2002-Fez conference on Partial Differential Equations,
Electronic Journal of Differential Equations, Conference 09, 2002, pp 93–106.
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Existence and multiplicity of nontrivial solutions for double resonance semilinear elliptic problems *

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Abstract

We consider resonance problems at an arbitrary eigenvalue of the Laplacien. We prove the existence of nontrivial solutions for some semilinear elliptic Dirichlet boundary values problems. We also obtain two nontrivial solutions by using Morse theory.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, and let $g : \Omega\times\mathbb{R} \to \mathbb{R}$ be a nonlinear function satisfying the Carathéodory conditions. We consider the Dirichlet problem

$$\begin{gathered} -\Delta u = \lambda_k u + f(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.1}$$

where $0 < \lambda_1 < \lambda_2 \leq \dots \lambda_k \leq \dots$ is the sequence of eigenvalues of the problem

$$\begin{gathered} -\Delta u = \lambda u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega\,. \end{gathered}$$

Let $F(x,s)$ be the primitive $\int_0^s f(x,t)\,dt$, and

$$l_\pm(x) = \liminf_{s\to\pm\infty} \frac{f(x,s)}{s}, \quad k_\pm(x) = \limsup_{s\to\pm\infty} \frac{f(x,s)}{s}\,.$$

Let us assume that $0 \leq l_\pm(x) \leq k_\pm(x) \leq \lambda_{k+1} - \lambda_k$ uniformly for a.e. $x \in \Omega$.

There have been many papers concerning problem (1.1) at resonance in the situation where $l_\pm(x) \equiv 0$ or $k_\pm(x) \equiv \lambda_{k+1} - \lambda_k$; see for example [11, 1, 4, 9, 8, 17]. Some multiplicity theorems are obtained by using the topological degree technique and the variational methods, [2, 12, 13, 14, 5, 10, 7, 16].

In this paper, we are interested in finding nontrivial solutions of (1.1). First, we prove the existence of a nontrivial solution when $f(x,s)/s$ stays between 0

* *Mathematics Subject Classifications:* 58E05, 35J65, 56J20.
Key words: Variational elliptic problem, resonance, critical group, Morse theory, minimax method.

Published December 28, 2002.

and $\lambda_{k+1} - \lambda_k$ for large values of $|s|$. We replace the non-resonance conditions of Costa-Oliviera [8] by classical resonance conditions of Ahmad-Lazer-Paul.

Let us denote by $E(\lambda_j)$ the λ_j-eigenspace, and state the following hypotheses:

(F0) For all $R > 0$, $\sup_{|s|\leq R} |f(x,s)| \in L^\infty(\Omega)$.

(F1) $0 \leq f(x,s)s \leq (\lambda_{k+1} - \lambda_k)s^2$ for $|s| \geq r > 0$ and a.e. $x \in \Omega$

(F2) $\lim_{\|u\|\to\infty, u\in E(\lambda_k)} \int F(x,u(x))\,dx = +\infty$.

(F3) $$\lim_{\|u\|\to+\infty, u\in E(\lambda_{k+1})} \int [\frac{1}{2}(\lambda_{k+1} - \lambda_k)u^2(x) - F(x,u(x))]\,dx = +\infty.$$

(F4) $\limsup_{s\to\pm\infty} \dfrac{2F(x,s)}{s^2} \leq \beta < \lambda_1 - \lambda_k$ uniformly for a.e. $x \in \Omega$.

Now, we state the following result.

Theorem 1.1 *Under the assumptions* (F0)–(F4)*, Problem* (1.1) *has at least one nontrivial solution.*

Note that (F4) implies $f(x,0) = 0$ for a.e. $x \in \Omega$, so that (1.1) has the trivial solution in this case.

The second purpose of this paper is to study the existence of at least two nontrivial solutions of (1.1) when f is C^1, $f(x,s)s$ lies between 0 and $(\lambda_{k+1} - \lambda_k)s^2$ for large values of $|s|$, and

(F5) $f(x,s)s \to +\infty$, uniformly on Ω, as $s \to +\infty$ or as $s \to -\infty$.

(F6) $(\lambda_{k+1} - \lambda_k)s^2 - sf(x,s) \to +\infty$, uniformly on Ω, as $s \to +\infty$ or as $s \to -\infty$.

(F7) $f'(x,0) + \lambda_k \leq \lambda_1$ on Ω, with strict inequality $f'(x,0) + \lambda_k < \lambda_1$ holding on subset of positive measure.

Our main results now read as the follows.

Theorem 1.2 *Suppose that $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ such that $f(x,0) = 0$ for a.e. $x \in \Omega$, and* (F5)–(F7) *are satisfied. Then (1.1) has at least two nontrivial solutions.*

As an immediate consequence we obtain the corollary below, under the assumption that

(F8) $0 < f'(s) < \lambda_{k+1} - \lambda_k$ for $|s| \geq r > 0$.

Corollary 1.3 *Assume that $f(x,s) = f(s) \in C^1(\mathbb{R})$, $f(0) = 0$, $f'(0) + \lambda_k < \lambda_1$ and* (F8) *is satisfied. Then, (1.1) has at least two nontrivial solutions.*

Theorem 1.4 *Suppose that $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ such that $f(x,0) = 0$ for a.e. $x \in \Omega$. Assume that f satisfies (F5), (F6), $k = 1$, and (F8). Also assume that there is an $m \geq 2$ such that*

$$\lambda_m \leq f'(x,0) + \lambda_1 \leq \lambda_{m+1} \quad \textit{on } \Omega,$$

with strict inequality $\lambda_m < f'(x,0) + \lambda_1 < \lambda_{m+1}$ holding on subset of positive measure. Then, (1.1) has at least two nontrivial solutions.

Remarks

1. It is obvious to see that the conditions (F5)-(F6) imply the conditions (F2)-(F3).
2. Note that our multiplicity results are not covred by the results mentioned in [2, 12, 13, 14, 5, 10]. In fact, the condition $\sup_{t\in\mathbb{R}} f'(t) < \lambda_{k+1} - \lambda_k$ cited in [13, 5, 10] implies (F6).

The proof of Theorem 1.1 uses a variational argument and the general minimax theorem proved by Bartolo in [3]. In section 4, using Morse theory we compare the computed critical groups of Φ at the trivial critical point and the first nontrivial critical point given by minimax method. The existence of the second nontrivial solution is deduced from the calculation of the Leray-Schauder index of critical points.

2 Preliminaries

By a solution of (1.1) we mean a function $u \in H_0^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \nabla v - \lambda_k \int_\Omega uv - \int_\Omega f(x,u)v = 0, \mathrm{forall} v \in H_0^1(\Omega)$$

where $H_0^1(\Omega)$ is the usual Sobolev space obtained through completion of $C_c^\infty(\Omega)$ with respect to the norm induced by the inner product

$$\langle u, v\rangle = \int_\Omega \nabla u \nabla v, \quad u, v \in H_0^1(\Omega)\,.$$

For $u \in H_0^1(\Omega)$ define the functional

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2}\lambda_k \int u^2 - \int F(x,u).$$

It is well know that under a sub-critical growth condition on f, Φ is well defined on $H_0^1(\Omega)$, weakly lower semi-continuous and continuously Fréchet differentiable, with derivative

$$\Phi'(u)v = \int_\Omega \nabla u \nabla v - \lambda_k \int uv - \int f(x,u)v, \quad \text{for } u, v \in H_0^1(\Omega)$$

Thus, finding solutions of (1.1) is equivalent to finding critical points of the functional Φ.

To apply minimax methods for finding critical points of Φ, we need to verify that Φ satisfies a compactness condition of the Palais-Smale type which was introduced by Cerami.

Definition A functional $\Phi \in C^1(E,\mathbb{R})$, with E a real Banach space, is said to satisfy condition $(C)_c$, at the level $c \in \mathbb{R}$, if:
Every sequence $(u_n) \subset E$ such that

$$\Phi(u_n) \to c, \|u_n\|\Phi'(u_n) \to 0$$

possesses a convergent subsequence

It was shown in [3] that condition $(C)_c$ actually suffices to get a deformation theorem. Then, by standard minimax arguments [3], the following result was proved.

Theorem 2.1 *Suppose that $\Phi \in C^1(E,\mathbb{R})$, E a real Banach space, satisfies condition $(C)_c \forall c \in \mathbb{R}$ and that there exist a closed subset $S \subset E$ and $Q \subset E$ with boundary ∂Q satisfying the following conditions*

i) $\sup_{u\in\partial Q} \Phi(u) \leq \alpha < \beta \leq \inf_{u\in S} \Phi(u)$ *for some* $0 \leq \alpha < \beta$

ii) The intersection of S and ∂Q is not empty and for every $h \in C(E,E)$ such that $h/\partial Q = Id$, we have $h(Q) \cap S \neq \emptyset$.

iii) $\sup_{u\in Q} \Phi(u) < \infty$.

Then Φ possesses a critical value $c \geq \beta$.

Since we are going to apply the variational characterization of the eigenvalues, we shall decompose the space $H_0^1(\Omega)$ as $E = E_- \oplus E_k \oplus E_{k+1} \oplus E_+$, where E_- is the subspace spanned by the λ_j- eigenfunctions with $j < k$ and E_j is the eigenspace generated by the λ_j-eigenfunctions and E_+ is the orthogonal complement of $E_- \oplus E_k \oplus E_{k+1}$ in $H_0^1(\Omega)$ and we shall decompose for any $u \in H_0^1(\Omega)$ as following $u = u^- + u^k + u^+$ where $u^- \in E_-$, $u^k \in E_k$, $u^{k+1} \in E_{k+1}$ and $u^+ \in E_+$. We can verify easily that

$$\int |\nabla u|^2\,dx - \lambda_i \int |u|^2\,dx \geq \delta_i \|u\|^2 \quad \forall u \in \oplus_{j\geq i+1} E_j \tag{2.1}$$

$$\int |\nabla u|^2\,dx - \lambda_i \int |u|^2\,dx \leq -\delta_i \|u\|^2 \quad \forall u \in \oplus_{j\leq i} E_j. \tag{2.2}$$

where $\delta_i = \min\{1 - \frac{\lambda_i}{\lambda_{i+1}}, \frac{\lambda_i}{\lambda_{i-1}} - 1\}$.

Now, we present some technical lemmas.

Lemma 2.1 *Let $(u_n) \subset H_0^1(\Omega)$ and $(p_n) \subset L^\infty(\Omega)$ be sequences, and let A a nonnegative constant such that*

$$0 \le p_n(x) \le A \quad \text{a.e. in } \Omega \text{ and for all } n \in \mathbb{N}$$

and $p_n \rightharpoonup 0$ in the weak topology of L^∞, as $n \to \infty$. Then, there are subsequences $(u_n), (p_n)$ satisfying the above conditions, and there is a positive integer n_0 such that for all $n \ge n_0$,*

$$\int p_n u_n((u_n^- + u_n^k) - (u_n^{k+1} + u_n^+))\,dx \ge \frac{-\delta_k}{2}\|u_n^+ + u_n^{k+1}\|^2. \tag{2.3}$$

Proof: Since $p_n \ge 0$ a.e. in Ω, we see that

$$\begin{aligned} &\int p_n u_n((u_n^- + u_n^k) - (u_n^{k+1} + u_n^+)) \\ &\ge -\int p_n (u_n^+ + u_n^{k+1})^2\,dx \\ &\ge -\Big[\int p_n \big(\frac{u_n^+ + u_n^{k+1}}{\|u_n^+ + u_n^{k+1}\|}\big)^2\,dx\Big]\|u_n^+ + u_n^{k+1}\|^2. \end{aligned} \tag{2.4}$$

Moreover, by the compact imbedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ and $p_n \rightharpoonup 0$ in the weak* topology of L^∞, when $n \to \infty$, then there are subsequences $(u_n), (p_n)$ such that

$$\int p_n \Big(\frac{u_n^+ + u_n^{k+1}}{\|u_n^+ + u_n^{k+1}\|}\Big)^2\,dx \to 0.$$

Therefore, there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$ we have

$$\int p_n \Big(\frac{u_n^+ + u_n^{k+1}}{\|u_n^+ + u_n^{k+1}\|}\Big)^2\,dx \le \frac{\delta_k}{2}. \tag{2.5}$$

Combining inequalities (2.4) and (2.5), we get inequality (2.3). $\square$

Definition A sequence (u_n) is said to be a $(C)_c$ sequence, at the level $c \in \mathbb{R}$, if there is a sequence $\epsilon_n \to 0$, such that

$$\Phi(u_n) \to c \tag{2.6}$$

$$\|u_n\|\langle \Phi'(u_n), v\rangle_{H_0^1, H^{-1}} \le \epsilon_n\|v\| \quad \forall v \in H_0^1. \tag{2.7}$$

Lemma 2.2 *Let $(u_n) \subset H_0^1(\Omega)$ be a (C) sequence.*

1. *If $f_n(x) = \frac{f(x,u_n(x))}{u_n(x)}\chi_{[|u_n(x)| \ge r_\epsilon]} \rightharpoonup 0$ in the weak* topology of L^∞, as $n \to \infty$. Then, there is subsequence (u_n) such that $(\|u_n^- + (u_n^+ + u_n^{k+1})\|)_n$ is uniformly bounded in n.*

2. *If $f_n(x) = \frac{(\lambda_{k+1} - \lambda_k)u_n(x) - f(x,u_n(x))}{u_n(x)}\chi_{[|u_n(x)| \ge r_\epsilon]} \rightharpoonup 0$ in the weak* topology of L^∞, as $n \to \infty$. Then, there is subsequence (u_n) such that $(\|u_n^- + (u_n^+ + u_n^k)\|)_n$ is uniformly bounded in n.*

Proof: Since $(u_n)_n \subset H^1_0$ be a (C) sequence, (2.6) and (2.7) are satisfied. Now, we prove that the sequence $(\|u_n^- + u_n^+ + u_n^{k+1}\|)_n$ is uniformly bounded in n. Take $v = (u_n^- + u_n^k) - (u_n^+ + u_n^{k+1})$ in (8), $p_n(x) = f_n(x)$, and

$$\begin{aligned}\Lambda = \Big\{ &- \int |\nabla u_n^-|^2 + \lambda_k \int |u_n^-|^2\,dx + \int |\nabla(u_n^+ + u_n^{k+1})|^2 \\ &- \lambda_k \int |u_n^+ + u_n^{k+1}|^2\,dx + \int p_n u_n((u_n^- + u_n^k) - (u_n^{k+1} + u_n^+))\,dx \Big\} \\ \Gamma = \Big\{ &\epsilon_n + \int h((u_n^+ + u_n^{k+1}) - u_n^-)\,dx \\ &+ \int_{|u_n(x)|\le r_\epsilon} |f(x,u_n(x))||(u_n^+ + u_n^{k+1}) - (u_n^- + u_n^k)|\,dx \Big\}.\end{aligned}$$

Then $\Lambda \le \Gamma$. By the Poincaré inequality, from (2.1), (2.2), (2.3), and $\Lambda \le \Gamma$, it follows that there exist constants A_ϵ and B_ϵ such that

$$\frac{\delta_k}{2}\|u_n^- + (u_n^+ + u_n^{k+1})\|^2 \le \epsilon_n + A_\epsilon\|u_n^- + (u_n^+ + u_n^{k+1})\| + B_\epsilon.$$

This gives that $(\|u_n^- + (u_n^+ + u_n^{k+1})\|)_n$ is uniformly bounded in n. The same proof of eventuality 2 are given by taking $v = (u_n^+ + u_n^{k+1}) - (u_n^- + u_n^k)$ and

$$p_n(x) = f_n(x) = \frac{(\lambda_{k+1} - \lambda_k)u_n(x) - f(x,u_n(x))}{u_n(x)}\chi_{[|u_n(x)|\ge r_\epsilon]}.$$

Lemma 2.3 *1. Let $(u_n) \subset H^1_0(\Omega)$ such that $\|u_n^- + (u_n^+ + u_n^{k+1})\|$ is uniformly bounded in n and there exists A such that if $A \le \Phi(u_n)$, then*

$$\int F(x, \frac{u_n^k}{2})\,dx \le M.$$

2. Let $(u_n) \subset H^1_0(\Omega)$ such that $\|u_n^- + u_n^+ + u_n^k\|$ is uniformly bounded in n and there exists A such that if $\Phi(u_n) \le A$, then

$$\int [\frac{\lambda_{k+1} - \lambda_k}{2}(\frac{u_n^{k+1}}{2})^2 - F(x, \frac{u_n^{k+1}}{2})]\,dx \le M.$$

Proof: ¿From (2.6), and Poincaré inequality, we have

$$\begin{aligned}\int F(x, \frac{u_n^k}{2})\,dx \le &A + \int [F(x, \frac{u_n^k}{2}) - F(x,u_n)]\,dx \\ &+ \frac{1}{2}\|u_n^+ + u_n^{k+1} + u_n^-\|^2 + \frac{1}{\sqrt{\lambda_1}}\|h\|_{L^2}\|u_n^+ + u_n^{k+1} + u_n^-\|.\end{aligned} \tag{2.8}$$

However, by the mean value theorem, we get for a.e. $x \in \Omega$ an $t = t(x) \in [0,1]$ such that

$$\begin{aligned}
&\int [F(x, \frac{u_n^k}{2}) - F(x, u_n)]\, dx \\
&= \int f(x, t\frac{u_n^k}{2} + (1-t)u_n)(\frac{u_n^k}{2} - u_n)\, dx \\
&= \int_{|t\frac{u_n^k}{2}+(1-t)u_n| \leq r\epsilon} f(x, t\frac{u_n^k}{2} + (1-t)u_n)\, dx \\
&\quad + \int_{|t\frac{u_n^k}{2}+(1-t)u_n| \geq r\epsilon} \frac{f(x, t\frac{u_n^k}{2} + (1-t)u_n)}{t\frac{u_n^k}{2} + (1-t)u_n} \big[t(\frac{u_n^k}{2} - u_n)^2 + (\frac{u_n^k}{2} - u_n)u_n\big]\, dx.
\end{aligned} \tag{2.9}$$

So that using (2.9) and the Poincaré inequality again we have

$$\begin{aligned}
&\int [F(x, \frac{u_n^k}{2}) - F(x, u_n)]\, dx \\
&\leq \frac{2}{\sqrt{\lambda_1}} \| \sup_{|s|\leq r\epsilon} |f(x,s)| \|_{L^2} \|u_n^- + u_n^+ + u_n^{k+1}\| \\
&\quad + r_\epsilon \| \sup_{|s|\leq r\epsilon} |f(x,s)| \|_{L^1} + \frac{\lambda_{k+1} - \lambda_k + \epsilon}{4\lambda_1} \|u_n^- + u_n^+ + u_n^{k+1}\|^2.
\end{aligned} \tag{2.10}$$

¿From (2.8) and (2.10), there exists $M > 0$ such that

$$\int F(x, \frac{u_n^k}{2})\, dx \leq M.$$

3 Proof of Theorem 1.1

To apply Theorem 2.1, we shall do separate studies of the "compactness" of Φ and its "geometry". First, we prove that Φ satisfies the Cerami condition.

Lemma 3.1 *Φ satisfies the $(C)_c$ condition on $H_0^1(\Omega)$, for all $c \in \mathbb{R}$.*

Proof: Let $(u_n)_n \subset H_0^1$ be a $(C)_c$ sequence, i.e

$$\Phi(u_n) \to c$$
$$\|u_n\| \langle \Phi'(u_n), v \rangle_{H_0^1, H^{-1}} \leq \epsilon_n \|v\| \quad \forall v \in H_0^1,$$

where A is a constant and $\epsilon_n \to 0$. It clearly suffices to show that $(u_n)_n$ remains bounded in H_0^1. Assume by contradiction. Defining $z_n = \frac{u_n}{\|u_n\|}$, we have $\|z_n\| = 1$ and, passing if necessary to a subsequence, we may assume that $z_n \rightharpoonup z$ weakly in H_0^1, $z_n \to z$ strongly in $L^2(\Omega)$ and $z_n(x) \to z(x)$ a.e. in Ω. As in the proof of Lemma 2.1, we put

$$f_n(x) = \frac{f(x, u_n(x))}{u_n(x)} \chi_{[|u_n(x)| \geq r]}$$

and $l \in L^\infty$ such that $f_n \to l$ in the weak* topology of L^∞, as $n \to \infty$. where the L^∞-function l satisfies

$$0 \le l(x) \le \lambda_{k+1} - \lambda_k. \tag{3.1}$$

Set $m(x) = l(x) + \lambda_k$, by (2.6), we have

$$\frac{\langle \Phi'(u_n), u_n\rangle}{\|u_n\|^2} \to 1 - \int m(x)z(x)\,dx = 0.$$

So that, $z \not\equiv 0$. In other words , we verify easily that z satisfies

$$\begin{aligned} -\Delta z = m(x)z \quad \text{in } \Omega \\ z = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.2}$$

We now distinguish three cases: i)$\lambda_k < m(x)$ and $m(x) < \lambda_{k+1}$ on subset of positive measure; ii) $m(x) \equiv \lambda_k$; iii) $m(x) \equiv \lambda_{k+1}$.

In case i), we have $\lambda_k(m) < 1$ and $\lambda_{k+1}(m) > 1$. This contradicts that 1 is an eigenvalue of problem (3.2). On the other hand, by (F2), (F3), lemmas 2.2 and 2.3 the cases ii) and iii) are not possible. The proof is complete. □

Lemma 3.2 *Under hypothesis of Theorem 1.1, the functional Φ has the following properties:*

i) $\Phi(v) \to -\infty$*, as* $\|v\| \to \infty$*,* $v \in E_k \oplus E_- = V$

ii *There is an $r > 0$ such that $\Phi \le \alpha$ on $\partial B_r(0)$.*

Proof i) Assume by contradiction there exist a constant B and a sequence $(v_n) \subset V$ with $\|v_n\| \to \infty$ such that

$$B \le \Phi(v_n) \le -\delta\|v_n^-\|^2.$$

Therefore, $\|v_n^-\|$ is bounded and by Lemma 2.3, we obtain

$$\liminf_{n\to\infty} \int F(x, \frac{v_n^k}{2})\,dx \le \text{constant}.$$

This is a contradiction with assumption (F2).
ii) By (F4), there exists $\delta > 0$ such that

$$F(x,s) \le \frac{\beta'}{2}|s|^2 \quad \text{for } |s| \le \delta \text{ and a.e. } x \in \Omega \tag{3.3}$$

where $\beta' \in]\beta, \lambda_1 - \lambda_k[$.

On the other hand, (F0) and (F1) implies that there exist $\mu > 2$ and $A > 0$ such that

$$F(x,s) \le A|s|^\mu \quad \text{for } |s| \le \delta \text{ and a.e. } x \in \Omega \tag{3.4}$$

Thus, from (3.3) and (3.4) we deduce

$$F(x,s) \leq \frac{\beta'}{2}|s|^2 + A|s|^\mu \quad \text{for } |s| \leq \delta \text{ and a.e. } x \in \Omega \tag{3.5}$$

Let $u \in H_0^1(\Omega)$, via (3.5) and using the Poincaré inequality $\lambda_1 \int u^2 \leq \|u\|^2$ and the Sobolev inequality $\int u^\mu \leq K\|u\|^\mu$, we obtain

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda_k}{2}\int u^2 - \int F(x,u)\,dx \\
&\geq \frac{\lambda_1 - \lambda_k}{2}\int u^2 - \frac{\beta'}{2}\int u^2 - A\int u^\mu \\
&\geq \frac{\lambda_1 - \lambda_k - \beta'}{2\lambda_1}\|u\|^2 - AK\|u\|^\mu
\end{aligned}$$

Since $\mu > 2$, we obtain the estimates

$$\Phi(u) \geq (\frac{\lambda_1 - \lambda_k - \beta'}{2\lambda_1} - AK\|u\|^{\mu-2})\|u\|^2 \geq \frac{\lambda_1 - \lambda_k - \beta'}{4\lambda_1}r^2 = \nu \geq 0 \quad \forall \|u\| = r$$

with $r = (\frac{\lambda_1 - \lambda_k - \beta'}{4\lambda_1 AK})^{\frac{1}{\mu-2}}$, wich proves the lemma. □

Proof of Theorem 1.1 The first assertion of lemma 3.2 implies that there exists $v \in H_0^1$ such that $\Phi(tv) \to -\infty$ as $t \to \infty$. In view of lemmas 3.1 and 3.2, we may apply Theorem 2.1 letting $S = \{u \mid \|u\| = r\}$ and $Q = \{tv \mid v \in E_- \oplus E_k, 0 \leq t \leq t_0\}$, with $t_0 > 0$ being such that $\Phi(tv) \leq 0$. It follows that the functional Φ has a critical value $c \geq \beta > 0$ and, hence, problem (1.1) has a nontrivial solution $u \in H_0^1$.

4 Existence of multiple nontrivial solutions

In this section, we consider the existence of multiple nontrivial solutions of problem (1.1). It is well known That, under the conditions of f in Theorem 1.2 or Theorem 1.4, Φ is a C^2 functional with

$$\begin{aligned}
\Phi'(u)v &= \int_\Omega \nabla u \nabla v - \lambda_k \int uv - \int f(x,u)v, \quad \text{for } u, v \in H_0^1(\Omega), \\
\Phi''(u).v.w &= \int_\Omega \nabla w \nabla v - \lambda_k \int wv - \int f'(x,u)wv, \quad \text{for } u, w, v \in H_0^1(\Omega).
\end{aligned}$$

Set, $\Phi^c = \{u \in H_0^1(\Omega) \mid \Phi(u) \leq c\}$. Denote by $H_q(X,Y)$ the q-th relative singular homology group with integer coefficient. The critical groups of Φ at an isolated critical point u with $\Phi(u) = c$ are defined by

$$C_q(\Phi, u) = H_q(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\}); q = 0, 1, 2, \ldots,$$

where U is a closed neighborhood of u.

We will use the notation $\deg(\Phi', U, 0)$ for the Leray-Schauder degree of Φ with respect to the set U and the value 0. Denote also by $\text{index}(\Phi', u)$ the Leray-Schauder index of Φ' at critical point u. Recall that this quantity is defined as $\lim_{r\to 0}\deg(\Phi', B_r(u), 0)$, if this limit exists, where $B_r(u)$ is the ball of radius r centered at u.

Now, we will prove the following lemmas.

Lemma 4.1 *There is an $r > 0$ and an $\alpha > 0$ such that $\Phi \geq \alpha$ on $\partial B_r(0)$.*

Proof: Using the Poincare's inequality, we have for every $v \in H_0^1(\Omega)$,

$$\Phi''(0).v.v = \int_\Omega |\nabla v|^2 - \int [\lambda_k + f'(x,0)]v^2\, dx.$$

Put $m(x) = \lambda_k + f'(x,0)$, and so in the case where $m(x) \leq 0$, then

$$\Phi''(0).v.v \geq \|v\|^2.$$

In the case where $m(x) > 0$ on subset of positive measure we have $\lambda_1(m) > 1$. It follows from the Poincaré's inequality that

$$\Phi''(0).v.v \geq (1 - \frac{1}{\lambda_1(m)}\|v\|^2.$$

It follows that 0 is a non-degenerate critical point of Φ with the Morse Index of Φ at 0 is 0, and so it is well known that $C_q(\Phi, 0) = \delta_{q0}\mathbb{Z}$.

For sufficiently small $\rho > 0$ we have for $\|u\| \leq \rho$

$$\Phi(u) = \frac{1}{2}\Phi"(0).u.u + o(\|u\|^2) \leq \frac{1}{2}[\frac{\Phi"(0).u.u + o(\|u\|^2)}{\|u\|^2}]\|u\|^2$$

with $\frac{o(\|u\|^2)}{\|u\|^2} \to 0$ as $\|u\| \to 0$. For $r < \rho$, there holds

$$\Phi(u) \geq \frac{1}{4}(1 - \xi(m))r^2, \|u\| = r$$

with

$$\xi(m) = \begin{cases} \frac{1}{\lambda_1(m)} & \text{if } m > 0 \text{ on subset of positive measure} \\ 0 & \text{if } m \leq 0 \end{cases}$$

Since $\lambda_1(m) > 1$, clearly there exists $\alpha > 0$ such that $\Phi(u) \geq \alpha$ on $\partial B_r(0)$.

Lemma 4.2 *There exists at least one nontrivial critical point u_0 of Φ such that $C_q(\Phi, u_0) = \delta_{q1}\mathbb{Z}$.*

Proof: It easy to see that the conditions (F4), (F5) follow from conditions (F2), (F3). Then, by lemma 3.2, Φ is anticoercive on $E^- \oplus E^k$ we can find a w such that $\|w\| > R$ and $\Phi(w) \leq 0$. The compactness condition and geometry of Φ assure that $c = \inf_{h\in\Gamma} \max_{0\leq t\leq 1} \Phi(h(t))$ is a critical value for Φ, where $\Gamma = \{h \in C([0,1], H_0^1) \mid h(0) = 0, h(1) = w\}$, and $c \geq \alpha$. Then, there exists a critical point nontrivial, u_0, of mountain pass type such that $C_1(\Phi, u_0) \neq 0$.

If u_0 is a non-degenerate critical point with its Morse index is k and then $C_q(\Phi, u_0) = \delta_{qk}\mathbb{Z}$. Since $C_1(\Phi, u_0) \neq 0$ it follows that $C_q(\Phi, u_0) = \delta_{q1}\mathbb{Z}$.

If u_0 is degenerate, by Gromoll-Meyers theorem (cf. [16]), we have

$$C_q(\Phi, u_0) = 0 \quad \text{for } q < k \ \text{ and } \ q > j + k$$

with $j = \dim Ker\Phi''(u_0)$. Since $C_1(\Phi, u_0) \neq 0$, it follows that $k \leq 1$.

If $k = 1$, by the shifting theorem and the critical group characterization of the local minimum, we have $C_q(\Phi, u_0) = \delta_{q1}\mathbb{Z}$.

If $k = 0$, we have

$$\Phi''(0).v.v = \int_\Omega |\nabla v|^2 - \int [\lambda_k + f'(x,0)]v^2\, dx \geq 0.$$

for every $v \in H_0^1(\Omega)$ and according to a result of Manes-Micheletti (cf. [15]), we have $j = \dim Ker\Phi''(u_0) = 1$. Consequently, from the shifting theorem and the critical group characterization of the local maximum, we have $C_q(\Phi, u_0) = \delta_{q1}\mathbb{Z}$.

Proof of Theorem 1.2 By the Riesz representation theorem we can write

$$\langle \Phi'(u), v\rangle = \langle u, v\rangle - \langle Nu, v\rangle, \quad \text{for all } u, v \in H_0^1(\Omega)$$

where $\langle u, v\rangle = \int_\Omega \nabla u \nabla v$ and $\langle Nu, v\rangle = \int [\lambda_k u + f(x,u)]v\, dx$. So that, $\Phi' = I - N$ and By the Sobolev embedding theorem, N is compact. We see that Φ' has the form Identity-compact, so that leary-shauder techniques are applicable. In a similar way we can define a compact map $T : H_0^1(\Omega) \to H_0^1(\Omega)$ by $< Tu, v >= \int(\lambda_k + \mu)uv$. It is well know for $0 < \mu < \lambda_{k+1} - \lambda_k$ that

$$\deg(I - T, B_R(0), 0) = (-1)^m$$

where m represents the dimension of $V^- \oplus V^k$ and $B_R(0)$ the ball in $H_0^1(\Omega)$ of radius $R > 0$. Suppose $\{0, u_0\}$ is the critical set of Φ and let $R > 0$ such that $\{0, u_0\} \subset B_R(0)$. According to [7, theorem 3.2, capt. II] and the addition property of Leray-Schauder degree imply

$$\begin{aligned}
\deg(\Phi', B_R(0), 0) &= \text{index}(\Phi', 0) + \text{index}(\Phi', u_0)\\
&= \sum_{q=0}^{\infty}(-1)^q \dim C_q(\Phi, 0) + \sum_{q=0}^{\infty}(-1)^q \dim C_q(\Phi, u_0)\\
&= \dim C_0(\Phi, 0) + \dim C_1(\Phi, u_0)\\
&= 1 - 1 = 0.
\end{aligned}$$

But this contradicts the fact that

$$\deg(\Phi', B_R(0), 0) = (-1)^m.$$

Indeed, in [17] Omari and Zanolin establish an a priori bound for the solution set of the family equation

$$-(\Delta u + \lambda_k u) = (1-t)\mu u + tf(x,u), \quad x \in \Omega, t \in [0,1]$$
$$u = 0 \quad \text{on } \partial\Omega$$

with, $0 < \mu < \lambda_{k+1} - \lambda_k$. The homotopy invariance of Leray-Schauder degree implies that $\deg(\Phi', B_R(0), 0) = \deg(I - T, B_R(0), 0) = (-1)^m$.

Proof of corollary 1.1 ¿From (F8), there are s_1, s_2 such that $0 < f'(s_1)$, $f'(s_2) < \lambda_{k+1} - \lambda_k = \mu$. Thus, by continuity, there exist $s_3 \in \mathbb{R}$, $\varepsilon > 0$, $\delta > 0$ such that

$$\varepsilon \le f'(s) \le \mu - \varepsilon$$

for every $s \in [s_3 - \delta, s_3 + \delta]$. Let $s_3 \ge 0$ be and take $s > s_3 + \delta$, we have

$$\varepsilon\delta \le f(s) = \int_0^{s_3} f(t)\,dt + \int_{s_3}^{s_3+\delta} f(t)\,dt + \int_{s_3+\delta}^{s} f(t)\,dt \le \mu s - \varepsilon\delta.$$

Hence, the corollary follows. Similarly, we have F_5) and F_6), if $s_3 < 0$. □

For the proof of Theorem 1.4, we will need the following lemma.

Lemma 4.3 *The functional Φ is coercive in $E(\lambda_1)^\perp = W$. Moreover, the functional Φ has at least nontrivial critical point of mountain pass type.*

Proof. Suppose by contradiction that Φ is not coercive in W. Thus, there is some constant B and some sequence $(w_n) \subset W$, with $\|w_n\| \to \infty$, such that

$$\Phi(w_n) = \frac{1}{2}\int \left[|\nabla w_n|^2\,dx - \lambda_{k+1} w_n^2\right] + \int \left[\frac{1}{2}(\lambda_{k+1} - \lambda_k) w_n^2 - F(x, w_n)\right] dx \le B$$

This implies that $\|w_n^+\|$ is bounded, and so Lemma 2.3 gives us a contradiction from (F3). Therefore, Φ is bounded from below in W. Hence, since Φ is weakly lower semi-continuous and coercive on W, Φ attains the infimum $\beta = \inf_W \Phi$.

On the other hand, since Φ is anticoercive on $E(\lambda_1)$, we can find $t_0 > 0$ that $\Phi(\pm t_0 \varphi_1) < \beta$. In view of Lemma 3.1, we may apply Theorem 2.1 to get that

$$c = \inf_{h \in \Gamma} \max_{0 \le t \le 1} \Phi(h(t)) \ge \beta$$

is a critical value for Φ, where $\Gamma = \{h \in C([0,1], H_0^1) \mid h(0) = -t_0\varphi_1, h(1) = t_0\varphi_1\}$. As in the proof of Theorem 1.2, there exists a critical point u_0 of Φ such that $C_q(\Phi, u_0) = \delta_{q1}\mathbb{Z}$.

Now we will prove that 0 is a nondegenerate critical point with Morse index of Φ at 0 equal d.

Claim. The Morse index of Φ at 0 is d, with $d \geq 2$.

In fact, for every $v \in \oplus_{j \leq m} E^j$ we have

$$\Phi''(0).v.v = \int_\Omega |\nabla v|^2 - \int [\lambda_1 + f'(x,0)]v^2\, dx \leq \int \lambda_m - [\lambda_1 + f'(x,0)]v^2\, dx.$$

On the other hand, for every $v \in \oplus_{j \geq m+1} E^j$ we have

$$\Phi''(0).v.v \geq \int \lambda_{m+1} - [\lambda_1 + f'(x,0)]v^2\, dx.$$

¿From (F8) we obtain that 0 is a nondegenerate critical point of Φ with Morse index is d, where d is the the dimension of $\oplus_{j \leq m} E^j$ and clearly d is larger than 2. The last claim implies that $C_q(\Phi, 0) = \delta_{qd}\mathbb{Z}$, and so u_0 is a nontrivial critical point of mountain pass type.

Proof of Theorem 1.4 As in the proof of Theorem 1.2, we assume $\{0, u_0\}$ is the critical set of Φ. Let $R > 0$ such that $\{0, u_0\} \subset B_R(0)$. According to [7, Theorem 3.2, Capt.II] and the addition property of Leray-Schauder degree imply

$$\begin{aligned}
\deg(\Phi', B_R(0), 0) &= \operatorname{index}(\Phi', 0) + \operatorname{index}(\Phi', u_0) \\
&= \sum_{q=0}^{\infty} (-1)^q \dim C_q(\Phi, 0) + \sum_{q=0}^{\infty} (-1)^q \dim C_q(\Phi, u_0) \\
&= \dim C_0(\Phi, 0) + \dim C_1(\Phi, u_0) \\
&= (-1)^d - 1.
\end{aligned}$$

Observe that the right hand side of this equality is even, which contradicts the claim given in the proof of Theorem 1.2. Hence, problem (1.1) has at least two nontrivial solutions. □

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