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Function spaces of BMO and Campanato type *

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Abstract

To obtain the Littlewood-Paley characterization for Campanato spaces $\mathcal{L}^{2,\lambda}$ modulo polynomials (which contain as special case the John and Nirenberg space BMO), we define and study a scale of function spaces on $\mathbb{R}^n$. We discuss the real interpolation of these spaces and some embeddings between these spaces and the classical spaces. These embeddings cover some classical results obtained by Campanato, Strichartz, Stein and Zygmund.

1 Introduction

In this work, we introduce and study a scale of function spaces on $\mathbb{R}^n$. The homogeneous version of these spaces contains Campanato spaces $\mathcal{L}^{2,\lambda}$ and John and Nirenberg space $BMO = \mathcal{L}^{p,n}$. It is classical that the homogeneous space of Triebel-Lizorkin $\dot{F}^s_{p,q}(\mathbb{R}^n)$ coincides with BMO modulo polynomials for some values of p, q and s. Namely, $BMO = \dot{F}^0_{\infty,2}$ [13, chapter 5] and $I^s(BMO) = \dot{F}^s_{\infty,2}$, where $I^s = \mathcal{F}^{-1}(|.|^{-s}\mathcal{F})$ is the Riesz potential operator. The spaces $I^s(BMO)$ were studied by Strichartz [12]. We use a Littlewood-Paley partition to define these spaces denoted by $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ and their homogeneous version $\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)$. These spaces allow us to give the Littlewood-Paley characterization of Campanato spaces $\mathcal{L}^{2,\lambda}$ and more generally of $I^s(\mathcal{L}^{2,\lambda})$ modulo polynomials (cf. Theorem 2.3). If we denote L^s_p the local approximation Campanato spaces defined for instance in the book [14, Definition 1.7.2. (5)] for $s \geq -n/p$ and $1 \leq p < +\infty$, then we recall that $L^s_p = C^s$ for any $s > 0$, $L^{-n/p}_p = L^p$ and $L^0_p = bmo$ the local version of BMO, cf. [4], [10], [15] and [14] for the proof and more references. The spaces of Campanato $\mathcal{L}^{p,\lambda}$ considered here (Definition 1.4) coincide with the local approximation Campanato spaces L^s_p with $s = (\lambda - n)/p$ for $-\frac{n}{p} < s < 0$ (ie. $0 < \lambda < n$) which are themselves equal to Morrey spaces. The characterization given here is of interest for L^s_2 spaces in the case $-n/2 < s < 0$.

Next we give a result concerning the real interpolation of these spaces, and we extend some injections due to Strichartz [12] and Stein and Zygmund [11]

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by showing some embeddings between the spaces $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin ones $F^s_{p,q}(\mathbb{R}^n)$ and Besov-Peetre ones $B^s_{p,q}(\mathbb{R}^n)$, and on the other hand between the same spaces $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ and Hölder-Zygmund ones $C^s(\mathbb{R}^n)$. Such embeddings shed some light on duals of the closure of Schwartz space $\mathcal{S}(\mathbb{R}^n)$ in BMO and in Campanato spaces $\mathcal{L}^{2,\lambda}$ (Corollary 2.13). To define the spaces we will need the following partition of unity: we denote $x \in \mathbb{R}^n$ and ξ its dual variable. Let $\varphi \in C^\infty_0(\mathbb{R}^n)$, $\varphi \geq 0$, φ equal to 1 on $|\xi| \leq 1$, and equal to 0 on $|\xi| \geq 2$. Let $\theta(\xi) = \varphi(\xi) - \varphi(2\xi)$, $\operatorname{supp}\theta \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$. For $j \in \mathbf{Z}$ we set $\dot{\Delta}_j u = \theta(2^{-j} D_x)u$, $\Delta_0 u = \varphi(D_x)u$ and if $j \geq 1$ we set also $\Delta_j u = \dot{\Delta}_j u$.

Remark 1.1 We recall that if $u \in \mathcal{S}'(\mathbb{R}^n)$ then $u = \sum_{k\geq 0} \Delta_k u$ and $u = \sum_{k\in\mathbf{Z}} \dot{\Delta}_k u$ modulo polynomials.

Now we give the definition of the nonhomogeneous spaces.

Definition 1.2 Let $s \in \mathbb{R}$, $\lambda \geq 0$, $1 \leq p < +\infty$ and $1 \leq q < +\infty$. The space $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ denotes the set of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|u\|_{\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)} = \Big(\sup_B \frac{1}{|B|^{\lambda/n}} \sum_{j\geq J^+} 2^{jqs} \|\Delta_j u\|^q_{L^p(B)}\Big)^{1/q} < +\infty \qquad (1.1)$$

where $J^+ = \max(J, 0)$, $|B|$ is the measure of B and the supremum is taken over all $J \in \mathbf{Z}$ and all balls B of $\mathbb{R}^n$ of radius 2^{-J}.

When $p = q$, the space $\mathcal{L}^{\lambda,s}_{p,p}(\mathbb{R}^n)$ is denoted $\mathcal{L}^{p,\lambda,s}(\mathbb{R}^n)$. Note that the space $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ equipped with the norm (1.1) is a Banach space.

To define the homogeneous spaces we recall the notation of [13]: $Z'(\mathbb{R}^n) := \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ is the space of all tempered distributions modulo the set $\mathcal{P}$ of polynomials of $\mathbb{R}^n$ with complex coefficients.

Definition 1.3 Let $s \in \mathbb{R}$, $\lambda \geq 0$, $1 \leq p < +\infty$ and $1 \leq q < +\infty$. The dotted space $\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ denotes the set of all $u \in Z'(\mathbb{R}^n)$ such that

$$\|u\|_{\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)} = \Big(\sup_B \frac{1}{|B|^{\lambda/n}} \sum_{j\geq J} 2^{jqs} \|\dot{\Delta}_j u\|^q_{L^p(B)}\Big)^{1/q} < +\infty \qquad (1.2)$$

where the supremum is taken over all $J \in \mathbf{Z}$ and all balls B of $\mathbb{R}^n$ of radius 2^{-J}.

The space $\dot{\mathcal{L}}^{\lambda,s}_{p,p}(\mathbb{R}^n)$ will be denoted $\dot{\mathcal{L}}^{p,\lambda,s}(\mathbb{R}^n)$. If P is a polynomial of $\mathcal{P}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, it follows immediately that

$$\|u + P\|_{\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)} = \|u\|_{\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)}$$

This shows that the norm (1.2) is well defined. Further, the space $\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ equipped with this norm is a Banach space.

Now we recall the definition of Campanato spaces and BMO.

Definition 1.4 Let $\lambda \geq 0$ and $1 \leq p < +\infty$. (i) We say $u \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ if $u \in L^p_{\text{loc}}(\mathbb{R}^n)$ and

$$\|u\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} = \Big(\sup_B \frac{1}{|B|^{\lambda/n}} \int_B |u - m_B u|^p dx\Big)^{1/p} < +\infty$$

where $m_B u = \frac{1}{|B|} \int_B u(y)dy$ is the mean value of u and the supremum is taken over all the balls B of $\mathbb{R}^n$. The space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ is a Banach space modulo constants and is equal to $\{0\}$ for $\lambda > n + p$.
Let us denote BMO the space $\mathcal{L}^{2,n}(\mathbb{R}^n)$. Note that BMO is equal to $\mathcal{L}^{p,n}(\mathbb{R}^n)$ for any $1 \leq p < +\infty$, cf. [10].
(ii) For $0 \leq \lambda < n + p$, we define the space $\dot{\mathcal{L}}^{p,\lambda}(\mathbb{R}^n)$ as the set of all equivalence classes modulo $\mathcal{P}$ of elements of $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$, equipped with the norm $\|U\|_{\dot{\mathcal{L}}^{p,\lambda}(\mathbb{R}^n)} = \|u\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)}$ where u is the unique (modulo constants) element of $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ belonging to the class U.

2 Results

The following proposition yields the dyadic characterization of $\dot{\mathcal{L}}^{2,\lambda}(\mathbb{R}^n)$.

Proposition 2.1 *Let $0 \leq \lambda < n + 2$. The space $\dot{\mathcal{L}}^{2,\lambda,0}(\mathbb{R}^n)$ coincides algebraically and topologically with Campanato space $\dot{\mathcal{L}}^{2,\lambda}(\mathbb{R}^n)$.*

This proposition allows us to deduce the link between the discrete scale built on $\dot{\mathcal{L}}^{2,\lambda}(\mathbb{R}^n)$ and the continuous scale.

Corollary 2.2 *Let $0 \leq \lambda < n + 2$ and $m \in \mathbb{N}$.*

(i) $\dot{\mathcal{L}}^{2,\lambda,m}(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{H}}^{2,\lambda,m} := \{u \in \dot{\mathcal{L}}^{2,\lambda}(\mathbb{R}^n);\, D^\alpha u \in \dot{\mathcal{L}}^{2,\lambda}(\mathbb{R}^n), |\alpha| \leq m\}$.

(ii) $\dot{\mathcal{H}}^{2,\lambda,m} \bigcap \dot{H}^{m+\lambda/2}(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{L}}^{2,\lambda,m}(\mathbb{R}^n)$*, here* $\dot{H}^{m+\lambda/2}(\mathbb{R}^n)$ *is the classical homogeneous Sobolev space.*

Therefore, $\dot{\mathcal{H}}^{2,\lambda,m} \bigcap \dot{H}^{m+\lambda/2}(\mathbb{R}^n) \equiv \dot{\mathcal{L}}^{2,\lambda,m}(\mathbb{R}^n) \bigcap \dot{H}^{m+\lambda/2}(\mathbb{R}^n)$.

To state a more general result than the proposition 2.1, we recall the definition of the Riesz potential operator

$$I^s f = \mathcal{F}^{-1}\{|.|^{-s}\mathcal{F}f\}, \quad f \in Z'(\mathbb{R}^n) \text{ and } s \in \mathbb{R}$$

Theorem 2.3 *Let $s \in \mathbb{R}$ and $0 \leq \lambda < n + 2$. The space $\dot{\mathcal{L}}^{2,\lambda,s}(\mathbb{R}^n)$ coincides algebraically and topologically with the space $I^s(\dot{\mathcal{L}}^{2,\lambda}(\mathbb{R}^n))$ image of $\dot{\mathcal{L}}^{2,\lambda}(\mathbb{R}^n)$ under I^s.*

Remark 2.4 These results are not true in general for the spaces $\dot{\mathcal{L}}^{p,\lambda}$, $p \neq 2$. For this, G. Bourdaud notes that $\dot{\mathcal{L}}^{p,0,0} = \dot{B}^0_{p,p}$ for any $1 \leq p < +\infty$, and it is classical that $\dot{\mathcal{L}}^{p,0} = L^p$.

The following lemma shows that these spaces are independent of the partition $(\Delta_j)_j$.

Lemma 2.5 *Let $R > 1$. Let $(u_j)_{j\geq 0}$ be a sequence of $L^p_{\text{loc}}(\mathbb{R}^n)$ satisfying the following assumptions:*

(i) $\operatorname{supp}\mathcal{F}u_0 \subset \{|\xi| \leq R\}$ *and* $\operatorname{supp}\mathcal{F}u_j \subset \{\frac{1}{R}2^j \leq |\xi| \leq R2^j\}$ *for $j \geq 1$.*

(ii)
$$M := (\sup_B \frac{1}{|B|^{\lambda/n}} \sum_{j\geq J^+} 2^{jsq} \|u_j\|^q_{L^p(B)})^{1/q} < +\infty$$

where the supremum is taken over all $J \in \mathbf{Z}$ and all balls B of $\mathbb{R}^n$ of radius 2^{-J}.

Then the series $\sum_j u_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$, and its sum u belongs to $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ with $\|u\|_{\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)} \leq CM$, where the constant C depends only from s, p, n, R and the partition $(\Delta_j)_{j\geq 0}$. We have an analogous result for the dotted spaces.

Corollary 2.6 *The derivation D^α_x is a bounded operator from the sapce $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ to the space $\mathcal{L}^{\lambda,s-|\alpha|}_{p,q}(\mathbb{R}^n)$ and from $\dot{\mathcal{L}}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ to $\dot{\mathcal{L}}^{\lambda,s-|\alpha|}_{p,q}(\mathbb{R}^n)$.*

For this it suffices to note that $D^\alpha_x u = \sum_{j\geq 0} \Delta_j D^\alpha_x u = \sum_{j\geq 0} 2^{j|\alpha|} L_j u$, where $\mathcal{F}L_j u(\xi) = \theta_\alpha(2^{-j}\xi)\mathcal{F}u(\xi)$, with $\theta_\alpha(\xi) = \xi^\alpha \theta(\xi)$. We apply lemma 2.5 then.

We can remove the spectral assumption (i) of lemma 2.5 by giving a result dealing with the real interpolation of these spaces:

Theorem 2.7 (Interpolation) *Let N, be an integer ≥ 1, $0 < s < N$, $\lambda \geq 0$ and $p, q \in [1, +\infty[$. Let $(u_j)_j$ be a sequence of functions belonging to $C^\infty(\mathbb{R}^n) \cap L^p_{\text{loc}}(\mathbb{R}^n)$. We assume that there is a sequence $(\varepsilon_j)_j \in l^q$ such that for any ball B of $\mathbb{R}^n$ of radius 2^{-J}, $J \in \mathbf{Z}$,,*

$$\|D^\alpha_x u_j\|_{L^p(B)} \leq \varepsilon_j 2^{j(|\alpha|-s)} |B|^{\lambda/(qn)} \inf\{1, 2^{-JN}\} \text{ for any } j \geq 0 \text{ and } |\alpha| \leq N \tag{2.1}$$

Then, the series $\sum_{j\geq 0} u_j$ converges in $L^p_{\text{loc}}(\mathbb{R}^n)$ and its sum u belongs to $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$ with

$$\|u\|_{\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)} \leq C\|(\varepsilon_j)_j\|_{l^q},$$

where C depends only on N, s, p, n, λ and the partition defining the norm of $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$.

The following lemma gives the inclusion property among these spaces in dependance of their parameters:

Lemma 2.8 *Let $1 \leq p \leq p' < +\infty$, $1 \leq q' \leq q < +\infty$ and $s \in \mathbb{R}$. Further, let λ and $\mu \geq 0$ such that $\frac{n}{p'} - \frac{\mu}{q'} \geq \frac{n}{p} - \frac{\lambda}{q}$. Then we have the continuous embedding*

$$\mathcal{L}^{\mu,s+\frac{n}{p'}-\frac{\mu}{q'}}_{p',q'}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{\lambda,s+\frac{n}{p}-\frac{\lambda}{q}}_{p,q}(\mathbb{R}^n)$$

We have the same result for the dotted spaces $\dot{\mathcal{L}}$.

In particular, if $p = p'$ and $q = q'$ then $\mathcal{L}_{p,q}^{\mu,s-\frac{\mu}{q}}(\mathbb{R}^n) \hookrightarrow \mathcal{L}_{p,q}^{\lambda,s-\frac{\lambda}{q}}(\mathbb{R}^n)$ holds for any $\mu \le \lambda$. Furthermore if $p = p' = q = q'$ we get $\mathcal{L}^{p,\lambda,s+\frac{\alpha}{p}}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{p,\lambda+\alpha,s}(\mathbb{R}^n)$ for any $\alpha \ge 0$ and $\lambda \ge 0$. Now we give the connection between $\mathcal{L}_{p,q}^{\lambda,s}(\mathbb{R}^n)$ and $\dot{\mathcal{L}}_{p,q}^{\lambda,s}(\mathbb{R}^n)$.

Lemma 2.9 *Let $1 \le p, q < +\infty$, $\lambda \ge 0$ and $s \in \mathbb{R}$.*

(i) If the class of u modulo $\mathcal{P}$ belongs to $\dot{\mathcal{L}}_{p,q}^{\lambda,s}(\mathbb{R}^n)$ and if $\Delta_0 u \in L^p(\mathbb{R}^n)$, then $u \in \mathcal{L}_{p,q}^{\lambda,s}(\mathbb{R}^n)$.

(ii) $L^p(\mathbb{R}^n) \cap \dot{\mathcal{L}}_{p,q}^{\lambda,s}(\mathbb{R}^n) \subset \mathcal{L}_{p,q}^{\lambda,s}(\mathbb{R}^n)$ with the same meaning as (i).

(iii) $\mathcal{L}_{p,q}^{\lambda,s}(\mathbb{R}^n) \subset \dot{\mathcal{L}}_{p,q}^{\lambda,s}(\mathbb{R}^n)$ provided $s > 0$.

Remark 2.10 It follows that if $s > 0$ then

$$L^p(\mathbb{R}^n) \cap \dot{\mathcal{L}}_{p,q}^{\lambda,s}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \cap \mathcal{L}_{p,q}^{\lambda,s}(\mathbb{R}^n).$$

Finally we give the connection between these spaces and the classical spaces. For the definitions of the spaces $B_{p,q}^s, C^s, F_{p,q}^s$ and the dotted ones we refer to [13].

Theorem 2.11 *Let $s \in \mathbb{R}$, $1 \le p < +\infty$, $1 \le q < +\infty$ and $\lambda \ge 0$. We have the following continuous embeddings*

$$\begin{aligned} \mathcal{L}_{p,q}^{\lambda,s+\frac{n}{p}-\frac{\lambda}{q}}(\mathbb{R}^n) &\hookrightarrow C^s(\mathbb{R}^n) \\ F_{p,q}^{s+\frac{n}{p}}(\mathbb{R}^n) &\hookrightarrow \mathcal{L}_{p,q}^{\lambda,s+\frac{n}{p}-\frac{\lambda}{q}}(\mathbb{R}^n) \quad \text{provided } q \ge p \\ F_{p,q}^{s+\frac{n}{p}}(\mathbb{R}^n) &\hookrightarrow \mathcal{L}^{p,\lambda,s-\frac{\lambda-n}{p}}(\mathbb{R}^n) \quad \text{provided } p \ge q \\ B_{\infty,q}^{s-\frac{n}{p}+\frac{\lambda}{q}}(\mathbb{R}^n) &\hookrightarrow \mathcal{L}_{p,q}^{\lambda,s}(\mathbb{R}^n) \quad \text{provided } \lambda \ge n\frac{q}{p} \end{aligned}$$

and finally

$$\sum_{j\ge 0} 2^{jq(s+\frac{\lambda-n}{q})}|\Delta_j u|^q \in L^\infty(\mathbb{R}^n) \text{ implies } u \in \mathcal{L}^{q,\lambda,s}(\mathbb{R}^n) \text{ provided } \lambda \ge n$$

We have also the same continuous embeddings if we replace B, C, F and $\mathcal{L}$ respectively by the dotted spaces $\dot{B}, \dot{C}, \dot{F}$ and $\dot{\mathcal{L}}$.

Remark 2.12 (i) These embeddings cover theorem 2.1 of [5] and theorem 3.4 of [12] which asserts that $\dot{B}_{\infty,2}^{s}(\mathbb{R}^n) \hookrightarrow I^s(BMO) \hookrightarrow \dot{C}^{s}(\mathbb{R}^n)$, where I^s is the Riesz potential operator and $I^s(BMO) = \dot{\mathcal{L}}^{2,n,s}(\mathbb{R}^n)$, BMO is defined modulo polynomials.

(ii) In the case $s = 0$, S. Campanato [3] and [4] showed that if $n < \lambda < n + p$ we have $\mathcal{L}^{p,\lambda} \cong C^{\frac{\lambda-n}{p}}$ and $\mathcal{L}^{p,n+p} = Lip$ (we refer also to [9]).

(iii) If we do $s = 0,\ p = q = 2$ and $\lambda = n$ in the third embedding, then we find again a result due to Stein and Zygmund [11]

$$\dot{H}^{\frac{n}{2}} = \dot{F}^{\frac{n}{2}}_{2,2} \hookrightarrow \dot{\mathcal{L}}^{2,n,0}(\mathbb{R}^n) = BMO \text{ modulo polynomials}$$

From this theorem we deduce a partial result on the topological dual of $\overset{\circ}{\mathcal{L}}{}^{\lambda,s}_{p,q}(\mathbb{R}^n)$, the closure of Schwartz space $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{L}^{\lambda,s}_{p,q}(\mathbb{R}^n)$.

Corollary 2.13 *Let $s \in \mathbb{R}$, $\lambda \geq 0$, $1 \leq p < +\infty$, $1 \leq q < +\infty$, $1 < p' \leq +\infty$ and $1 < q' \leq +\infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. We have*

$$F_{1,1}^{-s-\frac{\lambda}{q}+\frac{n}{p}}(\mathbb{R}^n) \hookrightarrow (\overset{\circ}{\mathcal{L}}{}^{\lambda,s}_{p,q}(\mathbb{R}^n))' \hookrightarrow F_{p',q'}^{-s-\frac{\lambda}{q}}(\mathbb{R}^n) \text{ provided } p \leq q \tag{2.2}$$

$$F_{1,1}^{-s-\frac{\lambda}{p}+\frac{n}{p}}(\mathbb{R}^n) \hookrightarrow (\overset{\circ}{\mathcal{L}}{}^{p,\lambda,s}(\mathbb{R}^n))' \hookrightarrow F_{p',q'}^{-s-\frac{\lambda}{p}}(\mathbb{R}^n) \text{ provided } q \leq p \tag{2.3}$$

in particular

$$F_{1,1}^{\frac{n}{2}-\frac{\lambda}{2}}(\mathbb{R}^n) \hookrightarrow (\overset{\circ}{\mathcal{L}}{}^{2,\lambda}(\mathbb{R}^n))' \hookrightarrow F_{2,q'}^{-\frac{\lambda}{2}}(\mathbb{R}^n) \quad \text{for any } q' \geq 2$$

We have the same injections for the dotted spaces.

All the previous results are proved in [6] and [7].

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