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A polyharmonic analogue of a Lelong theorem and polyhedric harmonicity cells *

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Abstract

We prove a polyharmonic analogue of a Lelong theorem using the topological method presented by Siciak for harmonic functions. Then we establish the harmonicity cells of a union, intersection, and limit of domains of $\mathbb{R}^n$. We also determine explicitly all the extremal points and support hyperplanes of polyhedric harmonicity cells in $\mathbb{C}^2$.

1 Introduction

Throughout this paper, D denotes a domain (a connected open) in $\mathbb{R}^n$ with $n \geq 2$, where D and ∂D are not empty. Since 1936, p-polyharmonic functions in D have been used in elasticity calculus [14]. These functions are C^∞-solutions of the partial differential equation

$$\Delta^p f(x) = \sum_{|\alpha|=p} \frac{p!}{\alpha!} \frac{\partial^{2|\alpha|} f(x)}{\partial x_1^{2\alpha_1} \dots \partial x_n^{2\alpha_n}} = 0, \quad p \in N^*, \quad x \in D.$$

To study the singularities of these functions in D, Aronzajn [1, 2] considered the connected component $\mathcal{H}(D)$, containing D, of the open set $\mathbb{C}^n \setminus \cup_{t\in\partial D}\Gamma(t)$, where $\Gamma(t) = \{w \in \mathbb{C}^n : \sum_{j=1}^n (w_j - t_j)^2 = 0\}$. $\mathcal{H}(D)$ is called the harmonicity cell of D. Lelong [12, 13] proved that $\mathcal{H}(D)$ coincides with the set of points $w \in \mathbb{C}^n$ such that there exists a path γ satisfying: $\gamma(0) = w$, $\gamma(1) \in D$ and $T[\gamma(\tau)] \subset D$ for every τ in $[0, 1]$, where T is the Lelong transformation, mapping points $w = x + iy \in \mathbb{C}^n$ to $(n-2)$-spheres $\mathbb{S}^{n-2}(x, \|y\|)$ of the hyperplane of $\mathbb{R}^n$ defined by: $\langle t - x, y\rangle = 0$. This work can be divided into three sections: the first one treats a result on polyharmonic functions, the second some general properties on $\mathcal{H}(D)$, and the last one deals with a geometrical description of polyhedric harmonicity cells in $\mathbb{C}^2$.

Pierre Lelong [11] proved in addition that for every bounded domain D of $\mathbb{R}^n$, there exists a harmonic function f in D such that its domain of holomorphy

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(X_f, Φ) over $\mathbb{C}^n$satisfies $\Phi(X_f) = \mathcal{H}(D)$, see also [4]. A concise proof of this result is given in Siciak's paper [16] in the case of the Euclidean ball $B_n^r = \{x \in \mathbb{R}^n; \|x\| < 1\}$. In [5], we established that the former method can be applied to arbitrary domains. Also, V.Avanissian noted in [4] that the equality: $(X_f, \Phi) = (\mathcal{H}(D), Id)$ holds in the following cases: D is starshaped with respect to some point x_0 of D, or D is a C-domain (that is D contains the convex hull of any $(n-2)$ dimensional-sphere included in D), or $D \subset \mathbb{R}^n$ with n even and $n \geq 4$. The object of Section 2 is to use a topological argument [16] to prove an analogous result for polyharmonic functions in D. As a consequence of this generalization we shall get

> For every integer $1 \leq p \leq [\frac{n}{2}]$ and suitable domain D (say D is a C-domain, or in particular a convex domain), the harmonicity cell $\mathcal{H}(D)$ is nothing else but the greatest (in the inclusion sense) domain of $\mathbb{C}^n$ whose trace on $\mathbb{R}^n$ is D and to which all p-polyharmonic functions in D extends holomorphically.

In Section 3, we establish the harmonicity cell of an intersection, a union, and a limit of domains of $\mathbb{R}^n$, $n \geq 2$. We give next in Section 4 some results about plane domains, prove the existence of polyhedric harmonicity cells in $\mathbb{C}^2$, and we calculate all extremal points of the harmonicity cell of a regular polygon. For an arbitrary convex polygon P_n, with n edges, we show that $\mathcal{H}(P_n)$ has exactly $2n$ faces in $\mathbb{R}^4$ completely determined by means of the n support lines of P_n. It is well known by [10] that if we are given a complex analytic homeomorphism $f : D_1 \to D_2$, where D_1, D_2 are domains of $\mathbb{R}^2$, D_1, D_2 not equal to $\mathbb{R}^2$ and $\mathbb{R}^2 \simeq \mathbb{C}$, then $\mathcal{H}(D_1)$ and $\mathcal{H}(D_2)$ are analytically homeomorphic in $\mathbb{C}^2$. The holomorphic map $Jf : \mathcal{H}(D_1) \to \mathcal{H}(D_{\mathbf{2}})$ defined by $w \mapsto w'$ with:

$$w_1' = \frac{f(w_1 + iw_2) + \overline{f(\overline{w}_1 + i\overline{w_2})}}{2}, \quad w_2' = \frac{f(w_1 + iw_2) - \overline{f(\overline{w}_1 + i\overline{w_2})}}{2i}$$

realizes this homeomorphism.

In proposition 4.4, we show the continuity, according to the compact uniform topology, of the above Jarnicki extension $f \mapsto Jf$ and estimate $\|(Jf)(w)\|, w \in \mathcal{H}(D)$ by means of $\sup_{z\in D} |f(z)|$. As applications, we find the harmonicity cells of half strips and arbitrary convex plane polygonal domains (owing to an explicit calculation of their support function).

2 A polyharmonic analogue of Lelong theorem

Recall that any polyharmonic function u in D, being in particular analytic in D, has a holomorphic continuation $\widetilde{u}$ in a corresponding domain D^u of $\mathbb{C}^n$ whose trace with $\mathbb{R}^n$ is D. Therefore, given any integer p $(0 < p < +\infty)$ and any domain D of $\mathbb{R}^n$, one can associate a domain $\mathcal{NH}(D)$ of $\mathbb{C}^n$ (depending on D only) such that the whole class $H^p(D)$, of all p-polyharmonic functions in D, extends holomorphically to $\mathcal{NH}(D)$. This last complex domain, called the kernel of $\mathcal{H}(D)$, coincides with the set of all $z \in \mathcal{H}(D)$ satisfying $C_h[T(z)] \subset D$, where $C_h[T(z)]$ denotes the convex hull of $T(z)$. For more details see [4].

Making use of a topological argument appearing in [16], we will show now the following theorem.

Theorem 2.1 *Let D be a bounded domain of $\mathbb{R}^n$, $n \geq 2$, $D \neq \emptyset$, $\partial D \neq \emptyset$, and $\mathcal{H}(D)$ its harmonicity cell. Then for all integer $1 \leq p \leq [\frac{n}{2}]$, ($[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$) and all domains $\widetilde{\Omega} \supset \mathcal{H}(D)$ the problem for the $2p$-order linear partial differential operator Δ^p*

$$\begin{aligned} \Delta^p u = 0 \quad & \textit{in } D \\ \overline{D_1}\widetilde{u} = \cdots = \overline{D_n}\widetilde{u} = 0 \quad & \textit{in } \mathcal{H}(D) \end{aligned}$$

has a solution $h \in H^p(D)$ which cannot be holomorphically continued in $\widetilde{\Omega}$. Here $\Delta = \Delta_x = \partial_{x_1x_1} + \cdots + \partial_{x_nx_n}$ is the usual Laplacian of $\mathbb{R}^n$, $\overline{D_j} = \frac{\partial}{\partial \overline{z_j}}$, $j = 1, 2, \ldots, n$.

Proof Let $\xi \in \partial\mathcal{H}(D)$ and $1 \leq p \leq [\frac{n}{2}]$. Firstly, we will construct explicitly a p-polyharmonic function h_ξ in D whose holomorphic continuation $\widetilde{h_\xi}$ in $\mathfrak{N}\mathcal{H}(D)$ extends to the whole of $\mathcal{H}(D)$; however $\widetilde{h_\xi}$ cannot be holomorphically continued in a neighborhood of ξ. Next, we will deduce by a topological reasoning the existence of a p-polyharmonic function h in D such that $\mathcal{H}(D)$ is the domain of holomorphy of $\widetilde{h}$.

Construction of h_ξ. 1) $D \subset \mathbb{C} \simeq \mathbb{R}^2$: by [12], the boundary point ξ belongs to some isotropic cone of vertex a $t \in \partial D$, i.e. $\xi \in \Gamma(t)$, or $t \in T(\xi) = \{\xi_1 + i\xi_2, \overline{\xi_1} + i\overline{\xi_2}\}$.

a) If $t = \xi_1 + i\xi_2$, we consider the function

$$\widetilde{h_\xi}(z) = \mathrm{Ln}\left\{[(\xi_1 + i\xi_2) - (z_1 + iz_2)]\overline{[(\xi_1 + i\xi_2) - (\overline{z_1} + i\overline{z_2})]}\right\},$$

where the branch is chosen such that $\widetilde{h_\xi}$ is real in D. Note that $\widetilde{h_\xi}$ is holomorphic in $\mathcal{H}(D)$, its restriction $(\widetilde{h_\xi}|D)(x) = 2\,\mathrm{Ln}\,\|x - t\|$, where $x = x_1 + ix_2$ is harmonic in D, and $\lim_{z\to\xi} |\widetilde{h_\xi}(z)| = \infty$. Hence $\widetilde{h_\xi}$ cannot be holomorphically continued beyond ξ.

b) If $t = \overline{\xi_1} + i\overline{\xi_2}$, the function

$$\widetilde{h_\xi}(z) = \mathrm{Ln}\left\{[(\overline{\xi_1} + i\overline{\xi_2}) - (z_1 + iz_2)]\overline{[(\overline{\xi_1} + i\overline{\xi_2}) - (\overline{z_1} + i\overline{z_2})]}\right\},$$

satisfies the same requirements of (a). 2) $D \subset \mathbb{R}^n$, $n \geq 3$:

a) Suppose n even ≥ 4. There exists by [12] a point $t \in \partial D$ such that $\sum_{j=1}^{n}(\xi_j - t_j)^2 = 0$. Consider then $\widetilde{h^p_\xi} : \mathcal{H}(D) \to \mathbb{C}$, $z = (z_1, \ldots, z_n) \mapsto \widetilde{h^p_\xi}(z)$ with

$$\widetilde{h^p_\xi}(z) = \begin{cases} \frac{1}{[(z_1-t_1)^2+\ldots(z_n-t_n)^2]^{\frac{n}{2}-p}} & \text{when } 1 \leq p \leq \frac{n}{2} - 1 \\ \mathrm{Ln}\sum_{j=1}^{n}(z_j - t_j)^2 & \text{when } p = \frac{n}{2} \end{cases}$$

(The branch is chosen in the complex logarithm such that $(\widetilde{h_\xi}|D)$ is real in D). Since $\mathcal{H}(D$ is the connected component containing D of $\mathbb{C}^n \setminus \cup_{t\in\partial D} \{z \in \mathbb{C}^n \sum_{j=1}^{n}(z_j - t_j)^2 = 0\}$, we see that $\widetilde{h_\xi^p}$ is defined and holomorphic in $\mathcal{H}(D)$ and that $\lim_{z\to\xi} |\widetilde{h_\xi^p}(z)| = \infty$. It remains thus to prove that the restriction $\widetilde{h_\xi^p}|D$ is actually p-polyharmonic in D.

2.a.i: $1 \leq p \leq \frac{n}{2} - 1$. Since for every $x \in D : (\widetilde{h_\xi^p}|D)(x) = 1/(r^{n-2p})$ depends only on $r = \|x - t\|$ the proof can be carried out directly. Indeed, it is simplest to introduce polar coordinates with t as origin and to use

$$\Delta^p = \big(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}B\big)^p,$$

where B is the Beltrami operator containing only derivatives with respect to the angles variables. Now by induction on $q = 1, 2, \ldots.$ we find after some calculus that for an arbitrary (complex) α,

$$\Delta^q(r^\alpha) = \alpha(\alpha+n-2)(\alpha-2)(\alpha+n-4)\ldots(\alpha-2q+2)(\alpha+n-2q)r^{\alpha-2q}.$$

Observe that if $\alpha = 2p - n$ we obtain

$$\Delta^q(r^{2p-n}) = (2p-n)(2p-2)\ldots(2p-n-2q+2)(2p-2q)r^{2p-n-2q},$$

which gives respectively for $q = p$ and $q = p - 1$:

$$\Delta^p(r^{2p-n}) = 0,$$
$$\Delta^{p-1}(r^{2p-n}) = (2p-n)(2p-2)(2p-n-2)(2p-4)\ldots(4-n)2r^{2-n}.$$

Note that $\Delta^{p-1}(r^{2p-n}) \neq 0$ if n is even and greater than or equal to 6; in addition, the case $n = 4$ involves $p = 1$, and so the last equality holds since $\Delta(\frac{1}{r^2}) = 0$, $\Delta^0(\frac{1}{r^2}) = 1$.

2.a.ii: $p = \frac{n}{2}$: Since for every $x \in D$, the restriction $\widetilde{h_\xi^p}|D : x \mapsto 2\,\mathrm{Ln}\, r$, where $r = \|x - t\|$, is a radial function, we use the same process to verify that Ln r is a $\frac{n}{2}$-polyharmonic function in $\mathbb{R}^n - \{0\}$ for all $n = 2p \geq 4$. As $\Delta(\mathrm{Ln}\, r) = (2p-2)r^{-2}$, and $\Delta^q(\mathrm{Ln}\, r) = (2q-2)\Delta^{q-1}(r^{-2})$, we can make use of the corresponding formula of 2.a.(i) with $\alpha = -2$ to have

$$\Delta^q(\mathrm{Ln}\, r) = (-1)^{q-1}2^q(q-1)[(q-1)!](n-4)(n-6)\ldots(n-2q)\frac{1}{r^{2q}}.$$

The last equality holds actually for all $n \geq 2$ and $q \geq 1$ since by the case (1) above this result is true for $n = 2$. Observe that if $n = 2p \geq 4$ one obtains

$$\Delta^q(\mathrm{Ln}\, r) = (-1)^{q-1}2^q(q-1)[(q-1)!](2p-4)\ldots(2p-2q)\frac{1}{r^{2q}} \quad \text{in} \quad \mathbb{R}^{2p} - \{0\}.$$

Thus $\Delta^q(\mathrm{Ln}\, r) \neq 0$ for $q = 1, 2, \ldots, p-1$, and $\Delta^q(\mathrm{Ln}\, r) = 0$ if $q = p$.

b) Suppose n is odd, $n \geq 3$. We consider again

$$\widetilde{h_\xi^p}(z) = \frac{1}{[[\xi_1 - z_1)^2 + \cdots + [\xi_n - z_n)^2]^{\frac{n}{2}-p}}$$

with $1 \leq p \leq [\frac{n}{2}] - 1$, where the chosen branch is such that $\widetilde{h^p_\xi}|D$, $(x) > 0$ in D. Note that $\widetilde{h^p_\xi}(z)$ is holomorphic in $\mathcal{H}(D)$ and infinite in any neighborhood of ξ. By a similar calculus, we find for every $x \in D$,

$$\Delta^p[\widetilde{h^p_\xi}|D(x)] = 0$$
$$\Delta^{p-1}[\widetilde{h^p_\xi}|D(x)] \neq 0\,.$$

Existence of h: In the following we shall make use of the lemma.

Lemma 2.2 *Let $\mathcal{O}[\mathcal{H}(D)]$ denote the Fréchet space of all holomorphic functions on $\mathcal{H}(D)$, if it is endowed with the topology (τ) of uniform convergence on compact subsets of $\mathcal{H}(D)$. Then for all integer $p = 1, 2, \dots$, the set*

$$\mathcal{O}^p[\mathcal{H}(D)] = \{F \in \mathcal{O}[\mathcal{H}(D)];\ F|D \in H^p(D)\}$$

is a close subspace of $\mathcal{O}[\mathcal{H}(D)]$, and therefore it is itself a Fréchet space.

Proof Let us consider $F_1, F_2, \dots.$ a sequence in $\mathcal{O}^p[\mathcal{H}(D)] \subset \mathcal{O}[\mathcal{H}(D)]$ converging to a function F , uniformly on every compact K' of $\mathcal{H}(D)$. It is well known by a theorem of Weierstrass that F is also holomorphic in $\mathcal{H}(D)$, it remains thus to verify that $\Delta^p(F|D) = 0$, $p = 1, 2, \dots.$ By [7], page 161, for all multi-index $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$: $D^\beta F_j \to D^\beta F$, uniformly on every compact K' of $\mathcal{H}(D)$; in particular we also have $(D^\beta F_j)|D \to (D^\beta F)|D$ uniformly on any compact $K \subset D$ since we may treat all $K' \cap \mathbb{R}^n \neq \emptyset$ as compact subsets of the real subspace in the complex $(z_1, \dots, z_n)-$space. Now, note that

$$\begin{aligned}(D^\beta F_j)|D &= (D^\beta_z F_j)|D\\ &= (\frac{\partial^{|\beta|} F_j}{\partial z_1^{\beta_1} \dots \partial z_n^{\beta_n}})|D = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}(F_j|D) = D^\beta_x(F_j|D),\end{aligned}$$

where $z_j = x_j + iy_j$, $j = 1, \dots, n$. Then for $q = 1, 2, \dots, p-1$, the sequence

$$\begin{aligned}(\Delta^q_z F_j)|D = [(\sum_{j=1}^n \frac{\partial^2}{\partial z_j^2})^q F_j]|D &= (\sum_{|\alpha|=q} \frac{q!}{\alpha!} D^{2\alpha}_z F_j)|D\\ &= \sum_{|\alpha|=q} \frac{q!}{\alpha!} D^{2\alpha}_x (F_j|D) = \Delta^q_x(F_j|D),\end{aligned}$$

being a finite sum of derivatives $(D^\beta F_j)|D$, we have $\Delta^q_x(F_j|D) \to \Delta^q_x(F|D)$, uniformly on every compact K of D. Putting $F_j|D = f_j$ and $F|D = f$, we have also for every $x \in D$: $\lim_{j\to\infty}[\Delta^q f_j(x)] = \Delta^q f(x)$, $q = 1, 2, \dots, p-1$. Since each f_j is supposed p-polyharmonic in D for $1 \leq p \leq [\frac{n}{2}]$, we have $f_j \in \mathbf{C}^{2p}_{\mathbb{R}}(D)$ and f_j satisfies the appropriate mean value property, see [4]:

$$\lambda(f_j, x, R) = f_j(x) + \sum_{q=1}^{p-1} a_q R^{2q} \Delta^q f_j(x) \tag{2.1}$$

for all x $\in D$, and $R > 0$ so small that $B_n^r(x,R) = \{y \in \mathbb{R}^n; \|y-x\| < R\} \subset D$, where $\lambda(f_j, x, R)$ denotes the integral mean values over the surface $\partial B_n^r(x,R)$:

$$\lambda(f_j, x, R) = \frac{\Gamma(\frac{n}{2})}{2\sqrt{\pi^n}} \int_{\|a\|=1} f_j(x+Ra)d\sigma(a)\,,$$

with $d\sigma(a)$ an element of surface differential on the sphere $S^{n-1}(O,1)$ and $a_q = \Gamma(\frac{n}{2})/(2^{2q}q!\Gamma(q+\frac{n}{2}))$. As f_j converges to f uniformly on the compact set $S^{n-1}(x,R)$, the limit process applied to (2.1) yields of course (2.1) for f:

$$\lambda(f, x, R) = f(x) + \sum_{q=1}^{p-1} a_q R^{2q} \Delta^q f(x);$$

that is, $f = \lim_{j\to\infty} f_j$ has the mean value property (2.1) in D. Thus f is p-polyharmonic in D.

To prove the existence of the aforesaid function h, let $z^{(1)}, \ldots, z^{(j)}.$ be a denumerable dense subset of the compact set $\partial\mathcal{H}(D)$. For every $(j,k) \in {\mathbb{N}^*}^2$, let $B_n^c(z^{(j)}, \frac{1}{k}) = \{w \in \mathbb{C}^n; \|w - z^{(j)}\| < \frac{1}{k}\}$ denote the hermitian ball of $\mathbb{C}^n$ centered at $z^{(j)}$ and of radius$\frac{1}{k}$, and put

$$\mathcal{H}_{j,k}(D) = \mathcal{H}(D) \cup B_n^c(z^{(j)}, \frac{1}{k}).$$

Due to the density of $\{z^{(j)}\}_{j\in N^*}$ in $\partial\mathcal{H}(D)$, it is enough to prove the existence of a function belonging to $\mathcal{O}^p[\mathcal{H}(D)]$ which cannot be holomorphically continued beyond $\mathcal{H}(D)$. This amounts to show the existence of a functions belonging to $\mathcal{O}^p[\mathcal{H}(D)]$ which cannot be holomorphically continued to any domain $\mathcal{H}_{j,k}(D)$. Thanks to the construction step, for all $j,k \in \mathbb{N}^*$ and every $p = 1, 2, \ldots, [\frac{n}{2}]$, we have:

$$\mathcal{O}^p[\mathcal{H}(D)] \setminus \mathbf{R}_{j,k}\{\mathcal{O}^p[\mathcal{H}_{j,k}(D)]\} \neq \emptyset,$$

where $\mathbf{R}_{j,k}$ denotes the restriction mapping from $\mathcal{O}^p[\mathcal{H}_{j,k}(D)]$ to $\mathcal{O}^p[\mathcal{H}(D)]$, and $\mathcal{O}^p$ $[\mathcal{H}_{j,k}(D)]$ the space of all holomorphic functions in $\mathcal{H}_{j,k}(D)$ whose trace on $\mathbb{R}^n$ is p-polyharmonic in D. The spaces $\mathcal{O}^p[\mathcal{H}(D)]$ and $\mathcal{O}^p[\mathcal{H}_{j,k}(D)]$ being Fréchet spaces by the Lemma above, and the linear and continuous mapping $\mathbf{R}_{j,k}$ being not onto, we deduce owing to a Banach Theorem [14] that the range of $\mathbf{R}_{j,k}$ is a subset of the first category of $\mathcal{O}^p[\mathcal{H}(D)]$; that is,

$$\mathbf{R}_{j,k}\{\mathcal{O}^p[\mathcal{H}_{j,k}(D)]\} = \cup_{m=1}^{\infty} X_{j,k}^m,$$

where $X_{j,k}^m$, $m = 1, 2, \ldots$ are subsets of $\mathcal{O}^p[\mathcal{H}(D)]$ satisfying $(\overline{X_{j,k}^m})^0 = \emptyset$, with respect to the topology (τ). Observe that

$$\cup_{j,k=1}^{\infty} \mathbf{R}_{j,k}\{\mathcal{O}^p[\mathcal{H}_{j,k}(D)]\} = \cup_{j,k=1}^{\infty}(\cup_{m=1}^{\infty} X_{j,k}^m) = \cup_{j,k,m=1}^{\infty} X_{j,k}^m$$

is also of the first category in $\mathcal{O}^p[\mathcal{H}(D)]$. Since $\mathcal{O}^p[\mathcal{H}(D)]$ is in particular a Baire space, we have, of course,

$$\mathcal{O}^p[\mathcal{H}(D)] \setminus \cup_{j,k,m=1}^{\infty} X_{j,k}^m = \mathcal{O}^p[\mathcal{H}(D)] \setminus \cup_{j,k=1}^{\infty} \mathbf{R}_{j,k}\{\mathcal{O}^p[\mathcal{H}_{j,k}(D)]\} \neq \emptyset,$$

so we can pick up an element h of $\mathcal{O}^p[\mathcal{H}(D)]$ which cannot be continued holomorphically through $\partial\mathcal{H}(D)$. □

Corollary 2.3 *Let D be a C-domain of $\mathbb{R}^n (n \geq 2)$, or D be a convex domain of $\mathbb{R}^n$. Then for every integer p, $(1 \leq p \leq [\frac{n}{2}]$, the integer part of $\frac{n}{2})$, the harmonicity cell of D satisfies*

$$\mathcal{H}(D) = [\cap_{u \in H^p(D)} D^u]^0 \tag{2.2}$$

where $H^p(D) = \{u \in \mathbf{C}^\infty(D);\ \Delta^p u = 0$ in $D\}$ and D^u is the complex domain of $\mathbb{C}^n$ to which a polyharmonic function u extends holomorphically.

Proof By [4] Lemma 1.1.2, each p-polyharmonic function u in D, $p \in \mathbb{N}^*$, is the restriction of a holomorphic function $\widetilde{u}$ in $D^u \subset \mathbb{C}^n$ such that $D^u \cap \mathbb{R}^n = D$. The former property is actually a consequence of the analyticity of u. In addition, the p-polyharmonicity of u implies more precisely that the kernel of $\mathcal{H}(D)$ is included in D^u(see [4] Theorem 5.2.6). If u wanders through the whole class $H^p(D)$, we obtain: $\mathcal{NH}(D) \subset [\cap_{u \in H^p(D)} D^u]^0$ (note here that the kernel of a harmonicity cell is a connected open of $\mathbb{C}^n$).

Inversely, due to Theorem 2.1 above, we can associate to every $1 \leq p_0 \leq [\frac{n}{2}]$ a function $f_{p_0} \in H^{p_0}(D)$ satisfying $D^{f_{p_0}} = \mathcal{H}(D)$. So, if $1 \leq p \leq [\frac{n}{2}]$ we get obviously the inclusion: $[\cap_{u \in H^p(D)} D^u]^0 \subset D^{f_{p_0}}$. Hence, one deduces $\mathcal{NH}(D) \subset [\cap_{u \in H^p(D)} D^u]^0 \subset \mathcal{H}(D)$. Now the assumption on D guarantees that $\mathcal{NH}(D) = \mathcal{H}(D)$; the desired equality follows. □

Observe that it is an unexpected result that the right-hand side of Equality (2.2) does not depend on the choice of p. This allows us thus to give the following result.

Corollary 2.4 *For every bounded domain D of $\mathbb{R}^n$, $n \geq 2$, with $D \neq \emptyset$, and $\partial D \neq \emptyset$, we have*

$$\mathcal{H}(D) = \cap_{1 \leq p \leq [\frac{n}{2}]} [\cap_{u \in H^p(D)} D^u]^0.$$

Remark 2.5 Putting $p = 1$ in Corollary 2.3, we find again an Avanissian s' result (cf. [4] p.67): Let $\mathbf{A}(D)$ $(\mathbf{H}a(D))$ be the class of all real analytic (harmonic) functions on $D \subset \mathbb{R}^n$. For $f \in \mathbf{A}(D)$, we denote $\widetilde{f} : D^f \to \mathbb{C}$ the holomorphic extension of f to the maximal domain D^f of $\mathbb{C}^n$ (in the inclusion meaning). Then the sets: $A = \cap_{f \in \mathbf{A}(D)} D^f$ and $B = \cap_{f \in \mathbf{Ha}(D)} D^f$ satisfy $\overset{\circ}{A} = \emptyset$, $\overset{\circ}{B} = \mathcal{H}(D)$.

3 Some properties of harmonicity cells

In [4], Avanissian established the following general results about the operation $D \mapsto \mathcal{H}(D)$; see also [13].

Proposition 3.1 *The harmonicity cells of domains of $\mathbb{R}^n, n \geq 2$, satisfy*

a) If $D_1 \cap D_2 = \emptyset$, $\mathcal{H}(D_1) \cap \mathcal{H}(D_2) = \emptyset$; if $D_1 \subset D_2$, $\mathcal{H}(D_1) \subset \mathcal{H}(D_2)$.

b) $\mathcal{H}(\cup_{\nu\in J}D_\nu) = \cup_{\nu\in J}\mathcal{H}(D_\nu)$ *for every exhaustive increasing family of domains* D_ν*.*

c) $\mathcal{H}(D)\cap\mathbb{R}^n = D$*;* $\mathcal{H}(D)$ *is symmetric with respect to* $\mathbb{R}^n$*; and if* D *is convex then so is* $\mathcal{H}(D)$*.*

d) If D *is starshaped at* a_0*, then* $\mathcal{H}(D)$ *is starshaped at* a_0 *and* $\mathcal{H}(D) = \{z \in \mathbb{C}^n; T(z) \subset D\}$*.*

e) $\partial D \subset \partial\mathcal{H}(D)$ *; if* $z \in \overline{\mathcal{H}(D)}$ *,* $T(z) \subset \overline{D}$ *; and if* $z \in \partial\mathcal{H}(D)$ *,* $T(z)\cap\partial D \neq \emptyset$*.*

f) $\delta[\mathcal{H}(D)] \leq 2[\frac{n}{2n+2}]^{\frac{1}{2}}\delta(D)$*, where* $\delta(D)$ *denotes the diameter of* D*.*

g) $\mathcal{H}(D)$ *may be explicitly obtained when* D *is a ball, a cube, or a difference of two balls.*

h) If $n = 2$ *and* $\mathbb{R}^2 \simeq \mathbb{C}$*,* $\mathcal{H}(D) = \{z \in \mathbb{C}^2;\ z_1 + iz_2 \in D,\ \overline{z_1} + i\overline{z_2} \in D\}$*.*

In the following, we establish supplementary results. Let $\mathfrak{D}^n$ denote henceforth the family of all domains D of $\mathbb{R}^n$, $D \neq \emptyset$, $\partial D \neq \emptyset$, and $\mathfrak{C}^n_s$ the family of all domains of $\mathbb{C}^n$ which are symmetric with respect to $\mathbb{R}^n = \{x + iy \in \mathbb{C}^n; y = 0\}$.

Proposition 3.2 *The mapping* $\mathcal{H} : D \in \mathfrak{D}^n \mapsto \mathcal{H}(D) \in \mathfrak{C}^n_s$ *satisfies:*

a) $\mathcal{H}$ *is injective ;* $\mathcal{H}(D)$ *is bounded if and only if* D *is bounded.*

b) For every compact set $K \subset \mathcal{H}(D)$*, there exists a domain* $D_1 \in \mathfrak{D}^n$ *such that* D_1 *is relatively compact in* D *and* $K \subset \mathcal{H}(D_1)$*.*

c) If $D_1, D_2 \in \mathfrak{D}^2$ *are such that* $D_1 \cap D_2$ *is connected then* $\mathcal{H}(D_1 \cap D_2) = \mathcal{H}(D_1)\cap\mathcal{H}(D_2)$*. If* $(D_j)_{j\in J}$ *is a family of starshaped domains in* $\mathbb{R}^n (n \geq 2)$ *such that* $\cap_{j\in J}D_j$ *is a starshaped domain, or if* $(D_j)_{j\in J}$ *is a family of convex domains in* $\mathbb{R}^n$ *such that* $\cap_{j\in J}D_j$ *is open, then* $\mathcal{H}(\cap_{j\in J}D_j) = \cap_{j\in J}\mathcal{H}(D_j)$*.*

d) If $D_1, D_2 \in \mathfrak{D}^2$ *with* $D_1\cap D_2 \neq \emptyset$ *then* $\mathcal{H}(D_1\cup D_2) \supset \mathcal{H}(D_1)\cup\mathcal{H}(D_2)$*, and the equality holds if and only if* $D_1 \subset D_2$ *or* $D_2 \subset D_1$*. More generally, if* $(D_j)_{j\in J} \subset \mathfrak{D}^n (n \geq 2)$ *is such that* $D_i \cap D_j \neq \emptyset$ *for all* $i, j \in J$ *then* $\mathcal{H}(\cup_{j\in J}D_j) \supset \cup_{j\in J}\mathcal{H}(D_j)$*. The equality holds if* $\cup_{j\in J}D_j = D_{j_0}$ *for a certain* $j_0 \in J$*.*

Proof a) By Proposition 3.1, $\mathcal{H}$ is well defined on $\mathfrak{D}^n$ with values in $\mathfrak{C}^n_s$; and if two harmonicity cells $\mathcal{H}(D)$ and $\mathcal{H}(D')$ coincide in $\mathbb{C}^n$, then their traces on $\mathbb{R}^n$ coincide also, that is $D = D'$. Besides, suppose that for some $R > 0$, $D \subset B^r_n(0, R) = \{x \in \mathbb{R}^n; \|x\| < R\}$. Then $\mathcal{H}(D) \subset \mathcal{H}[B^r_n(0, R)] = LB(0, R) = \{z \in \mathbb{C}^n; L(z) < R\}$ (the Lie ball of $\mathbb{C}^n$), see [3, 5, 9], where

$$L(z) = \left[\|z\|^2 + \sqrt{\|z\|^4 - |\sum_{j=1}^n z_j^2|^2}\,\right]^{1/2}.$$

Since $\|z\| \leq L(z)$, we have $LB(0,R) \subset B_n^c(0,R) = \{z \in \mathbb{C}^n; \|z\| < R\}$ and $\mathcal{H}(D)$ is bounded in $\mathbb{C}^n$. The converse is obvious since $D \subset \mathcal{H}(D)$.
b) Let us consider an increasing exhaustive family $(D_\nu)_{\nu\in J}$ (J is a fixed indices set) of bounded domains $D_\nu \in \mathfrak{D}^n$ such that $D = \cup_{\nu\in J} D_\nu$. Due to 3.1.b, the family $(\mathcal{H}(D_\nu))_{\nu\in J}$ of harmonicity cells of $(D_\nu)_{\nu\in J}$ is also increasing, exhaustive and satisfies $\mathcal{H}(D) = \cup_{\nu\in J}\mathcal{H}(D_\nu)$. We then have: $K \subset \cup_{\nu\in J}\mathcal{H}(D_\nu)$. Since K is a compact set, we can extract from this open covering of K, a finite sub-covering of K: $K \subset \cup_{k=1}^n \mathcal{H}(D_{\nu_k})$. Afterwards we have by 3.1.a: $\cup_{k=1}^n \mathcal{H}(D_{\nu_k}) \subset \mathcal{H}(\cup_{k=1}^n D_{\nu_k})$ and $K \subset \mathcal{H}(\cup_{k=1}^n D_{\nu_k})$. Seeing that D' is a relatively compact domain in D and taking $D' = \cup_{k=1}^n D_{\nu_k}$, we obtain the desired result.
c) The inclusion $\mathcal{H}(D_i \cap D_j) \subset \mathcal{H}(D_i) \cap \mathcal{H}(D_j)$ is obvious from $D_i \cap D_j \subset D_i$, $D_i \cap D_j \subset D_j$. If $w \in \mathcal{H}(D_i) \cap \mathcal{H}(D_j)$, $T(w) \subset D_i \cap D_j$, that is $w \in \mathcal{H}(D_i \cap D_j)$. By similar arguments we obtain the general case.
d) $D_1 \cap D_2 \neq \emptyset$ guarantees that $D_1 \cup D_2 \in \mathfrak{D}^n$. Since $D_i \subset D_1 \cup D_2$, $i = 1, 2$, $\mathcal{H}(D_1) \cup \mathcal{H}(D_2) \subset \mathcal{H}(D_1 \cup D_2)$. Suppose now that D_1 is neither included in D_2, nor D_2 in D_1. If a and b are arbitrarily chosen in $D_1 \setminus D_2$ and $D_2 \setminus D_1$ respectively, and if $\mathbb{R}^2 \simeq \mathbb{C}$, then the point $w = (\frac{a+\bar{b}}{2}, \frac{a-\bar{b}}{2i})$ of $\mathbb{C}^2$ satisfies $T(w) = \{a, b\} \subset D_1 \cup D_2$. Now, the last hypothesis on D_1 and D_2 involves that $w \notin \mathcal{H}(D_1) \cup \mathcal{H}(D_2)$. Besides, as $D_i \cap D_j \neq \emptyset$ we have $\cup_{j\in J} D_j \in \mathfrak{D}^n$ and thus this union does possess a harmonicity cell in $\mathbb{C}^n$. The given inclusion is evident since $D_i \subset \cup_{j\in J} D_j$. Suppose in addition that $D_{j_0} = \cup_{j\in J} D_j$. From $\mathcal{H}(D_{j_0}) \subset \cup_{j\in J}\mathcal{H}(D_j)$ and $\mathcal{H}(\cup_{j\in J} D_j) \supset \cup_{j\in J}\mathcal{H}(D_j)$, we deduce the equality $\mathcal{H}(\cup_{j\in J} D_j) = \cup_{j\in J}\mathcal{H}(D_j)$. □

Corollary 3.3 *If $(D_j)_{j\geq 1}$ is a monotonous sequence in $\mathfrak{D}^n$, so is $(\mathcal{H}(D_j))_{j\geq 1}$ in $\mathfrak{C}_s^n$; and writing $D = \lim_{j\to\infty} D_j$, we have $\lim_{n\to\infty} \mathcal{H}(D_n) = \mathcal{H}(D)$ under the assumptions that: $\cup_{j\geq 1} D_j \neq \mathbb{R}^n$ if the sequence $(D_j)_{j\geq 1}$ is increasing, and that $\cap_{j\in J} D_j \in \mathfrak{D}^n$ in the decreasing case.*

Proof If the sequence is increasing then $\liminf_{n\to\infty} D_n = \cup_{n\geq 1}(\cap_{k\geq n} D_k) = \cup_{n\geq 1} D_n$, $\limsup_{n\to\infty} D_n = \cap_{n\geq 1}(\cup_{k\geq n} D_k) = \cap_{n\geq 1}(\cup_{k\geq n} D_k) = \cup_{k\geq 1} D_k$, so $\lim_{n\to\infty} D_n = \cup_{n\geq 1} D_n$. Since $\cup_{n\geq 1} D_n \neq \emptyset$ and $\cup_{n\geq 1} D_n \neq \mathbb{R}^n$, we deduce that $\lim_{n\geq 1} D_n$ is an element of $\mathfrak{D}^n$.

Next, $(\mathcal{H}(D_n))_{n\geq 1}$ being also increasing, $\lim_{n\to\infty} \mathcal{H}(D_n) = \cup_{n\geq 1}\mathcal{H}(D_n)$. Now by 3.2.d: $\cup_{n\geq 1}\mathcal{H}(D_n) \subset \mathcal{H}(\cup_{n\geq 1} D_n)$. Moreover, if $w_0 \in \mathcal{H}(\cup_{n\geq 1} D_n)$ one has $T(w_0) \subset \cup_{n\geq 1} D_n$; then by 3.2.b and the fact that $T(z)$ is a compact set for every $z \in \mathbb{C}^n$, there exists $n_0 \geq 0$ such that $T(w_0) \subset D_{n_0}$ i.e. $w_0 \in \mathcal{H}(D_{n_0}) \subset \cup_{n\geq 1}\mathcal{H}(D_n)$. Thus $\mathcal{H}(\lim_{n\to\infty} D_n) \subset \lim_{n\to\infty} \mathcal{H}(D_n)$, which involves the aforesaid equality. In case of a decreasing sequence $(D_n)_{n\geq 1}$ one has $\liminf_{n\to\infty} D_n = \cup_{n\geq 1}(\cap_{k\geq n} D_k) = \cup_{n\geq 1}(\cap_{k\geq 1} D_k) = \cap_{k\geq 1} D_k$, and $\limsup_{n\to\infty} D_n = \cap_{n\geq 1}(\cup_{k\geq n} D_k) = \cap_{n\geq 1} D_n$. So $\lim_{n\to\infty} D_n = \cap_{n\geq 1} D_n$, which is in $\mathfrak{D}^n$ by hypothesis. Now, $(\mathcal{H}(D_n))_{n\geq 1}$ being decreasing one also has: $\lim_{n\to\infty} \mathcal{H}(D_n) = \mathcal{H}(\lim_{n\to\infty} D_n)$. □

Corollary 3.4 *The mapping $D \mapsto \mathcal{H}(D)$ is not a surjective operator: The unit hermitian ball $B_n^c = \{z \in \mathbb{C}^n; \|z\| < 1\}$ does not represent a harmonicity cell in*

$\mathbb{C}^n$.

Proof. Since $B_n^c \cap \mathbb{R}^n = B_n^r = \{x \in \mathbb{R}^n; \|x\| < 1\}$ is a convex domain of $\mathbb{R}^n$ (according to the induced topology), we have to find a point $w_0 \in B_n^c$ for which the Lelong sphere $T(w_0)$ is not contained into B_n^r. For, put $w_0 = \rho\,(i, 1, \ldots, 1) \in \mathbb{C}^n$ where $\rho > 0$ is small enough for w_0 to belong at B_n^c and for $T(w_0)$ to contain a certain $\xi_0 \in \mathbb{R}^n$ with $\|\xi_0\| \geq 1$. Taking $[n + 2\,\sqrt{n-1}]^{-1/2} < \rho < 1/n$ and writing $w_0 = x_0 + i\,y_0$ we see that a ξ_0 satisfying

$$[\langle \xi_0 - x_0, y_0 \rangle = 0, \quad \|\xi_0 - x_0\| = \|y_0\|, \quad \text{and} \quad \|\xi_0\| \geq 1];$$

that is,

$$\rho\xi_1 = 0, \quad \xi_1^2 + (\xi_2 - \rho)^2 + \ldots (\xi_n - \rho)^2 = \rho^2 \quad \text{and} \quad \xi_1^2 + \cdots + \xi_n^2 \geq 1$$

is given by: $\xi_0 = \rho[1 + (n-1)^{\frac{-1}{2}}](0, 1, \ldots, 1)$.

Remark 3.5 Due to propositions 3.1 and 3.2 above, the definition of a harmonicity cell may be naturally extended to arbitrary open sets of $\mathbb{R}^n$ for $n \geq 1$ as follows $\mathcal{H}(\emptyset) = \emptyset$, $\mathcal{H}(\mathbb{R}^n) = \mathbb{C}^n$, $\mathcal{H}(]a, b[) = \mathbb{C}$ for $]a, b[\subset \mathbb{R}$, and $\mathcal{H}(O) = \cup_{i \in I} \mathcal{H}(O_i)$, where O is an open set of $\mathbb{R}^n$, $(O_i)_{i \in I}$ the family of the connected components of O.

Remark 3.6 Some properties are not always preserved by $D \mapsto \mathcal{H}(D)$; this is especially the case if:

(i) D is simply connected in $\mathbb{R}^n$ with $n \geq 3$. Indeed, the two domains $D = \mathbb{R}^n - \{0\}$ and $\mathcal{H}(D) = \mathbb{C}^n - \{z \in \mathbb{C}^n; z_1^2 + \cdots + z_n^2 = 0\}$, having 0 and $\mathbb{Z}$ respectively as fundamental groups, they offer then an example of a not simply connected harmonicity cell corresponding to a real simply connected domain; for $\pi_1[\mathcal{H}(D)] = \mathbb{Z}$, see [6].

(ii) D is strictly convex in $\mathbb{R}^n$with $n \geq 2$. An example is given by the harmonicity cell of the unit ball B_n^r of $\mathbb{R}^n$. If $\mathcal{E}(\overline{V})$ denotes the set of all extremal points of a convex V we have $\mathcal{E}(\overline{B_n^r}) = \partial B_n^r$ since these two sets coincide with the unit Euclidean sphere S^{n-1} of $\mathbb{R}^n$. Nevertheless, by [9]: $\mathcal{E}(\overline{\mathcal{H}(B_n^r)}) = \partial^\vee[\mathcal{H}(B_n^r)] = \{w = xe^{i\theta} \in \mathbb{C}^n; x \in S^{n-1}, \theta \in \mathbb{R}\}$, where $\partial^\vee U$ denotes the Ŝilov boundary of $U \subset \mathbb{C}^n$; thus: $\mathcal{E}(\overline{\mathcal{H}(B_n^r)}) \overset{\neq}{\subset} \partial[\mathcal{H}(B_n^r)]$.

(iii) D is partially - circled in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, $n \geq 2$, that is (for instance): $z \in D \Rightarrow (z_1, \ldots, z_{n-1}, e^{i\theta} z_n) \in D$, for all $\theta \in \mathbb{R}$. Indeed if $D = B_n^c = \{z \in \mathbb{C}^n; \|z\| < 1\}$, $\mathcal{H}(B_n^c)$ is not partially - circled in $\mathbb{C}^{2n}$ with respect to w_{2n} since $w_0 = \sqrt{1+2n}(1, \ldots, 1) \in \mathbb{C}^{2n}$ satisfies $L(w_0) = \sqrt{2n/(1+2n)} < 1$, but $L[(2n+1)^{\frac{-1}{2}}, \ldots, (2n+1)^{\frac{-1}{2}}, i(2n+1)^{\frac{-1}{2}}] = [2n + 2\sqrt{2n-2}]^{\frac{1}{2}}(2n+1)^{\frac{-1}{2}} > 1$. On the other hand, B_n^c is even circled (at the origin).

4 Harmonicity cells of polygonal plane domains

The case $n = 2$ is rather special since the Lelong map T is given by: $T(z) = \{z_1 + iz_2, \overline{z_1} + i\overline{z_2}\}$, where $z \in \mathbb{C}^2$ and $\mathbb{R}^2 \simeq \mathbb{C}$. So, in [5], we have determined explicitly the harmonicity cells of some plane domains and shed light on the close connection between the set $\mathcal{E}(\overline{D})$, of all the extremal points of a convex domain D of $\mathbb{R}^2$, and the set $\mathcal{E}(\overline{\mathcal{H}(D)})$, see also [4]. We will give now some properties and constructions which are proper to the complex plane. More precisions on the Jarnicki extension given in Section 1 will also be established.

Proposition 4.1 *The operator $\mathcal{H} : \mathfrak{D}^2 \to \mathfrak{C}_s^2$ satisfies*

a) If D is circled at $z_0 \in \mathbb{C}$, balanced at $z_0 \in D$, or simply connected, then so is $\mathcal{H}(D)$ respectively.

b) If P_n^a is an arbitrary convex polygon with n edges, then the harmonicity cell $\mathcal{H}(P_n^a)$ is of polyhedric form in $\mathbb{C}^2$ with $2n$ faces and n^2 vertices. Furthermore, identifying $\mathbb{C}^2$ with $\mathbb{R}^4$ by writing $y = (x_3, x_4)$ and $x + iy = (x_1, x_2, x_3, x_4)$, each support line of P_n^a defined, for a certain $j = 1, \ldots, n$, by $a_j x_1 + b_j x_2 - \alpha_j = 0$, $(a_j, b_j, \alpha_j \in \mathbb{R})$, generates two support hyperplanes of $\mathcal{H}(P_n^a)$ of respective equations:

$$a_j x_1 + b_j x_2 + b_j x_3 - a_j x_4 - \alpha_j = 0 \quad \text{and} \quad a_j x_1 + b_j x_2 - b_j x_3 + a_j x_4 - \alpha_j = 0.$$

c) Let P_n^r denote the regular polygon which vertices are $\omega_k = e^{2ik\pi/n}$, $k = 0, \ldots, n-1$. Then

$$\begin{aligned}\mathcal{H}(P_n^r) = \Big\{ & w = x + iy \in \mathbb{C}^2 : x_1 \cos(2k+1)\frac{\pi}{n} + x_2 \sin(2k+1)\frac{\pi}{n} \\ & + \sqrt{\|y\|^2 - [y_1 \cos(2k+1)\frac{\pi}{n} + y_2 \sin(2k+1)\frac{\pi}{n}]^2} < \cos\frac{\pi}{n}, \\ & k = 0, \ldots, n-1 \Big\}.\end{aligned}$$

d) The n^2 vertices of $\overline{\mathcal{H}(P_n^r)}$ are given by $\omega_{km} = x_{km} + iy_{km}$ and $\overline{\omega_{km}} = x_{km} - iy_{km}$, $(0 \leq k \leq m \leq n-1)$, where

$$\begin{aligned} x_{km} &= \frac{1}{2}(\cos\frac{2k\pi}{n} + \cos\frac{2m\pi}{n}, \sin\frac{2k\pi}{n} + \sin\frac{2m\pi}{n}), \\ y_{kk} &= 0, \ k = 0, \ldots, n-1, \\ y_{km} &= \frac{\sin \pi(m-k)\ /n}{\sqrt{2}[1 - \cos 2\pi(m-k)\ /n]^{1/2}} \\ &\times (\sin\frac{2\pi m}{n} - \sin\frac{2\pi k}{n}, \cos\frac{2\pi k}{n} - \cos\frac{2\pi m}{n}). \end{aligned}$$

Proof a) For $\theta \in \mathbb{R}$, $z_0 = a + ib \in \mathbb{C}$, and $w = (w_1, w_2) \in \mathcal{H}(D)$, we see that $z_0 + e^{i\theta}w$ remains in $\mathcal{H}(D)$. Since $T(z_0 + e^{i\theta}w) = \{a + e^{i\theta}w_1 + i(b + e^{i\theta}w_2), a+e^{-i\theta}\overline{w_1}+i(b+e^{-i\theta}\overline{w_2})\} = \{z_0+e^{i\theta}(w_1+iw_2), z_0+e^{-i\theta}(\overline{w_1}+i\overline{w_2})\}$, and as D is circled with respect to z_0, we have $T(z_0+e^{i\theta}w) \subset D$. If the above circled domain D is supposed starshaped at z_0 too, then $\mathcal{H}(D)$ is also starshaped at z_0 (by 3.1.d) that is, $\mathcal{H}(D)$ is balanced at z_0. Let $D \in \mathcal{D}^2$ be a simply connected domain and f a holomorphic one-one map sending D onto $B = \{z \in \mathbb{C}; |z| < 1\}$. By Jarnicki Theorem , f extends to a holomorphic homeomorphism $Jf : \mathcal{H}(D) \to \mathcal{H}(B)$. Now, by [4], $\mathcal{H}(B)$ is the unit disk of $(\mathbb{C}^2, L)$, where L is the Lie norm; this means that $\mathcal{H}(B)$ is convex and in particular simply connected. Since Jf is a homeomorphism, $\mathcal{H}(D)$ is also simply connected.
b) Suppose that P_n^a is defined by:

$$P_n^a = \{x = x_1 + ix_2 \in \mathbb{R}^2; \langle x, V^j\rangle < \alpha_j, j = 1, \ldots, n\},$$

with given vectors $V^j = (a_j, b_j) \in \mathbb{R}^2$ and scalars $\alpha_j \in \mathbb{R}$. By 3.1.d, one has $w = x + iy \in \mathcal{H}(P_n^a) \iff x + T(iy) \subset P_n^a \iff x + \xi \in P_n^a, \forall\xi \in T(iy) \iff \langle x, V^j\rangle + \max_{\xi\in T(iy)}\langle \xi, V^j\rangle < \alpha_j$, $j = 1, \ldots, n$. Since $T(iy) = \{(-y_2, y_1), (y_2, -y_1)\}$, we have

$$\mathcal{H}(P_n^a) = \{w = x + iy \in \mathbb{C}^2; \langle w, U^j\rangle < \alpha_j \text{ and } \langle w, W^j\rangle < \alpha_j,\ j = 1, \ldots, n\},$$

where $w = (x_1, x_2, x_3, x_4)$, $y = (x_3, x_4)$, $U^j = (a_j, b_j, -b_j, a_j)$, and $W^j = (a_j, b_j, b_j, -a_j)$, while $\langle , \rangle$ denotes the usual scalar product in $\mathbb{R}^4$. From the expression above, we deduce that the harmonicity cell of an arbitrary convex polygon (not necessarily bounded) with n edges is a polyhedron of $\mathbb{C}^2 \simeq \mathbb{R}^4$ having $2n$ faces and by [5], n^2 vertices.
c) For the regular polygon P_n^r, we have also another expression of its harmonicity cell. Indeed, if $\mathbb{C} \simeq \mathbb{R}^2$, we put $\omega_n = \omega_0, \omega_k = (\cos\frac{2k\pi}{n}, \sin\frac{2k\pi}{n})$, and $V^k = \omega_{k+1} - \omega_k = (a_k, b_k)$, $k = 0, \ldots, n-1$. By (b) we have

$$\mathcal{H}(P_n^r) = \big\{x \in \mathbb{R}^2; \langle x, V^k\rangle + \max_{\xi\in T(iy)}\langle \xi, V^k\rangle < \cos\frac{\pi}{n}, k = 0, \ldots, n-1\big\}.$$

By the method of Lagrange multipliers [4], we find $\max_{\xi\in T(iy)}\langle \xi, V^k\rangle = [\|y\|^2 - \langle y, V^k\rangle^2]^{1/2}$; the announced expression of $\mathcal{H}(P_n^r)$ follows.
d) Applying the following two lemmas proved in [5], (see also [4]) we obtain all the extremal points of $\overline{\mathcal{H}(P_n^r)}$ by means of those of $\overline{P_n^r}$ □

Lemma 4.2 *If D is a non empty convex domain of $\mathbb{R}^n$, $n \geq 2$, $\partial D \neq \emptyset$, then $\mathcal{E}(\overline{D}) \subset \mathcal{E}(\overline{\mathcal{H}(D)})$.*

Lemma 4.3 *Let D be a non empty convex domain, $\partial D \neq \emptyset$, in $\mathbb{C} \simeq \mathbb{R}^2$.*
a) Every point $w \in \mathcal{E}(\overline{\mathcal{H}(D)})$ satisfies $T(w) \subset \mathcal{E}(\overline{D})$.
b)Conversely, given arbitrary points a and b of $\mathcal{E}(\overline{D})$, there exists $w \in \mathcal{E}(\overline{\mathcal{H}(D)})$ such that $T(w) = \{a, b\}$.

Let U, V be two domains of $\mathbb{C}^n$, $n \geq 1$. we denote $\hom(U,V)$ the set of all holomorphic homeomorphisms $F : U \to V$, and $\hom_r(\mathcal{H}(D), \mathcal{H}(D'))$ the set of all $F \in \hom(\mathcal{H}(D), \mathcal{H}(D'))$ of which the restriction $F|_D$ belongs to $\hom(D, D')$, where $D, D' \in \mathcal{D}^2$ and $\mathbb{C} \simeq \mathbb{R}^2$.

Proposition 4.4 *Let $D, D' \subset \mathbb{C}$ be two non empty domains with $D \neq \mathbb{C}$, $D' \neq \mathbb{C}$. The Jarnicki extension J is an injective continuous mapping from* $\hom(D, D')$ *onto* $\hom_r(\mathcal{H}(D), \mathcal{H}(D'))$ *according to the compact uniform topology (τ).*

Furthermore, $\hom_r(\mathcal{H}(D), \mathcal{H}(D')) \simeq \hom(D, D')$ *(topologically homeomorphic); and for a holomorphic homeomorphism $f : D \to D'$ we have the estimate*

$$\|Jf(w)\| \leq \sup_{z \in D} |f(z)|, \quad \textit{for every } w \in \mathcal{H}(D).$$

Proof If f and f' are such that $Jf = Jf'$ on $\mathcal{H}(D)$ then by [10], $f = (Jf)|D = (Jf')|D = f'$ on D. Let $(f_n)_{n \geq 1}$ be a convergent sequence in $(\hom(D, D'), \tau)$. By 3.2.b,to test $(J\ f_n)_{n \geq 1}$ for compact uniform convergence in the harmonicity cell of D it is not really necessary to check uniform convergence on every compact set K in $\mathcal{H}(D)$ - checking it on the closed harmonicity cells $\overline{\mathcal{H}(D_0)}$ where D_0 is an arbitrary relatively compact domain in D is enough. Now if $w_0 \in \mathcal{H}(D_0)$ with $w_0 = (w_1^0, w_2^0)$:

$$\|Jf_n(w_0) - Jf(w_0)\|^2 = A_n^2(w) + B_n^2(w),$$

where $f = \lim_{n \to \infty} f_n$, and

$$A_n = \frac{1}{2}|[f_n(w_1^0 + iw_2^0) - f(w_1^0 + iw_2^0)] + [\overline{f_n(\overline{w_1^0} + i\overline{w_2^0})} - \overline{f(\overline{w_1^0} + i\overline{w_2^0})}]|,$$
$$B_n = \frac{1}{2}|[f_n(w_1^0 + iw_2^0) - f(w_1^0 + iw_2^0)] - [\overline{f_n(\overline{w_1^0} + i\overline{w_2^0})} - \overline{f(\overline{w_1^0} + i\overline{w_2^0})}]|.$$

Both A_n and B_n are bounded above by $\frac{1}{2} \sup_{w \in \mathcal{H}(D_0)} |f_n(w_1 + iw_2) - f(w_1 + iw_2)| + \frac{1}{2} \sup_{w \in \mathcal{H}(D_0)} |f_n(\overline{w_1} + i\overline{w_2}) - f(\overline{w_1} + i\overline{w_2})$. By 3.1.h: $w \in \mathcal{H}(D_0)$ if and only if $w_1 + iw_2 \in D_0$ and $\overline{w_1} + i\overline{w_2} \in D_0$. Thus:

$$A_n \leq \sup_{z \in D_0} |f_n(z) - f(z)|, \quad B_n \leq \sup_{z \in D_0} |f_n(z) - f(z)|,$$
$$\sup_{w \in \overline{\mathcal{H}(D_0)}} \|Jf_n(w) - Jf(w)\| \leq \sqrt{2} \sup_{z \in \overline{D_0}} |f_n(z) - f(z)|.$$

Since $\lim_{n \to \infty} \sup_{z \in \overline{D_0}} |f_n(z) - f(z)| = 0$, we have $Jf_n \to Jf$, according to (τ). The mapping $J : \hom(D, D') \to \hom_r(\mathcal{H}(D), \mathcal{H}(D'))$ is continuous and injective. To see that this mapping is onto, take $F \in \hom_r(\mathcal{H}(D), \mathcal{H}(D'))$ and observe that (by [10]) $J(F|D)$ and F are both holomorphic homeomorphisms from $\mathcal{H}(D)$ onto $\mathcal{H}(D')$ having the same restriction on $D : (J(F|D))|D = F|D$. So by the uniqueness principle of analytic extension in $\mathbb{C}^n : J(F|D) = F$. Conversely, putting: $R = J^{-1}$and making use of 3.1.c, e and 3.2.b, we have for every $D_0 \subset D$ with $\overline{D_o}$ compact: $\sup_{\overline{\mathcal{H}(D_0)}} \|F_n - F\| \geq \sup_{\overline{D_0}} |RF_n - RF|$, which implies that R is also continuous. Finally,we have

$$\begin{aligned}
\|Jf(w)\|^2 =& \frac{1}{4}|f(w_1 + iw_2) + \overline{f(\overline{w_1} + i\overline{w_2})}|^2 + \frac{1}{4}|f(w_1 + iw_2) - \overline{f(\overline{w_1} + i\overline{w_2})}|^2 \\
=& \frac{1}{2}[|f(w_1 + iw_2)|^2 + |f(\overline{w_1} + i\overline{w_2})|^2] \\
\leq& \frac{1}{2}\big[(\sup_{\overline{D}} |f|)^2 + (\sup_{\overline{D}} |f|)^2\big] = (\sup_{\overline{D}} |f|)^2.
\end{aligned}$$

□

Remark 4.5 The notion of harmonicity cells has a functorial aspects; indeed let $\mathfrak{D}^2$ still denote the category of all domains D of $\mathbb{R}^2 \simeq \mathbb{C}$, $D \neq \emptyset$, $\partial D \neq \emptyset$ with arrows in $\hom(D_1, D_2)$, and $\mathfrak{C}_s^2$ the category of all domains U of $\mathbb{C}^2$ which are symmetric with respect to $\mathbb{R}^2$, with arrows F in $\hom(U_1, U_2)$. Then, by the uniqueness theorem of holomorphic continuation in $\mathbb{C}^n$, to the composition: $D_1 \xrightarrow{f} D_2 \xrightarrow{g} D_3$ corresponds $\mathcal{H}(D_1) \xrightarrow{Jf} \mathcal{H}(D_2) \xrightarrow{Jg} \mathcal{H}(D_3)$ such that: $J(g \circ f) = (Jg) \circ (Jf)$; next $f = Id$ in Jarnicki Theorem (Section 1) gives:
$J\ Id_D = Id_{\mathcal{H}(D)}$. This means that the operator: $D \in \mathfrak{D}^2 \mapsto \mathcal{H}(D) \in \mathfrak{C}_s^2$ and $f \in \hom(D_1, D_2) \mapsto \mathcal{H}(f) = Jf \in \hom[\mathcal{H}(D_1), \mathcal{H}(D_2)]$ may be considered as a covariant functor between the said categories. The representability of this functor and its classifying object will be discussed in a further paper.

Example If V is an arbitrary half strip of $\mathbb{R}^2$, there exists an usual transformation f, mapping V onto $V' = \{x \in \mathbb{R}^2 : x_1 > a, k_1 < x_2 < k_2\}$, for some $a > 0$, $k_1, k_2 \in \mathbb{R}$. Now by [4, 7], we have for all convex domains U of $\mathbb{R}^n$ $(n \geq 2)$:

$$\mathcal{H}(U) = \big\{w = x + iy \in \mathbb{C}^n;\ \max_{t \in T(iy)} \big[\max_{\xi \in S^{n-1}} (\langle x + t, \xi\rangle - \sup_{u \in U}\langle \xi, u\rangle)\big] < 0\big\}.$$

This formula gives $\mathcal{H}(U)$ by means of the support function of $U : \delta_U(\xi) = \sup_{u \in U}\langle \xi, u\rangle$. Making use of the fact that the function $u \mapsto \xi_1 u_1 + \xi_2 u_2$, being harmonic in V', attains its supremum at some point of $\partial V'$. We find by simple calculations that

$$\delta_{V'}(\xi) = \begin{cases} +\infty & \text{if } \xi_1 > 0 \\ a\xi_1 + k_2\xi_2 & \text{if } \xi_1 \leq 0 \text{ and } \xi_2 \geq 0 \\ a\xi_1 + k_1\xi_2 & \text{if } \xi_1 \leq 0 \text{ and } \xi_2 \leq 0 \end{cases}$$

where $\xi \in \Gamma$, the unit circle of $\mathbb{C}$. Next, to search the supremum on Γ of the function $g(\xi_1, \xi_2) = \langle x + t, \xi\rangle - \delta_{V'}(\xi)$, we restrict the study to $\{\xi \in \Gamma : \xi_1 \leq 0\}$. Since $g(\xi_1, \xi_2) = g(\xi_1, \pm\sqrt{1 - \xi_1^2})$, with $\xi_1 \in [-1, 0]$, we put

$$g_1(\xi_1) = g(\xi_1, \sqrt{1 - \xi_1^2}) = \alpha_1\xi_1 + \alpha_2\sqrt{1 - \xi_1^2} \quad \text{and} \quad g_2(\xi_1) = \alpha_1\xi_1 - \beta\sqrt{1 - \xi_1^2},$$

where $\alpha_1 = x_1 + t_1 - a$, $\alpha_2 = x_2 - t_2 - k_2$, $\beta = x_2 - t_2 - k_1$. One obtains that $g_1'(\xi_1) = 0$ if $\xi_1 = \pm\alpha_1/\sqrt{\alpha_1^2+\alpha_2^2}$ (when $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$). In addition, the study of variations of $g_1(\xi_1)$, in $-1 \leq \xi_1 \leq 0$, in each of the three cases: $\alpha_1 \leq 0$, ($\alpha_1 \geq 0$ and $\alpha_2 \leq 0$), and ($\alpha_1 \geq 0$ and $\alpha_2 \geq 0$) leads to $\max_{-1\leq\xi_1\leq 0} g_1(\xi_1) = \max(-\alpha_1, \alpha_2)$. Obviously, this equality holds even if $\alpha_1 = \alpha_2 = 0$. A similar calculus for $g_2(\xi_1)$ gives $\max_{-1\leq\xi_1\leq 0} g_2(\xi_1) = \max(-\beta, -\alpha_2)$. Putting $\gamma = \max(-\alpha_1, \alpha_2)$, $\delta = -\min(\beta, \alpha_2)$, and as $T(iy) = \{(-y_2, y_1), (y_2, -y_1)\}$, we obtain the equivalence

$$\max(\gamma,\delta) < 0 \Leftrightarrow \begin{cases} a - x_1 + y_2 < 0, x_2 + y_1 - k_2 < 0, k_1 - x_2 - y_1 < 0, \\ a - x_1 - y_2 < 0, x_2 - y_1 - k_2 < 0, k_1 - x_2 + y_1 < 0. \end{cases}$$

At last, writing $\min(u,v) = \frac{1}{2}(u + v - |u - v|)$, and by the Jarnicki extension $f \mapsto Jf = \widetilde{f}$ (see section 1), we deduce $\mathcal{H}(V) = (\widetilde{f})^{-1}[\mathcal{H}(V')]$, where

$$\mathcal{H}(V') = \{w = x + iy \in \mathbb{C}^2; |y_1| < \frac{k_2 - k_1}{2} - |x_2 - \frac{k_1 + k_2}{2}|, |y_2| < x_1 - a\}.$$

Example The harmonicity cell of an arbitrary convex polygon P_n' may be explicited by means of the n vertices $\omega_0', \ldots, \omega_{n-1}'$. For, put $\alpha = \frac{\omega_0'+\omega_2'}{2}$ and consider the translation $\tau_{-\alpha} : z \mapsto z - \alpha$. The domain $P_n = \tau_{-\alpha}(P_n')$ is also a convex polygon, with $O \in P_n$ and n vertices $\omega_0, \ldots, \omega_{n-1}$, given by $\omega_k = \omega_k' - \alpha$. Making use of (d) and (h) in Proposition 3.1, we find after calculus and simplifications:

$$\begin{aligned} \mathcal{H}(P_n) = \Big\{ & w = x + iy \in \mathbb{C}^2 : \operatorname{sgn}(\operatorname{Im} \overline{\omega_k}\omega_{k+1}) \operatorname{Im} \overline{x}(\omega_{k+1} - \omega_k) \\ & + \sqrt{|y|^2|\omega_{k+1} - \omega_k|^2 - \operatorname{Im}^2 \overline{y}(\omega_{k+1} - \omega_k)} \\ & < |\operatorname{Im} \overline{\omega_k}\omega_{k+1}|,\ k = 0, 1, \ldots, n-1 \Big\} \end{aligned}$$

with $\mathbb{R}^2 \simeq \mathbb{C}$, $\operatorname{Im} z$ is the imaginary part of z, and $\operatorname{sgn} \alpha$ is the sign of α. Note that $P_n' = \tau_\alpha P_n$ means that $[w' \in \mathcal{H}(P_n')]$ if and only if $[w' - \alpha \in \mathcal{H}(P_n)]$. If now $P_{n,r}'$ is some regular polygon, it is enough to consider its circumscribed circle $\mathcal{C}(\beta, R)$, centered at $\beta \in \mathbb{R}^2$, with radius $R > 0$. Next, applying successively the translation $\tau_{-\beta}$, the homothety $h_{\frac{1}{R}}$ and a suitable rotation ρ_θ, we obtain $P_n^r = \rho_\theta h_{1/R} \tau_{-\beta} P_{n,r}'$ which is studied in Proposition 4.1.c. Note that the same process applies to arbitrary regular polyhedrons in $\mathbb{R}^n$, $n \geq 3$.

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