&\int_a^b v(x,t)\xi(x,t)dx \\
&\leq \int_a^b v(x,0)\xi(x,0)dx + \int_0^t \xi(a,\tau)\int_a^b \beta(x,\tau)v(x,\tau)\,dx\,d\tau \\
&\quad + \int_0^t\int_a^b [\xi_\tau(x,\tau) + g(x,\tau)\xi_x(x,\tau)]v(x,\tau)\,dx\,d\tau \\
&\quad - \int_0^t\int_a^b \xi(x,\tau)\,[m(x,\tau,\phi(u)(\tau)) + M\phi(u)(\tau) - M\phi(v)(\tau)]\,v(x,\tau)\,dx\,d\tau.
\end{aligned} \tag{4.2}$$

A function $u(x,t)$ is called a solution of (1.1) on D_T if u satisfies (4.1) with "$\geq$" replaced by "$=$" and $\phi(v)(\tau)$ by $\phi(u)(\tau)$. Using this definition, we establish the following comparison principle [6].

Theorem 4.2 *Let u and v be a nonnegative upper solution and a nonnegative lower solution of (1.1), respectively. Then $u \geq v$ a.e. in D_T.*

Furthermore, we prove the following uniqueness result.

Corollary 4.3 *Let $u(x,t)$ be a nonnegative solution of (1.1) with $\phi(u)(t) \in C([0,T])$. Then u is unique.*

We now construct a pair of nonnegative lower and upper solutions of (1.1). Let $\underline{u}^0(x,t) = 0$. Choose a constant γ large enough such that

$$\max_{\overline{D_T}} \beta(x,t)/ \min_{[0,T]} g(a,t) \leq \gamma/2.$$

Fix this γ and choose δ large enough such that

$$\|u_0\|_\infty \leq (\delta/2)\exp(-\gamma b).$$

Now choose σ large enough such that

$$\sigma \geq 2M\delta\|\eta\|_\infty \exp(-\gamma a)/\gamma + \gamma \max_{\overline{D_T}} g(x,t) + \max_{\overline{D_T}} |g_x(x,t)|.$$

Let $\overline{u}^0(x,t) = \delta\exp(\sigma t)\exp(-\gamma x)$. Then it can be easily shown that $\underline{u}^0$ and $\overline{u}^0$ are a pair of lower and upper solutions of (1.1) on $[a,b] \times [0,T_0]$ with $T_0 = \min\{T, (\ln 2)/\sigma\}$. We then define two sequences $\{\underline{u}^k\}_{k=0}^\infty$ and $\{\overline{u}^k\}_{k=0}^\infty$ as follows:
For $k = 1,2,\dots$

$$\begin{gathered} \underline{u}_t^k + (g(x,t)\underline{u}^k)_x = -D^{k-1}(x,t)\underline{u}^k \quad \text{in } D_{T_0}, \\ g(a,t)\underline{u}^k(a,t) = \underline{B}^{k-1}(t) \quad \text{on } (0,T_0), \\ \underline{u}^k(x,0) = u_0(x) \quad \text{in } [a,b], \end{gathered} \tag{4.3}$$

where

$$\begin{gathered} D^{k-1}(x,t) = m(x,t,\phi(\overline{u}^{k-1})) + M\phi(\overline{u}^{k-1}) - M\phi(\underline{u}^{k-1}), \\ \underline{B}^{k-1}(t) \equiv \int_a^b \beta(y,t)\underline{u}^{k-1}(y,t)dy, \end{gathered}$$

and

$$\begin{gathered} \overline{u}_t^k + (g(x,t)\overline{u}^k)_x = -E^{k-1}(x,t)\overline{u}^k \quad \text{in } D_{T_0}, \\ g(a,t)\overline{u}^k(a,t) = \overline{B}^{k-1}(t) \quad \text{on } (0,T_0), \\ \overline{u}^k(x,0) = u_0(x) \quad \text{in } [a,b], \end{gathered} \tag{4.4}$$

where

$$E^{k-1}(x,t) = m(x,t,\phi(\underline{u}^{k-1})) + M\phi(\underline{u}^{k-1}) - M\phi(\overline{u}^{k-1})$$

$$\overline{B}^{k-1}(t) \equiv \int_a^b \beta(y,t)\overline{u}^{k-1}(y,t)dy.$$

The existence of solutions to problems (4.3) and (4.4) follows from the fact that $\underline{B}^{k-1}$ and $\overline{B}^{k-1}$ are given functions.

By similar reasoning, we can show that $\underline{u}^k \le \underline{u}^{k+1} \le \overline{u}^{k+1} \le \overline{u}^k$ and that $\underline{u}^{k+1}$ and $\overline{u}^{k+1}$ are also a lower solution and an upper solution of (1.1), respectively. Thus by induction, we obtain two monotone sequences that satisfy

$$\underline{u}^0 \le \underline{u}^1 \le \cdots \le \underline{u}^k \le \overline{u}^k \le \cdots \le \overline{u}^1 \le \overline{u}^0 \quad \text{a.e. in } D_{T_0}.$$

Hence, it follows from the monotonicity of the sequences $\{\underline{u}^k\}_{k=0}^\infty$ and $\{\overline{u}^k\}_{k=0}^\infty$ that there exist functions $\underline{u}$ and $\overline{u}$ such that $\underline{u}^k \to \underline{u}$ and $\overline{u}^k \to \overline{u}$ pointwise in D_{T_0}. It is not too difficult to argue that $\underline{u} = \overline{u}$ a.e. in D_{T_0}. We denote this common limit by u.

Upon establishing the monotonicity of our sequences, we can also prove the following convergence result.

Theorem 4.4 *The sequences $\{\underline{u}^k\}_{k=0}^\infty$ and $\{\overline{u}^k\}_{k=0}^\infty$ converge uniformly along characteristic curves to a limit function $u(x,t)$. Moreover, the function u is the unique solution of problem (1.1) on $[a,b] \times [0,T_0]$.*

Remark 4.5 It is not too difficult to show that this local solution is indeed a global solution.

5 Unbounded Domains

This section is concerned with a special model which describes the aggregation of phytoplankton (see [9]). Here $a = 0$, $b = \infty$,

$$\phi(u)\,(x,t) = \frac{1}{2}\int_0^x \eta(x-y,y)u(x-y,t)u(y,t)dy - \int_0^\infty \eta(x,y)u(x,t)u(y,t)dy$$

and

$$F(x,t,u,\phi) = \phi + f(x,t)u.$$

We assume that η and f are bounded continuous functions. Let $C^1_{0,r}(D_T) = \{\psi \in C^1(D_T) : \exists\, x_\psi \in (0,\infty) \text{ such that } \psi \equiv 0 \text{ for } x \ge x_\psi\}$. We then introduce the following definition of coupled upper and lower solutions of problem (1.1).

Definition 5.1 A pair of functions $u(x,t)$ and $v(x,t)$ are called an upper solution and a lower solution of (1.1) on D_T, respectively, if all the following hold.

(i) $u, v \in L^\infty((0,T); L^1(0,\infty))$.

(ii) $u(x,0) \geq u_0(x) \geq v(x,0)$ a.e. in $(0,\infty)$.

(iii) For every $t \in (0,T)$ and every nonnegative $\xi \in C^1_{0,r}(D_T)$,

$$
\begin{aligned}
&\int_0^\infty u(x,t)\xi(x,t)dx \\
&\geq \int_0^\infty u(x,0)\xi(x,0)dx + \int_0^t \xi(0,\tau)\int_0^\infty \beta(x,\tau)u(x,\tau)\,dx\,d\tau \\
&\quad + \int_0^t\int_0^\infty [\xi_\tau(x,\tau) + g(x,\tau)\xi_x(x,\tau)]u(x,\tau)\,dx\,d\tau \\
&\quad + \int_0^t\int_0^\infty \xi(x,\tau)\mathcal{F}(u)(x,\tau)\,dx\,d\tau \\
&\quad - \int_0^t\int_0^\infty \xi(x,\tau)\int_0^\infty \eta(x,y)u(x,\tau)v(y,\tau)dydxd\tau
\end{aligned} \tag{5.1}
$$

and

$$
\begin{aligned}
&\int_0^\infty v(x,t)\xi(x,t)dx \\
&\leq \int_0^\infty v(x,0)\xi(x,0)dx + \int_0^t \xi(0,\tau)\int_0^\infty \beta(x,\tau)v(x,\tau)\,dx\,d\tau \\
&\quad + \int_0^t\int_0^\infty [\xi_\tau(x,\tau) + g(x,\tau)\xi_x(x,\tau)]v(x,\tau)\,dx\,d\tau \\
&\quad + \int_0^t\int_0^\infty \xi(x,\tau)\mathcal{F}(v)(x,\tau)\,dx\,d\tau \\
&\quad - \int_0^t\int_0^\infty \xi(x,\tau)\int_0^\infty \eta(x,y)v(x,\tau)u(y,\tau)dydxd\tau,
\end{aligned} \tag{5.2}
$$

where

$$
\mathcal{F}(w)(x,t) = \frac{1}{2}\int_0^x \eta(x-y,y)w(x-y,t)w(y,t)dy + f(x,t)w(x,t).
$$

A function $u(x,t)$ is called a solution of (1.1) on D_T if u satisfies (5.1) with "$\geq$" replaced by "$=$" and $v(y,\tau)$ in the last integral by $u(y,\tau)$.

The following comparison principle was established in [7].

Theorem 5.2 *Let u and v be a nonnegative upper solution and a nonnegative lower solution of (1.1), respectively. Then $u \geq v$ a.e. in D_T.*

Corollary 5.3 *Let $\underline{u}$ and $\overline{u}$ be a nonnegative lower solution and a nonnegative upper solution of (1.1), respectively. If u is a solution of (1.1), then $\underline{u} \leq u \leq \overline{u}$ a.e. in D_T.*

We now construct monotone sequences of upper and lower solutions. Suppose that $\underline{u}^0(x,t)$ and $\overline{u}^0(x,t)$ are a pair of lower and upper solutions of (1.1). Since f and η are bounded we can choose a positive constant M such that $M - \int_0^\infty \eta(x,y)u(y,t)dy + f(x,t) \geq 0$ for $(x,t) \in \overline{D}_T$ and $\underline{u}^0(x,t) \leq u(x,t) \leq \overline{u}^0(x,t)$. We then set up two sequences $\{\underline{u}^k\}_{k=0}^\infty$ and $\{\overline{u}^k\}_{k=0}^\infty$ by the following procedure:
For $k = 1, 2, \dots$ let $\underline{u}^k$ and $\overline{u}^k$ satisfy the systems

$$\begin{gathered}\underline{u}^k_t + (g\underline{u}^k)_x = \mathcal{F}(\underline{u}^{k-1}) - M(\underline{u}^k - \underline{u}^{k-1}) - \underline{u}^{k-1}\int_0^\infty \eta(x,y)\overline{u}^{k-1}(y,t)dy \text{ in } D_T,\\ g(0,t)\underline{u}^k(0,t) = \int_0^\infty \beta(y,t)\underline{u}^{k-1}(y,t)dy \quad \text{on } (0,T),\\ \underline{u}(x,0) = u_0(x) \text{ in } [0,\infty)\end{gathered}$$

and

$$\begin{gathered}\overline{u}^k_t + (g\overline{u}^k)_x = \mathcal{F}(\overline{u}^{k-1}) - M(\overline{u}^k - \overline{u}^{k-1}) - \overline{u}^{k-1}\int_0^\infty \eta(x,y)\underline{u}^{k-1}(y,t)dy \text{ in } D_T,\\ g(0,t)\overline{u}^k(0,t) = \int_0^\infty \beta(y,t)\overline{u}^{k-1}(y,t)dy \quad \text{on } (0,T),\\ \overline{u}(x,0) = u_0(x) \text{ in } [0,\infty).\end{gathered}$$

By induction, we can show that the sequences satisfy

$$\underline{u}^0 \leq \underline{u}^1 \leq \dots \leq \underline{u}^k \leq \overline{u}^k \leq \dots \leq \overline{u}^1 \leq \overline{u}^0 \quad \text{a.e. in } D_T.$$

Then, we have the following existence-uniqueness result.

Theorem 5.4 *Suppose that $\underline{u}^0(x,t)$ and $\overline{u}^0(x,t)$ are a nonnegative lower solution and a nonnegative upper solution of (1.1), respectively. Then there exist monotone sequences $\{\underline{u}^k(x,t)\}$ and $\{\overline{u}^k(x,t)\}$ which converge to the unique solution of (1.1).*

Remark 5.5 As an example, for a large class of initial data such as $u_0(x) = O(e^{-x})$ as $x \to \infty$, we can construct a pair of nonnegative lower and upper solutions of (1.1) as follows: Let $\underline{u}^0(x,t) = 0$ and $\overline{u}^0(x,t) = c_3e^{c_2t}/(1+c_1^2x^2)$ with c_1, c_2, c_3 positive constants. First choose c_1 so large such that

$$\pi \max_{\overline{D}_1} \beta(x,t)/\min_{[0,1]} g(0,t) \leq c_1.$$

Fix this c_1 and choose c_3 large enough such that $c_3/(1+c_1^2x^2) \geq u_0(x)$ for $0 \leq x < \infty$. We then determine c_2. Through a routine calculation, we find

$$\begin{aligned}\int_0^x \frac{dy}{[1+c_1^2(x-y)^2](1+c_1^2y^2)} &= \frac{2}{c_1^2x}\Big[\frac{c_1x\tan^{-1}(c_1x) + \log(1+c_1^2x^2)}{4+c_1^2x^2}\Big]\\ &\leq \frac{2(1+\pi)}{c_1(1+c_1^2x^2)}.\end{aligned}$$

Thus we can choose c_2 sufficiently large such that

$$c_2 \geq \frac{2c_3}{c_1}(1+\pi) + \max_{\overline{D}_1} g(x,t) + \max_{\overline{D}_1} |f(x,t) - g_x(x,t)|.$$

Then it follows that $\overline{u}^0$ is a desired upper solution of (1.1) on D_T with $T = \min\{1, \log 2/c_2\}$.

We now show that the solution of (1.1) has the following property.

Theorem 5.6 *For the solution $u(x,t)$ of (1.1), $P(t) = \int_0^\infty u(x,t)dx$ is continuous in the existence interval.*

Finally, we establish the existence of a global solution.

Theorem 5.7 *The unique solution of (1.1) exists for $0 \leq t < \infty$.*

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References

[1] A. S. Ackleh, Parameter estimation in size-structured coagulation-fragmentation phytoplankton population model, *Nonlinear Anal.,* **28** (1997), 837-854.

[2] A. S. Ackleh, H. T. Banks and K. Deng, A difference approximation for a coupled system of nonlinear size-structured populations, *Nonlinear Anal.,* **50** (2002), 727-748.

[3] A. S. Ackleh and K. Deng, A monotone method for first order nonlocal hyperbolic initial-boundary value problems, *Appl. Anal.*, **67** (1997), 173-183.

[4] A. S. Ackleh and K. Deng, A monotone approximation for a nonlinear nonautonomous size-structured population model, *Appl. Math. Comp.*, **108** (2000), 103-113.

[5] A. S. Ackleh and K. Deng, Monotone scheme for nonlinear first order hyperbolic initial-boundary value problems, *Appl. Math. Lett.*, **13** (2000), 111-119.

[6] A. S. Ackleh and K. Deng, Existence-uniqueness of solutions for a nonlinear nonautonomous size-structured population model: an upper-lower solution approach, *Canadian Appl. Math. Quart.*, **8** (2000), 1-15.

[7] A. S. Ackleh and K. Deng, On a first order hyperbolic coagulation model, *Math. Methods Appl. Sci.*, to appear.

[8] A. S. Ackleh and R. R. Ferdinand, A finite difference approximation for a nonlinear size-structured phytoplankton aggregation model, *Quart. Appl. Math.*, **57** (1999), 501-520.

[9] A. S. Ackleh and B. G. Fitzpatrick, Modeling aggregation and growth processes in an algal population model: analysis and computation, *J. Math. Biol.*, **35** (1997), 480-502.

[10] A. S. Ackleh and K. Ito, An implicit finite difference scheme for the nonlinear size-structured population model, *Num. Funct. Anal. Optim.*, **18** (1997), 865-884.

[11] H. T. Banks, F. Kappel and C. Wang, Weak solutions and differentiability for size structured population models, *Internat. Ser. Numer. Math.,* **100** (1991), 35-50.

[12] A. Calsina and J. Saldana, A model of physiologically structured population dynamics with a nonlinear individual growth rate, *J. Math. Biol.,* **33** (1995), 335-364.

[13] M. G. Crandall and A. Majda, Monotone difference approximations for scalar conservation laws*, Math. Comp.*, **34** (1980), 1-21.

[14] J. Li, Presistence and extinction in continuous age structured models, *Comp. Math. Appl.,* **15** (1988), 511-523.

[15] J. Smoller, Shock Waves and Reaction-Diffusion Equations, Springer-Verlag, New York, 1994.

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