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Strongly nonlinear degenerated elliptic unilateral problems via convergence of truncations *

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Abstract

We prove an existence theorem for a strongly nonlinear degenerated elliptic inequalities involving nonlinear operators of the form $Au+g(x,u,\nabla u)$. Here A is a Leray-Lions operator, $g(x,s,\xi)$ is a lower order term satisfying some natural growth with respect to $|\nabla u|$. There is no growth restrictions with respect to $|u|$, only a sign condition. Under the assumption that the second term belongs to $W^{-1,p'}(\Omega,w^*)$, we obtain the main result via strong convergence of truncations.

1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ and p a real number such that $1<p<\infty$. Let $w=\{w_i(x),\ 0\le i\le N\}$ be a vector of weight functions on Ω, i.e. each $w_i(x)$ is a measurable a.e. strictly positive function on Ω, satisfying some integrability conditions (see section 2). The aim of this paper, is to prove an existence theorem for unilateral degenerate problems associated to a nonlinear operators of the form $Au+g(x,u,\nabla u)$. Where A is a Leray-Lions operator from $W_0^{1,p}(\Omega,w)$ into its dual $W^{-1,p'}(\Omega,w^*)$, defined by,

$$Au=-\operatorname{div}(a(x,u,\nabla u))$$

and where g is a nonlinear lower order term having natural growth with respect to $|\nabla u|$. With respect to $|u|$ we do not assume any growth restrictions, but we assume a sign condition. Bensoussan, Boccardo and Murat have proved in the second part of [2] the existence of at least one solution of the unilateral problem

$$\begin{gathered}\langle Au,v-u\rangle+\int_\Omega g(x,u,\nabla u)(v-u)\,dx\ge\langle f,v-u\rangle\quad\text{for all } v\in K_\psi\\ u\in W_0^{1,p}(\Omega)\quad u\ge\psi \text{ a.e.}\\ g(x,u,\nabla u)\in L^1(\Omega)\quad g(x,u,\nabla u)u\in L^1(\Omega)\end{gathered}$$

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where $f \in W^{-1,p'}(\Omega)$ and $K_\psi = \{v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),\ v \geq \psi$ a.e. Here ψ is a measurable function on Ω such that $\psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. For that the authors obtain the existence results by proving that the positive part u_ε^+ (resp. u_ε^-) of u_ε strongly converges to u^+(resp. u^-) in $W_0^{1,p}(\Omega)$, where u_ε is a solution of the approximate problem. In the present paper, we study the variational degenerated inequalities. More precisely, we prove the existence of a solution for the problem (3.3) (see section 3), by using another approach based on the strong convergence of the truncations $T_k(u_\varepsilon)$ in $W_0^{1,p}(\Omega, w)$. Moreover, in this paper, we assume only the weak integrability condition $\sigma^{1-q'} \in L^1_{\text{loc}}(\Omega)$ (see (2.11) below) instead of the stronger $\sigma^{1-q'} \in L^1(\Omega)$ as in [1]. This can be done by approximating Ω by a sequence of compact sets Ω_ε. Note that, in the non weighted case the same result is proved in [3] where $f \in L^1(\Omega)$. Let us point out that other works in this direction can be found in [6, 1].

This paper is organized as follows: Section 2 contains some preliminaries and basic assumptions. In section 3 we state and prove our main results.

2 Preliminaries and basic assumption

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$, let $1 < p < \infty$, and let $w = \{w_i(x),\ 0 \leq i \leq N\}$ be a vector of weight functions, i.e. every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all our considerations that for $0 \leq i \leq N$,

$$w_i \in L^1_{\text{loc}}(\Omega) \tag{2.1}$$

$$w_i^{-\frac{1}{p-1}} \in L^1_{\text{loc}}(\Omega) \tag{2.2}$$

We define the weighted space $L^p(\Omega, \gamma)$ where γ is a weight function on Ω by,

$$L^p(\Omega, \gamma) = \{u = u(x),\ u\gamma^{1/p} \in L^p(\Omega)\}$$

with the norm

$$\|u\|_{p,\gamma} = \Big(\int_\Omega |u(x)|^p \gamma(x)\,dx\Big)^{1/p}.$$

We denote by $W^{1,p}(\Omega, w)$ the space of all real-valued functions $u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions satisfies

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \text{ for all } i = 1, \ldots, N,$$

which is a Banach space under the norm

$$\|u\|_{1,p,w} = \Big(\int_\Omega |u(x)|^p w_0(x)\,dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p}. \tag{2.3}$$

Since we shall deal with the Dirichlet problem, we shall use the space

$$X = W_0^{1,p}(\Omega, w) \tag{2.4}$$

defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.3). Note that, $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, w)$ and $(X, \|.\|_{1,p,w})$ is a reflexive Banach space.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-p'},\ \forall i = 0, \dots, N\}$, where p' is the conjugate of p i.e. $p' = \frac{p}{p-1}$ (for more details we refer to [5]).

Definition 2.1 Let Y be a separable reflexive Banach space, the operator B from Y to its dual Y^* is called of the calculus of variations type, if B is bounded and is of the form

$$B(u) = B(u, u), \tag{2.5}$$

where $(u, v) \to B(u, v)$ is an operator from $Y \times Y$ into Y^* satisfying the following properties:

$$\begin{gathered} \forall u \in Y,\ v \to B(u, v) \text{ is bounded hemicontinuous from } Y \text{ into } Y^* \\ \text{and } (B(u, u) - B(u, v), u - v) \geq 0, \end{gathered} \tag{2.6}$$

$$\forall v \in Y,\ u \to B(u, v) \quad \text{is bounded hemicontinuous from } Y \text{ into } Y^*, \tag{2.7}$$

$$\begin{gathered} \text{if } u_n \rightharpoonup u \text{ weakly in } Y \text{ and if } (B(u_n, u_n) - B(u_n, u), u_n - u) \to 0 \\ \text{then, } \ B(u_n, v) \rightharpoonup B(u, v) \text{ weakly in } Y^*,\ \forall v \in Y, \end{gathered} \tag{2.8}$$

$$\begin{gathered} \text{if } u_n \rightharpoonup u \text{ weakly in } Y \text{ and if } B(u_n, v) \rightharpoonup \psi \text{ weakly in } Y^*, \\ \text{then, } \ (B(u_n, v), u_n) \to (\psi, u). \end{gathered} \tag{2.9}$$

Definition 2.2 Let Y be a reflexive Banach space, a bounded mapping B from Y to Y^* is called pseudo-monotone if for any sequence $u_n \in Y$ with $u_n \rightharpoonup u$ weakly in Y and $\limsup_{n\to\infty} \langle Bu_n, u_n - u\rangle \leq 0$, one has

$$\liminf_{n\to\infty} \langle Bu_n, u_n - v\rangle \geq \langle Bu, u - v\rangle \quad \text{for all } v \in Y.$$

We start by stating the following assumptions:

Assumption (H1) The expression

$$|||u|||_X = \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\, dx \Big)^{1/p}$$

is a norm on X and it is equivalent to the norm (2.3). There exist a weight function σ on Ω and a parameter q, such that

$$1 < q < p + p', \tag{2.10}$$

$$\sigma^{1-q'} \in L^1_{\text{loc}}(\Omega), \tag{2.11}$$

with $q' = \frac{q}{q-1}$. The Hardy inequality,

$$\Big(\int_\Omega |u(x)|^q \sigma\, dx \Big)^{1/q} \leq c \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\, dx \Big)^{1/p}, \tag{2.12}$$

holds for every $u \in X$ with a constant $c > 0$ independent of u. Moreover, the imbedding

$$X \hookrightarrow\hookrightarrow L^q(\Omega, \sigma), \tag{2.13}$$

expressed by the inequality (2.12) is compact.

Note that $(X, |||.|||_X)$ is a uniformly convex (and thus reflexive) Banach space.

Remark 2.1 If we assume that $w_0(x) \equiv 1$ and in addition the integrability condition: There exists $\nu \in]\frac{N}{p}, \infty[\cap [\frac{1}{p-1}, \infty[$ such that

$$w_i^{-\nu} \in L^1(\Omega)$$

for all $i = 1, \dots, N$ (which is stronger than (2.2)). Then

$$|||u|||_X = \Big(\sum_{i=1}^{N} \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\, dx \Big)^{1/p}$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ and is equivalent to (2.3). Moreover

$$W_0^{1,p}(\Omega, w) \hookrightarrow\hookrightarrow L^q(\Omega),$$

for all $1 \le q < p_1^*$ if $p\nu < N(\nu + 1)$ and for all $q \ge 1$ if $p\nu \ge N(\nu + 1)$, where $p_1 = \frac{p\nu}{\nu+1}$ and $p_1^* = \frac{Np_1}{N-p_1} = \frac{Np\nu}{N(\nu+1)-p\nu}$ is the Sobolev conjugate of p_1 (see [5]). Thus the hypotheses (H1) is verified for $\sigma \equiv 1$ and for all $1 < q < \min\{p_1^*, p+p'\}$ if $p\nu < N(\nu + 1)$ and for all $1 < q < p + p'$ if $p\nu \ge N(\nu + 1)$.

Let A be a nonlinear operator from $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$ defined by,

$$Au = -\operatorname{div}(a(x, u, \nabla u)),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector function satisfying the following assumptions:

Assumption (H2)

$$|a_i(x, s, \xi)| \le \beta w_i^{1/p}(x)[k(x) + \sigma^{\frac{1}{p'}} |s|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}] \quad \text{for } i = 1, \dots, N, \tag{2.14}$$

$$[a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) > 0, \quad \text{for all } \xi \ne \eta \in \mathbb{R}^N, \tag{2.15}$$

$$a(x, s, \xi).\xi \ge \alpha \sum_{i=1}^{N} w_i |\xi_i|^p, \tag{2.16}$$

where $k(x)$ is a positive function in $L^{p'}(\Omega)$ and α, β are strictly positive constants.

Assumption (H3) Let $g(x,s,\xi)$ be a Carathéodory function satisfying the following assumptions:

$$g(x,s,\xi)s \geq 0 \tag{2.17}$$

$$|g(x,s,\xi)| \leq b(|s|)\Big(\sum_{i=1}^{N} w_i|\xi_i|^p + c(x)\Big), \tag{2.18}$$

where $b:\mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function and $c(x)$ is a positive function which lies in $L^1(\Omega)$. Now we recall some lemmas introduced in [1] which will be used later.

Lemma 2.1 (cf. [1]) *Let $g \in L^r(\Omega,\gamma)$ and let $g_n \in L^r(\Omega,\gamma)$, with $\|g_n\|_{r,\gamma} \leq c \quad (1 < r < \infty)$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ weakly in $L^r(\Omega,\gamma)$, where γ is a weight function on Ω.*

Lemma 2.2 (cf. [1]) *Assume that (H1) holds. Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly Lipschitzian, with $F(0)=0$. Let $u \in W_0^{1,p}(\Omega,w)$. Then $F(u) \in W_0^{1,p}(\Omega,w)$. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial(F\circ u)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & \text{a.e. in } \{x\in\Omega : u(x)\notin D\} \\ 0 & \text{a.e. in } \{x\in\Omega : u(x)\in D\}. \end{cases}$$

Lemma 2.3 (cf. [1]) *Assume that (H1) holds. Let $u \in W_0^{1,p}(\Omega,w)$, and let $T_k(u)$, $k\in\mathbb{R}^+$, is the usual truncation then $T_k(u) \in W_0^{1,p}(\Omega,w)$. Moreover, we have*

$$T_k(u) \to u \quad \text{strongly in } W_0^{1,p}(\Omega,w).$$

Lemma 2.4 *Assume that (H1) holds. Let (u_n) be a sequence of $W_0^{1,p}(\Omega,w)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega,w)$. Then, $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p}(\Omega,w)$*

Proof. Since $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega,w)$ and by (2.13) we have for a subsequence $u_n \to u$ strongly in $L^q(\Omega,\sigma)$ and a.e. in Ω. On the other hand,

$$\begin{aligned}|||T_k(u_n)|||_X^p &= \sum_{i=1}^{N}\int_\Omega |\frac{\partial T_k(u_n)}{\partial x_i}|^p w_i = \sum_{i=1}^{N}\int_\Omega |T_k'(u_n)\frac{\partial u_n}{\partial x_i}|^p w_i \\ &\leq \sum_{i=1}^{N}\int_\Omega |\frac{\partial u_n}{\partial x_i}|^p w_i = |||u_n|||_X^p.\end{aligned}$$

Then $(T_k(u_n))$ is bounded in $W_0^{1,p}(\Omega,w)$, hence by using (2.13), $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p}(\Omega,w)$.

Lemma 2.5 (cf. [1]) *Assume that (H1) and (H2) are satisfied, and let (u_n) be a sequence of $W_0^{1,p}(\Omega,w)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega,w)$ and*

$$\int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\,dx \to 0.$$

Then $u_n \to u$ strongly in $W_0^{1,p}(\Omega,w)$.

3 Main result

Let ψ be a measurable function with values in $\mathbb{R}$ such that

$$\psi^+ \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega). \tag{3.1}$$

Set

$$K_\psi = \{v \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega) \quad v \geq \psi \ a.e.\}. \tag{3.2}$$

Note that (3.1) implies $K_\psi \neq \emptyset$. Consider the nonlinear problem with Dirichlet boundary conditions,

$$\begin{gathered} \langle Au, v-u\rangle + \int_\Omega g(x,u,\nabla u)(v-u)\,dx \geq \langle f, v-u\rangle \text{for all } v \in K_\psi \\ u \in W_0^{1,p}(\Omega, w) \quad u \geq \psi \text{ a.e.} \\ g(x,u,\nabla u) \in L^1(\Omega), \quad g(x,u,\nabla u)u \in L^1(\Omega) \end{gathered} \tag{3.3}$$

Then, the following result can be proved for a solution u of this problem.

Theorem 3.1 *Assume that the assumptions (H1)–(H3) and (3.1) hold and let $f \in W^{-1,p'}(\Omega, w^*)$. Then there exists at least one solution of (3.3).*

Remark 3.1 1) Theorem 3.1 can be generalized in weighted case to an analogous statement in [2].

2) Note that in [1] the authors have assumed that $\sigma^{1-q'} \in L^1(\Omega)$ which is stronger than (2.11).

In the proof of theorem 3.1 we need the following lemma.

Lemma 3.1 *Assume that f lies in $W^{-1,p'}(\Omega, w^*)$. If u is a solution of $(\mathcal{P})$, then, u is also a solution of the variational inequality*

$$\begin{gathered} \langle Au, T_k(v-u)\rangle + \int_\Omega g(x,u,\nabla u)T_k(v-u)\,dx \geq \langle f, T_k(v-u)\rangle \quad \forall k > 0, \\ \textit{for all } v \in W_0^{1,p}(\Omega, w) \quad v \geq \psi \ a.e. \\ u \in W_0^{1,p}(\Omega, w) \quad u \geq \psi \ a.e. \\ g(x,u,\nabla u) \in L^1(\Omega). \end{gathered} \tag{3.4}$$

Conversely, if u is a solution of (3.4) then u is a solution of (3.3).

The proof of this lemma is similar to the proof of [3, Remark 2.2] for the non weighted case.

Proof of theorem 3.1 **Step (1) The approximate problem and a priori estimate.** Let Ω_ε be a sequence of compact subsets of Ω such that Ω_ε increases to Ω as $\varepsilon \to 0$. We consider the sequence of approximate problems,

$$\begin{aligned} \langle Au_\varepsilon, v-u_\varepsilon\rangle + \int_\Omega g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)(v-u_\varepsilon)\,dx &\geq \langle f, v-u_\varepsilon\rangle \\ v\in W_0^{1,p}(\Omega,w) \quad & v\geq\psi \text{ a.e.} \\ u_\varepsilon\in W_0^{1,p}(\Omega,w) \quad & u_\varepsilon\geq\psi \text{ a.e.} \end{aligned} \tag{3.5}$$

where,

$$g_\varepsilon(x,s,\xi) = \frac{g(x,s,\xi)}{1+\varepsilon|g(x,s,\xi)|}\chi_{\Omega_\varepsilon}(x),$$

and where $\chi_{\Omega_\varepsilon}$ is the characteristic function of Ω_ε. Note that $g_\varepsilon(x,s,\xi)$ satisfies the following conditions,

$$g_\varepsilon(x,s,\xi)s\geq 0, \quad |g_\varepsilon(x,s,\xi)|\leq |g(x,s,\xi)| \quad \text{and} \quad |g_\varepsilon(x,s,\xi)|\leq \frac{1}{\varepsilon}.$$

We define the operator $G_\varepsilon:\ X\to X^*$ by,

$$\langle G_\varepsilon u, v\rangle = \int_\Omega g_\varepsilon(x,u,\nabla u)v\,dx.$$

Thanks to Hölder's inequality we have for all $u\in X$ and $v\in X$,

$$\begin{aligned} \Big|\int_\Omega g_\varepsilon(x,u,\nabla u)v\,dx\Big| &\leq \Big(\int_\Omega |g_\varepsilon(x,u,\nabla u)|^{q'}\sigma^{-\frac{q'}{q}}\,dx\Big)^{1/q'}\Big(\int_\Omega |v|^q\sigma\,dx\Big)^{1/q} \\ &\leq \frac{1}{\varepsilon}\Big(\int_{\Omega_\varepsilon}\sigma^{1-q'}\,dx\Big)^{1/q'}\|v\|_{q,\sigma}\leq c_\varepsilon|||v|||. \end{aligned} \tag{3.6}$$

The last inequality is due to (2.11) and (2.13).

Lemma 3.2 *The operator $B_\varepsilon = A+G_\varepsilon$ from X into its dual X^* is pseudo-monotone. Moreover, B_ε is coercive, in the sense that: There exists $v_0\in K_\psi$ such that*

$$\frac{\langle B_\varepsilon v, v-v_0\rangle}{|||v|||}\to+\infty \quad \textit{as } |||v|||\to\infty, \quad v\in K_\psi.$$

The proof of this lemma will be presented below. In view of lemma 3.2, (3.5) has a solution by the classical result (cf. Theorem 8.1 and Theorem 8.2 chapter 2 [7]).

With $v=\psi^+$ as test function in (3.5), we deduce that $\int_\Omega g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)(u_\varepsilon-\psi^+)\geq 0$, then, $\langle Au_\varepsilon,u_\varepsilon\rangle\leq\langle f,u_\varepsilon-\psi^+\rangle+\langle Au_\varepsilon,\psi^+\rangle$, i.e.,

$$\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\,dx\leq\langle f,u_\varepsilon-\psi^+\rangle+\sum_{i=1}^N\int_\Omega a_i(x,u_\varepsilon,\nabla u_\varepsilon)\frac{\partial\psi^+}{\partial x_i}\,dx,$$

then,

$$\begin{aligned}
&\alpha \sum_{i=1}^{N} \int_\Omega w_i |\frac{\partial u_\varepsilon}{\partial x_i}|^p \, dx \\
&= \alpha |||u_\varepsilon|||^p \\
&\leq \|f\|_{X^*} (|||u_\varepsilon||| + |||\psi^+|||) + \\
&\quad + \sum_{i=1}^{N} \Big(\int_\Omega |a_i(x, u_\varepsilon, \nabla u_\varepsilon)|^{p'} w_i^{1-p'} \, dx \Big)^{\frac{1}{p'}} \Big(\int_\Omega |\frac{\partial \psi^+}{\partial x_i}|^p w_i \, dx \Big)^{1/p} \\
&\leq \|f\|_{X^*} (|||u_\varepsilon||| + |||\psi^+|||) + \\
&\quad + c \sum_{i=1}^{N} \Big(\int_\Omega (k^{p'} + |u_\varepsilon|^q \sigma + \sum_{j=1}^{N} |\frac{\partial u_\varepsilon}{\partial x_j}|^p w_j) \, dx \Big)^{1/p'} |||\psi^+|||.
\end{aligned}$$

Using (2.13) the last inequality becomes,

$$\alpha |||u_\varepsilon|||^p \leq c_1 |||u_\varepsilon||| + c_2 |||u_\varepsilon|||^{\frac{q}{p'}} + c_3 |||u_\varepsilon|||^{p-1} + c_4,$$

where c_i are various positive constants. Then, thanks to (2.10) we can deduce that u_ε remains bounded in $W_0^{1,p}(\Omega, w)$, i.e.,

$$|||u_\varepsilon||| \leq \beta_0, \tag{3.7}$$

where β_0 is some positive constant. Extracting a subsequence (still denoted by u_ε) we get

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } X \text{ and a.e. in } \Omega.$$

Note that $u \geq \psi$ a.e.

Step (2) Strong convergence of $T_k(u_\varepsilon)$. Thanks to (3.7) and (2.13) we can extract a subsequence still denoted by u_ε such that

$$\begin{aligned} u_\varepsilon \rightharpoonup u \quad &\text{weakly in } W_0^{1,p}(\Omega, w,) \\ u_\varepsilon \to u \quad &\text{a.e. in } \Omega. \end{aligned} \tag{3.8}$$

Let $k > 0$ by lemma 2.4 we have

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega, w) \text{ as } \varepsilon \to 0. \tag{3.9}$$

Our objective is to prove that

$$T_k(u_\varepsilon) \to T_k(u) \quad \text{strongly in } W_0^{1,p}(\Omega, w) \text{ as } \varepsilon \to 0. \tag{3.10}$$

Fix $k > \|\psi^+\|_\infty$, and use the notation $z_\varepsilon = T_k(u_\varepsilon) - T_k(u)$. We use, as a test function in (3.5),

$$v_\varepsilon = u_\varepsilon - \eta \varphi_\lambda(z_\varepsilon) \tag{3.11}$$

where $\varphi_\lambda(s) = s e^{\lambda s^2}$ and $\eta = e^{-4\lambda k^2}$. Then we can check that v_ε is admisible test function. So that

$$-\langle A u_\varepsilon, \eta \varphi_\lambda z_\varepsilon \rangle - \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \eta \varphi_\lambda(z_\varepsilon) \, dx \geq -\langle f, \eta \varphi_\lambda(z_\varepsilon) \rangle$$

which implies that

$$\langle Au_\varepsilon, \varphi_\lambda(z_\varepsilon)\rangle + \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\varphi_\lambda(z_\varepsilon)\, dx \le \langle f, \varphi_\lambda(z_\varepsilon)\rangle. \tag{3.12}$$

Since $\varphi_\lambda(z_\varepsilon)$ is bounded in X and converges a.e. in Ω to zero and using (2.13), we have $\varphi_\lambda(z_\varepsilon) \rightharpoonup 0$ weakly in X as $\varepsilon \to 0$. Then

$$\eta_1(\varepsilon) = \langle f, \varphi_\lambda(z_\varepsilon)\rangle \to 0, \tag{3.13}$$

and since $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\varphi_\lambda(z_\varepsilon) \ge 0$ in the subset $\{x \in \Omega : |u_\varepsilon(x)| \ge k\}$ hence (3.12) and (3.13) yield

$$\langle Au_\varepsilon, \varphi_\lambda(z_\varepsilon)\rangle + \int_{\{|u_\varepsilon| \le k\}} g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\varphi_\lambda(z_\varepsilon)\, dx \le \eta_1(\varepsilon). \tag{3.14}$$

We study each term in the left hand side of (3.14). We have,

$$\begin{aligned}
\langle Au_\varepsilon, \varphi_\lambda(z_\varepsilon)\rangle &= \int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon)\nabla(T_k(u_\varepsilon) - T_k(u))\varphi'_\lambda(z_\varepsilon)\, dx \\
&= \int_\Omega a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon))\nabla(T_k(u_\varepsilon) - T_k(u))\varphi'_\lambda(z_\varepsilon)\, dx \\
&\quad - \int_{\{|u_\varepsilon| > k\}} a(x, u_\varepsilon, \nabla u_\varepsilon)\nabla T_k(u)\varphi'_\lambda(z_\varepsilon)\, dx \\
&= \int_\Omega \left(a(x, T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) - a(x, T_k(u_\varepsilon), \nabla T_k(u))\right)\nabla(T_k(u_\varepsilon) \\
&\quad - T_k(u))\varphi'_\lambda(z_\varepsilon)\, dx + \eta_2(\varepsilon),
\end{aligned} \tag{3.15}$$

where,

$$\begin{aligned}
\eta_2(\varepsilon) = &\int_\Omega a(x, T_k(u_\varepsilon), \nabla T_k(u))\nabla(T_k(u_\varepsilon) - T_k(u))\varphi'_\lambda(z_\varepsilon)\, dx \\
&- \int_{\{|u_\varepsilon| > k\}} a(x, u_\varepsilon, \nabla u_\varepsilon)\nabla T_k(u)\varphi'_\lambda(z_\varepsilon)\, dx,
\end{aligned}$$

which converges to 0 as $\varepsilon \to 0$. On the other hand,

$$
\begin{aligned}
&|\int_{\{|u_\varepsilon|\le k\}} g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\varphi_\lambda(z_\varepsilon)\,dx| \\
\le& \int_{\{|u_\varepsilon|\le k\}} b(k)[c(x)+\sum_{i=1}^N |\frac{\partial u_\varepsilon}{\partial x_i}|^p w_i]|\varphi_\lambda(z_\varepsilon)|\,dx \\
\le& b(k)\int_{\{|u_\varepsilon|\le k\}} c(x)|\varphi_\lambda(z_\varepsilon)|\,dx + \frac{b(k)}{\alpha}\int_{\{|u_\varepsilon|\le k\}} a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon|\varphi_\lambda(z_\varepsilon)|\,dx \\
=& \eta_3(\varepsilon)+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon)|\varphi_\lambda(z_\varepsilon)|\,dx \\
=& \frac{b(k)}{\alpha}\int_\Omega \left(a(x,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon)) - a(x,T_k(u_\varepsilon),\nabla T_k(u))\right)\nabla(T_k(u_\varepsilon) \\
&- T_k(u))|\varphi_\lambda(z_\varepsilon)|\,dx + \eta_4(\varepsilon)
\end{aligned} \tag{3.16}
$$

where

$$\eta_3(\varepsilon) = b(k)\int_{\{|u_\varepsilon|\le k\}} c(x)|\varphi_\lambda(z_\varepsilon)|\,dx \to 0 \text{ as } \varepsilon\to 0$$

and

$$
\begin{aligned}
\eta_4(\varepsilon) \;=\;& \eta_3(\varepsilon)+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_\varepsilon),\nabla T_k(u))\nabla(T_k(u_\varepsilon)-T_k(u))|\varphi_\lambda(z_\varepsilon)|\,dx \\
&+\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))\nabla T_k(u)|\varphi_\lambda(z_\varepsilon)|\,dx \to 0 \quad \text{as } \varepsilon\to 0.
\end{aligned}
$$

Note that, when $\lambda \ge \left(\frac{b(k)}{2\alpha}\right)^2$ we have

$$\varphi'_\lambda(s) - \frac{b(k)}{\alpha}|\varphi(s)| \ge \frac{1}{2}.$$

Which combining with (3.14),(3.15) and (3.16) one obtains

$$
\begin{aligned}
&\int_\Omega \big(a(x,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon)) - a(x,T_k(u_\varepsilon),\nabla T_k(u))\big)\nabla(T_k(u_\varepsilon)-T_k(u))\,dx \\
&\qquad\le \eta_5(\varepsilon) = 2(\eta_1(\varepsilon)-\eta_2(\varepsilon)+\eta_4(\varepsilon)) \to 0 \quad \text{as } \varepsilon\to 0.
\end{aligned}
$$

Finally lemma 2.5 implies (3.10) for any fixed $k \ge \|\psi\|_\infty$.
Step (3) Passage to the limit. In view of (3.10) we have for a subsequence,

$$\nabla u_\varepsilon \to \nabla u \quad \text{a.e. in } \Omega, \tag{3.17}$$

which with (3.8) imply,

$$
\begin{gathered}
a(x,u_\varepsilon,\nabla u_\varepsilon) \to a(x,u,\nabla u) \quad \text{a.e. in } \Omega, \\
g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) \to g(x,u,\nabla u)\text{a.e. in } \Omega, \\
g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon \to g(x,u,\nabla u)u \quad \text{a.e. in } \Omega.
\end{gathered} \tag{3.18}
$$

On the other hand, thanks to (2.14) and (3.7) we have $a(x,u_\varepsilon,\nabla u_\varepsilon)$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega,w_i^*)$ then by lemma 2.1 we obtain

$$a(x,u_\varepsilon,\nabla u_\varepsilon) \rightharpoonup a(x,u,\nabla u) \quad \text{weakly in } \prod_{i=1}^N L^{p'}(\Omega,w_i^*). \tag{3.19}$$

We shall prove that,

$$g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) \to g(x,u,\nabla u) \quad \text{strongly in } \ L^1(\Omega). \tag{3.20}$$

By (3.18), to apply Vitali's theorem it suffices to prove that $g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)$ is uniformly equi-integrable. Indeed, thanks to (2.17), (3.6) and (3.7) we obtain,

$$0 \le \int_\Omega g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon\,dx \le c_0, \tag{3.21}$$

where c_0 is some positive constant. For any measurable subset E of Ω and any $m>0$ we have,

$$\int_E |g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)|\,dx = \int_{E\cap X_m^\varepsilon} |g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)|\,dx + \int_{E\cap Y_m^\varepsilon} |g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)|\,dx$$

where,

$$X_m^\varepsilon = \{x\in\Omega,\ |u_\varepsilon(x)|\le m\}, \quad Y_m^\varepsilon = \{x\in\Omega,\ |u_\varepsilon(x)|>m\}.$$

From these expressions, (2.18), and (3.21), we have

$$\begin{aligned} &\int_E |g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)|\,dx \\ &= \int_{E\cap X_m^\varepsilon} |g_\varepsilon(x,u_\varepsilon,\nabla T_m(u_\varepsilon))|\,dx + \int_{E\cap Y_m^\varepsilon} |g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)|\,dx \\ &\le \int_{E\cap X_m^\varepsilon} |g_\varepsilon(x,u_\varepsilon,\nabla T_m(u_\varepsilon))|\,dx + \frac{1}{m}\int_\Omega g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon\,dx \\ &\le b(m)\int_E (\sum_{i=1}^N w_i|\frac{\partial T_m(u_\varepsilon)}{\partial x_i}|^p + c(x)) + \frac{c_0}{m}. \end{aligned} \tag{3.22}$$

Since the sequence $(\nabla T_m(u_\varepsilon))$ strongly converges in $\prod_{i=1}^N L^p(\Omega,w_i)$, then (3.22) implies the equi-integrability of $g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)$.

Moreover, since $g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon \ge 0$ a.e. in Ω, then by (3.18), (3.21) and Fatou's lemma, we have $g(x,u,\nabla u)u \in L^1(\Omega)$. On the other hand, for $v\in L^\infty(\Omega)$, set $h = k + \|v\|_\infty$, then

$$\begin{aligned} |\frac{\partial T_k(v-u_\varepsilon)}{\partial x_i}|w_i^{1/p} &= \chi_{\{|v-u_\varepsilon|\le k\}}|\frac{\partial v}{\partial x_i} - \frac{\partial u_\varepsilon}{\partial x_i}|w_i^{1/p} \\ &\le \chi_{\{|u_\varepsilon|\le h\}}|\frac{\partial v}{\partial x_i} - \frac{\partial u_\varepsilon}{\partial x_i}|w_i^{1/p} \\ &\le |\frac{\partial v}{\partial x_i}|w_i^{1/p} + |\frac{\partial T_h(u_\varepsilon)}{\partial x_i}|w_i^{1/p} \end{aligned}$$

which implies, using Vitali's theorem with (3.10) and (3.17) that

$$\nabla T_k(v-u_\varepsilon) \to \nabla T_k(v-u) \quad \text{strongly in } \prod_{i=1}^N L^p(\Omega, w_i) \tag{3.23}$$

for any $v \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$. Thanks to lemma 3.1 and from (3.19), (3.20) and (3.23) we can pass to the limit in

$$\langle Au_\varepsilon, T_k(v-u_\varepsilon)\rangle + \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) T_k(v-u_\varepsilon) \geq \langle f, T_k(v-u_\varepsilon)\rangle$$

and we obtain,

$$\langle Au, T_k(v-u)\rangle + \int_\Omega g(x, u, \nabla u) T_k(v-u) \geq \langle f, T_k(v-u)\rangle \tag{3.24}$$

for any $v \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$ and for all $k > 0$.

Taking for any $v \in W_0^{1,p}(\Omega, w)$ and $v \geq \psi$ the test function $T_m(v)$ which belongs to $W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$ for $m \geq \|\psi^+\|_\infty$ and passing to the limit in (3.24) as $m \to \infty$, then u is a solution of (3.4). Using again lemma 3.1 we obtain the desired result, i.e., u is a solution of (3.3).

Proof of lemma 3.2 By proposition 2.6 chapter 2 [7], it is sufficient to show that B_ε is of the calculus of variations type in the sense of definition 2.1. Indeed put,

$$b_1(u, v, \tilde{w}) = \sum_{i=1}^N \int_\Omega a_i(x, u, \nabla v) \nabla \tilde{w}\, dx, \quad b_2(u, \tilde{w}) = \int_\Omega g_\varepsilon(x, u, \nabla u) \tilde{w}\, dx.$$

Then the mapping $\tilde{w} \mapsto b_1(u, v, \tilde{w}) + b_2(u, \tilde{w})$ is continuous in X. Then

$$b_1(u, v, \tilde{w}) + b_2(u, \tilde{w}) = b(u, v, \tilde{w}) = \langle B_\varepsilon(u, v), \tilde{w}\rangle, \quad B_\varepsilon(u, v) \in W^{-1,p'}(\Omega, w^*)$$

and we have

$$B_\varepsilon(u, u) = B_\varepsilon u.$$

Using (2.14) and Hölder's inequality we can show that A is bounded as in [4], and thanks to (3.6) B_ε is bounded. Then, it is sufficient to check (2.6)-(2.9).

Next we show that (2.6) and (2.7) are true. By (2.15) we have,

$$(B_\varepsilon(u, u) - B_\varepsilon(u, v), u - v) = b_1(u, u, u-v) - b_1(u, v, u-v) \geq 0.$$

The operator $v \to B_\varepsilon(u, v)$ is bounded hemicontinuous. Indeed, we have

$$a_i(x, u, \nabla(v_1 + \lambda v_2)) \to a_i(x, u, \nabla v_1) \quad \text{strongly in } L^{p'}(\Omega, w_i^*) \text{ as } \lambda \to 0. \tag{3.25}$$

On the other hand, $(g_\varepsilon(x, u_1 + \lambda u_2, \nabla(u_1 + \lambda u_2)))_\lambda$ is bounded in $L^{q'}(\Omega, \sigma^{1-q'})$ and $g_\varepsilon(x, u_1 + \lambda u_2, \nabla(u_1 + \lambda u_2)) \to g_\varepsilon(x, u_1, \nabla u_1)$ *a.e.* in Ω, hence lemma 2.1 gives

$$g_\varepsilon(x, u_1 + \lambda u_2, \nabla(u_1 + \lambda u_2)) \rightharpoonup g_\varepsilon(x, u_1, \nabla u_1) \quad \text{weakly in } L^{q'}(\Omega, \sigma^{1-q'}) \text{ as } \lambda \to 0. \tag{3.26}$$

Using (3.25) and (3.26) we can write

$$b(u, v_1 + \lambda v_2, \tilde{w}) \to b(u, v_1, \tilde{w}) \quad \text{as } \lambda \to 0 \ \ \forall u, v_i, \tilde{w} \in X.$$

Similarly we can prove (2.7).
Proof of assertion (2.8). Assume that $u_n \rightharpoonup u$ weakly in X and $(B(u_n, u_n) - B(u_n, u), u_n - u) \to 0$. We have,

$$\begin{aligned}&(B(u_n, u_n) - B(u_n, u), u_n - u)\\ &\qquad = \sum_{i=1}^{N} \int_\Omega (a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u))\, \nabla(u_n - u)\, dx \to 0,\end{aligned}$$

then, by lemma 2.5, $u_n \to u$ strongly in X, which gives

$$b(u_n, v, \tilde{w}) \to b(u, v, \tilde{w}) \quad \forall \tilde{w} \in X,$$

i.e., $B_\varepsilon(u_n, v) \rightharpoonup B_\varepsilon(u, v)$ weakly in X^*. It remains to prove (2.9). Assume that

$$u_n \rightharpoonup u \quad \text{weakly in } X \tag{3.27}$$

and that

$$B(u_n, v) \rightharpoonup \psi \quad \text{weakly in } X^*. \tag{3.28}$$

Thanks to (2.13), (2.14) and (3.27) we obtain,

$$a_i(x, u_n, \nabla v) \to a_i(x, u, \nabla v) \quad \text{in } L^{p'}(\Omega, w_i^*) \text{ as } n \to \infty,$$

then,

$$b_1(u_n, v, u_n) \to b_1(u, v, u). \tag{3.29}$$

On the other hand, by Hölder's inequality,

$$\begin{aligned}|b_2(u_n, u_n - u)| \leq& \left(\int_\Omega |g_\varepsilon(x, u_n, \nabla u_n)|^{q'} \sigma^{\frac{-q'}{q}}\, dx\right)^{1/q'} \left(\int_\Omega |u_n - u|^q \sigma\, dx\right)^{1/q}\\ \leq& \frac{1}{\varepsilon}\left(\int_{\Omega_\varepsilon} \sigma^{\frac{-q'}{q}}\, dx\right)^{1/q'} \|u_n - u\|_{L^q(\Omega,\sigma)} \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

i.e.,

$$b_2(u_n, u_n - u) \to 0 \text{ as } n \to \infty, \tag{3.30}$$

but in view of (3.28) and (3.29) we obtain

$$b_2(u_n, u) = (B_\varepsilon(u_n, v), u) - b_1(u_n, v, u) \to (\psi, u) - b_1(u, v, u)$$

and from (3.30) we have $b_2(u_n, u_n) \to (\psi, u) - b_1(u, v, u)$. Then,

$$(B_\varepsilon(u_n, v), u_n) = b_1(u_n, v, u_n) + b_2(u_n, u_n) \to (\psi, u).$$

Now show that B_ε is coercive. Let $v_0 \in K_\psi$. From Hölder's inequality, the growth condition (2.14) and the compact imbedding (2.13) we have

$$\begin{aligned}
\langle Av, v_0 \rangle &= \sum_{i=1}^{N} \int_\Omega a_i(x, v, \nabla v) \frac{\partial v_0}{\partial x_i}\, dx \\
&\leq \sum_{i=1}^{N} \Big(\int_\Omega |a_i(x, v, \nabla v)|^{p'} w_i^{\frac{-p'}{p}}\, dx \Big)^{\frac{1}{p'}} \Big(\int_\Omega |\frac{\partial v_0}{\partial x_i}|^p w_i\, dx \Big)^{1/p} \\
&\leq c_1 |||v_0||| \Big(\int_\Omega k(x)^{p'} + |v|^q \sigma + \sum_{j=1}^{N} |\frac{\partial v}{\partial x_j}|^p w_j\, dx \Big)^{\frac{1}{p'}} \\
&\leq c_2 (c_3 + |||v|||^{\frac{q}{p'}} + |||v|||^{p-1}),
\end{aligned}$$

where c_i are various constants. Thanks to (2.16), we obtain

$$\frac{\langle Av, v \rangle}{|||v|||} - \frac{\langle Av, v_0 \rangle}{|||v|||} \geq \alpha |||v|||^{p-1} - |||v|||^{p-2} - |||v|||^{\frac{q}{p'}-1} - \frac{c}{|||v|||}.$$

In view of (2.10) we have $p - 1 > \frac{q}{p'} - 1$. Then,

$$\frac{\langle Av, v - v_0 \rangle}{|||v|||} \to \infty \quad \text{as } |||v||| \to \infty.$$

Since $\langle G_\varepsilon v, v \rangle \geq 0$ and $\langle G_\varepsilon v, v_0 \rangle$ is bounded, we have

$$\frac{\langle B_\varepsilon v, v - v_0 \rangle}{|||v|||} \geq \frac{\langle Av, v - v_0 \rangle}{|||v|||} - \frac{\langle G_\varepsilon v, v_0 \rangle}{|||v|||} \to \infty \quad \text{as } |||v||| \to \infty.$$

Remark 3.2 Assumption (2.10) appears to be necessary to prove the boundedness of $(u_\varepsilon)_\varepsilon$ in $W_0^{1,p}(\Omega, w)$ and the coercivity of the operator B_ε. While Assumption (2.11) is necessary to prove the boundedness of G_ε in $W_0^{1,p}(\Omega, w)$. Thus, when $g \equiv 0$, we don't need to assume (2.11).

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