2001-Luminy conference on Quasilinear Elliptic and Parabolic Equations and Systems,
Electronic Journal of Differential Equations, Conference 08, 2002, pp 9–22.
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Semi-classical analysis and vanishing properties of solutions to quasilinear equations *

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Abstract

Let Ω be an open bounded subset of $\mathbb{R}^N$ and b a measurable nonnegative function in Ω. We deal with the time compact support property for

$$u_t - \Delta u + b(x)|u|^{q-1}u = 0$$

for $p \geq 2$ and

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + b(x)|u|^{q-1}u = 0$$

with $m \geq 1$ where $0 \leq q < 1$. We give criteria associated to the first eigenvalue of some quasilinear Schrödinger operators in semi-classical limits. We also provide a lower bound for this eigenvalue.

1 Introduction

Let Ω be a regular bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ and $q \in [0, 1)$. We consider the weak solution of the degenerate parabolic equations subject to the Neumann boundary condition:

$$\begin{gathered} u_t - \Delta u + b(x)|u|^{q-1}u = 0 \quad \text{in } \Omega \times (0, \infty), \\ \partial_\nu u = 0 \quad \text{on } \partial\Omega, \\ u(x, 0) = u_0(x) \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

and more generally,

$$\begin{gathered} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + b(x)|u|^{q-1}u = 0 \quad \text{in } \Omega \times (0, \infty), \\ \partial_\nu u = 0 \quad \text{on } \partial\Omega, \\ u(x, 0) = u_0(x) \quad \text{in } \Omega, \end{gathered} \tag{1.2}$$

* *Mathematics Subject Classifications:* 35K55, 35P15.
Key words: evolution equations, p-Laplacian, porous-medium, strong absorption, regularizing effects, semi-classical limits.

Published October 21, 2002.

with $p \geq 2$, or

$$\begin{aligned} u_t - \Delta(u^m) + b(x)|u|^{q-1}u = 0 \quad \text{in } \Omega \times (0,\infty), \\ \partial_\nu u = 0 \quad \text{on } \partial\Omega, \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{aligned} \tag{1.3}$$

with $m \geq 1$.

Many authors have already dealt with such equations giving a wide range of applications in physical mathematics. Now, our task is to describe a compact compact support property, in time.

Definition. A solution u satisfies the Time Compact Support property (for short **TCS** property) if there exists a time T such that for all $t \geq T$ and all $x \in \Omega$, $u(x,t) = 0$.

First, we study some simple cases for (1.1):
1) Suppose that there exists a real γ such as $b(x) \geq \gamma > 0$ a.e. in Ω. From the maximum principle, $u(x,t) \leq (1 - \gamma(1-q)t)^{\frac{1}{1-q}}$ in $\Omega \times (0,\infty)$. The nonlinear absorption is stronger than the diffusion and the **TCS** property holds.
2) We have a different feature if we assume that there exists a connected open set ω such as $b(x) = 0$ a.e. in ω (no absorption in ω). Then usually, u has not the compact support property. Indeed, if we denote by $\lambda(\omega)$ the first eigenvalue of $-\Delta$ in $W_0^{1,2}(\omega)$ and ζ the first eigenfunction with $\|\zeta\|_{L^\infty(\omega)} = 1$ and $\zeta \geq 0$, then from the maximum principle, $u(x,t) \geq \zeta(x)\, e^{-\lambda(\omega)t}$ for all x in ω and for all $t \geq 0$.

Up to some minor changes, the previous examples are also valid if u satisfies (1.2) and (1.3). The compact support property is related to $\{x : b(x) = 0\}$ and the behaviour of the function b in a neighbourhood of this set.

2 The time compact support property

The starting idea was in the article of Kondratiev and Véron [7]. They established this property for (1.1) with the help of the following quantities

$$\mu_n = \inf\Big\{ \int_\Omega (|\nabla v|^2 + 2^n b(x)|v|^2)dx : v \in W^{1,2}(\Omega), \int_\Omega |v|^2\, dx = 1 \Big\},$$

with n positive integer number. More precisely, up to a small change, they proved the following theorem.

Theorem 2.1 *Suppose that u is a solution of (1.1) and*

$$\sum_{n=0}^{+\infty} \frac{\ln \mu_n}{\mu_n} < +\infty,$$

then there exists some $T > 0$ such that $u(x,t) = 0$ for $(x,t) \in \Omega \times [T, +\infty)$.

We see that μ_n are linked to well-known questions in the semi-classical limit of Schrödinger operator of the type $-\Delta + 2^n b(.)$.

In [3], the authors give a first extension of this theorem by replacing the sequence 2^n by any decreasing sequence going to zero. For the sake of simplicity, we denote by $\mu(\alpha)$ the lowest eigenvalue of the Neumann realization of the Schrödinger operator $-\Delta + \alpha^{q-1} b(.)$ in $W^{1,2}(\Omega)$, that is,

$$\mu(\alpha) = \inf \Big\{ \int_\Omega (|\nabla v|^2 + \alpha^{q-1} b(x)|v|^2) dx : v \in W^{1,2}(\Omega), \int_\Omega |v|^2 \, dx = 1 \Big\}. \quad (2.1)$$

They proved the following theorem [3, page 50].

Theorem 2.2 *Assume that (α_n) is a decreasing sequence of positive numbers such that*

$$\sum_{n=1}^{+\infty} \frac{1}{\mu(\alpha_n)} \big(\ln(\mu(\alpha_n)) + \ln(\frac{\alpha_n}{\alpha_{n+1}}) + 1 \big) < +\infty, \quad (2.2)$$

then any solution of (1.1) satisfies the ***TCS*** *property.*

The proof is based on an iterative method using the following lemma.

Lemma 2.1 *Suppose that $b \geq 0$ a.e. in Ω, $0 \leq q < 1$ and u is a bounded weak solution of (1.1) such that $\|u_0\|_{L^\infty(\Omega)} \leq \alpha$ for some $\alpha > 0$. Then*

$$\|u(.,t)\|_{L^\infty(\Omega)} \leq \min \big(1, C(\mu(\alpha))^{N/4} e^{-t\mu(\alpha)} \big) \|u_0\|_{L^\infty(\Omega)}, \quad (2.3)$$

where $C = C(\Omega)$ is a positive real number.

Outline of the proof. We use u as test-function and since $u^{1-q} \geq \alpha^{1-q}$, we have

$$\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 \, dx + \int_\Omega (|\nabla u|^2 + b\alpha^{q-1} u^2) \, dx \leq 0.$$

The definition of $\mu(\alpha)$ and Hölder's inequality gives

$$\|u(.,s)\|_{L^2(\Omega)} \leq e^{-s\mu(\alpha)} |\Omega|^{1/2} \|u_0\|_{L^\infty(\Omega)},$$

for all positive real number s. The regularizing effects associated to this type of equation can be write under the following form [11, 12]:

$$\|u(.,t)\|_{L^\infty(\Omega)} \leq C(1 + \frac{1}{t-s})^{N/4} \|u(.,s)\|_{L^2(\Omega)},$$

for all $t > s$. Taking $t - s = 1/\mu(\alpha)$ completes the proof of the lemma. □

Sketch of the proof of the theorem 2.2. (α_n) is a decreasing sequence which tends to zero. We shall construct an increasing sequence (t_n) such that for all n,

$$\forall t \geq t_n,\ \|u(.,t)\|_{L^\infty(\Omega)} \leq \alpha_n.$$

If $\lim_{n\to+\infty} t_n = T < +\infty$ then u satisfies the **TCS** property. To do this, we use an iterative method to find an upper bound for $\sum_n t_{n+1} - t_n$ under the form of a convergent series. We set $t_0 = 0$ and $\alpha = \alpha_0 = \|u_0\|_{L^\infty(\Omega)}$. Applying Lemma 2.1 gives an upper bound for $\|u(.,t)\|_{L^\infty(\Omega)}$. t_1 is defined by

$$C(\mu(\alpha_0))^{N/4} e^{-(t_1-t_0)\mu(\alpha_0)} \alpha_0 = \alpha_1.$$

A this point, we apply Lemma 2.1 but for time $t \geq t_1$ with $\alpha = \alpha_1$. Iterating this process provide us the formula

$$C(\mu(\alpha_n))^{N/4} e^{-(t_{n+1}-t_n)\mu(\alpha_n)} \alpha_n = \alpha_{n+1}.$$

So we obtain an upper bound for the series $\sum_n t_{n+1} - t_n$. □

An analoguous result can be proved for (1.2). But before, we recall the regularizing effects for this type of equation [11, 12].

Theorem 2.3 *Let $p > 1$. Suppose that u is a weak solution of*

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + B(x,t,u) = 0 \quad \text{in } \Omega \times (0,\infty),$$
$$\partial_\nu u = 0 \quad \text{on } \partial\Omega,$$
$$u(x,0) = u_0(x) \in L^r(\Omega),$$

where B is a Caratheodory functions which satisfies $B(x,t,\rho)\rho \geq 0$ a.e. in $\Omega \times (0,\infty)$. If $r \geq 1$, $r > N(2/p - 1)$ then

$$\|u(.,t)\|_{L^\infty(\Omega)} \leq C\big(1 + \frac{1}{t}\big)^{\delta(r)} \|u(.,0)\|^{\sigma(r)}_{L^r(\Omega)},$$

with $C = C(\Omega, p)$, $\delta(r) = \frac{N}{rp+N(p-2)}$ and $\sigma(r) = \frac{rp}{rp+N(p-2)}$.

In a similar way, we introduce

$$\mu(\alpha,p) = \inf\Big\{ \int_\Omega (|\nabla v|^p + \alpha^{q-(p-1)}\, b(x)\, |v|^p)dx : v \in W^{1,p}(\Omega), \int_\Omega |v|^p\, dx = 1\Big\}.$$

Here $\mu(\alpha,p)$ is the first eigenvalue in $W^{1,p}(\Omega)$ for the Neumann boundary condition of

$$u \mapsto -\Delta_p u + \alpha^{q-(p-1)} b(.) u^{p-1}.$$

The theorem states as follows [1]:

Theorem 2.4 *Let $0 \leq q < 1$, $p > 2$ and assume that there exist two sequences of positive real numbers (α_n) and (r_n) such that (α_n) is decreasing and*

$$\sum_{n=0}^{\infty} \frac{r_n^{p-1}}{\alpha_{n+1}^{p-2}\mu(\alpha_n,p)^{\sigma(r_n)}} < +\infty. \tag{2.4}$$

Then any solution of (1.2) with initial bounded data satisfies the ***TCS*** *property.*

Consequently, if $r_n = \ln \mu(\alpha_n, p)$, we have the following statement.

Corollary 2.1 *Under the same assumptions on q and p, if there exists a decreasing sequence of positive real numbers* (α_n) *such that*

$$\sum_{n=0}^{\infty} \frac{(\ln \mu(\alpha_n, p))^{p-1}}{\alpha_{n+1}^{p-2} \mu(\alpha_n, p)} < +\infty, \tag{2.5}$$

then any solution of (1.2) satisfies the ***TCS*** *property.*

Theorem 2.4 comes from the following lemma.

Lemma 2.2 *Suppose there exists a measurable function u in* $\Omega \times \mathbb{R}^+$ *which satisfies weakly (1.2) with* $\|u_0\|_{L^\infty(\Omega)} \leq \alpha$ *for some* $\alpha > 0$. *Then*

$$\|u(.,t)\|_{L^r(\Omega)} \leq \Big(\frac{1}{\|u(.,0)\|_{L^r(\Omega)}^{2-p} + C_1 \mu(\alpha, p) t} \Big)^{\frac{1}{p-2}}, \tag{2.6}$$

where $C_1 = C_1(\Omega, r, p)$ *is a positive real constant and there exist two positive real numbers* $C = C(\Omega, p)$ *and* $C_2 = C_2(r, p)$ *such that*

$$\|u(.,t)\|_{L^\infty(\Omega)} \leq \min \Big(C(1 + \frac{2}{t})^{\delta(r)} \Big(\frac{1}{\|u(.,0)\|_{L^\infty(\Omega)}^{2-p} + C_2 \mu(\alpha, p) t} \Big)^{\frac{\sigma(r)}{p-2}}, 1 \Big),$$

with $\delta(r) = \frac{N}{rp + N(p-2)}$ *and* $\sigma(r) = \frac{rp}{rp+N(p-2)}$.

Idea in the proofs. The principle to prove them remains true. It is a bit more complicated because the term u_t is not homogenuous with u^{p-1} but it follows exactly the Kondratiev-Vron method as shown in the proof of Theorem 2.2. The main differences are technical. Instead of using u as test-function, we use $u|u|^{r_n - 1}$ at each step of the iteration. An estimate of the asymptotic behaviour when $r \to +\infty$ for the constant $C_2 = C_2(r, p)$ is needed. The proof of the theorem ends with sharp upper bounds for the series $\sum_n t_{n+1} - t_n$. □

Now, let us talk about equation 1.3. Formally, replacing $p-1$ by m give the same results [11, 12]:

Theorem 2.5 *Let* $m > 0$ *and u be a weak solution of*

$$u_t - \Delta(u^m) + B(x, t, u) = 0 \quad \textit{in } \Omega \times (0, \infty),$$
$$\partial_\nu u = 0 \quad \textit{on } \partial\Omega,$$
$$u(x, 0) = u_0(x) \in L^r(\Omega),$$

where B is a Caratheodory function satisfying $B(x,t,\rho)\rho \geq 0$ *a.e. in* $\Omega \times (0, \infty)$. *If* $r \geq 1$ *and* $r > N(1-m)/2$, *then*

$$\|u(.,t)\|_{L^\infty(\Omega)} \leq C(1 + \frac{1}{t})^{\delta(r)} \|u(.,0)\|_{L^r(\Omega)}^{\sigma(r)},$$

with $C = C(\Omega, m)$, $\delta(r) = \frac{N}{2r + N(m-1)}$ *and* $\sigma(r) = \frac{2r}{2r+N(m-1)}$.

We set quantities adapted to the problem

$$\mu'(\alpha,m)=\inf\Big\{\int_\Omega(|\nabla v|^2+\alpha^{q-m}b(x)|v|^2)dx : v\in W^{1,2}(\Omega),\int_\Omega |v|^2\,dx=1\Big\}.$$

Thus,

Theorem 2.6 ([1]) *Let $0\le q<1$, $m>1$ and assume that there exists two sequences of positive real numbers (α_n) and (r_n) such that (α_n) is decreasing and*

$$\sum_{n=0}^{\infty}\frac{r_n^m}{\alpha_{n+1}^{m-1}\mu'(\alpha_n,m)^{\sigma(r_n)}}<+\infty. \tag{2.7}$$

*Then any solution of (1.3) with initial bounded data satisfies the **TCS** property.*

With $r_n=\ln\mu'(\alpha_n,m)$, we deduce the following statement.

Corollary 2.2 *Under the above assumptions on q and m, if there exists a decreasing sequence of positive real numbers (α_n) such that*

$$\sum_{n=0}^{\infty}\frac{(\ln\mu'(\alpha_n,m))^m}{\alpha_{n+1}^{m-1}\mu'(\alpha_n,m)}<+\infty,$$

*then any solution of (1.3) satisfies the **TCS** property.*

The proof of Theorem 2.6 also comes from the following lemma.

Lemma 2.3 *We suppose there exists a measurable function u in $\Omega\times\mathbb{R}^+$ which satisfies weakly (1.3) with $\|u_0\|_{L^\infty(\Omega)}\le\alpha$ for some $\alpha>0$. Then*

$$\|u(.,t)\|_{L^r(\Omega)}\le\Big(\frac{1}{\|u(.,0)\|_{L^r(\Omega)}^{1-m}+C_1\mu'(\alpha,m)\,t}\Big)^{1/(m-1)}, \tag{2.8}$$

with $C_1=C_1(\Omega,r,m)$ and there exist two positive real numbers $C=C(\Omega,m)$ and $C_2=C_2(r,m)$ such that

$$\|u(.,t)\|_{L^\infty(\Omega)}\le\min\Big(C\big(1+\frac{2}{t}\big)^{\delta(r)}\big(\frac{1}{\|u(.,0)\|_{L^\infty(\Omega)}^{1-m}+C_2\mu'(\alpha,m)t}\big)^{\frac{\sigma(r)}{m-1}},1\Big),$$

where $\delta(r)$ and $\sigma(r)$ are defined in Theorem 2.5

The assumptions in Theorem 2.2 and Corollaries 2.1, 2.2 admit a simpler form. A comparaison between series and integral gives the following theorem.

Theorem 2.7 (Integral criterion [3, 1]) *Let $0\le q<1$. 1) If $p\ge2$ and*

$$\int_0^1\frac{(\ln\mu(t,p))^{p-1}}{t^{p-1}\mu(t,p)}dt<+\infty,$$

then all solutions of (1.2) satisfy the ***TCS*** *property.*
2) If $m \geq 1$ *and*

$$\int_0^1 \frac{(\ln \mu'(t,m))^m}{t^m \mu'(t,m)} dt < +\infty,$$

then all solutions of (1.3) satisfy the ***TCS*** *property.*

We remark that $\mu(t) = \mu(t,2)$ and that (1.1) is a particular case of (1.2) for $p = 2$ and (1.3) for $m = 1$. The proof is first establish for $p = 2$ [3, page 51] and then for $p > 2$ and $m > 1$ [1]. What is remarkable is that this criterion has a same simple form in all cases.

For applications, $\mu(t,p)$ and $\mu'(t,m)$ have to be linked directly to the function b. We recall that $\mu(\alpha,p)$ is the first eigenvalue in $W^{1,p}(\Omega)$ for the Neumann boundary condition of $u \mapsto -\Delta_p u + \alpha^{q-(p-1)} b(.) u^{p-1}$.

The aim of semi-classical analysis is to describe the behavior of the spectrum of the operator $u \mapsto -\Delta_p u + h^{-p} V(.) u^{p-1}$ in particular $\lambda_1(h)$ the lowest eigenvalue. V is a function which holds in our case

$$V \in L^\infty(\Omega), \quad \operatorname*{ess\,inf}_\Omega V = 0 \quad \text{and} \quad \int_\Omega V(x)\,dx > 0. \tag{2.9}$$

We denote by γ a positive number which satisfies:

$$\gamma \begin{cases} = \frac{N}{p} & \text{for } 1 < p < N, \\ \in (1, +\infty) & \text{for } p = N, \\ = 1 & \text{for } p > N, \end{cases} \tag{2.10}$$

Corollary 2.3 *If (2.9) holds then for h small enough,*

$$\lambda_1(h)(\text{meas}\{x : V(x) \leq h^p \lambda_1(h)\})^{1/\gamma} \geq C, \tag{2.11}$$

where $C = C(p, N, \gamma, \Omega, V)$ is a positive constant.

$\mu(t,p)$ can be written as $\mu(t,p) = \lambda_1(t^{\frac{(p-1)-q}{p}})$ which after a change of variables gives

$$\int_0^1 \frac{(\ln \mu(t,p))^{p-1}}{t^{p-1}\mu(t,p)} dt = \int_0^1 \frac{(\ln \lambda_1(h))^{p-1}}{h^{\frac{p(p-1)-(1+q)}{p-(1+q)}} \lambda_1(h)} dh.$$

If we have an estimate of the type

$$\lambda_1(h) \geq C \frac{1}{h^\theta},$$

where C and θ are two positive real numbers, then the integral criterion holds for $p > 2$ provided

$$\theta > \frac{p(p-2)}{p-(1+q)}. \tag{2.12}$$

Similar expressions can be found for $p = 2$ and $m > 1$. Finally, we obtain next theorem.

Theorem 2.8 ($1/b$ criterion [3, 1]) *Let $0 \leq q < 1$ and b be a bounded measurable function such that*

$$\operatorname*{ess\,inf}_{\Omega} b = 0 \quad \text{and} \quad \int_{\Omega} b(x)\,dx > 0.$$

1) If $p = 2$ and $\ln(1/b) \in L^s(\Omega)$ for some $s > N/2$ then equation (1.1) satisfies the ***TCS*** *property.*
2) If $p > 2$ and $(1/b)^s \in L^1(\Omega)$ for some s with

$$s > \begin{cases} \frac{p-2}{1-q}\left(\frac{N}{p}\right) & \text{for } p \leq N, \\ \frac{p-2}{1-q} & \text{for } p > N, \end{cases}$$

then equation (1.2) satisfies the ***TCS*** *property.*
3) If $m > 1$ and $(1/b)^s \in L^1(\Omega)$ for some s with

$$s > \begin{cases} \frac{m-1}{1-q}\left(\frac{N}{2}\right) & \text{for } N \geq 2, \\ \frac{m-1}{1-q} & \text{for } N = 1, \end{cases}$$

then equation (1.3) satisfies the ***TCS*** *property.*

Outline of the proof. the three cases are based on Marcinkiewicz type inequalities. For 1)

$$\text{meas}\left\{x \in \Omega \ : \ \ln \frac{1}{b(x)} \geq \ln \frac{1}{h^2 \lambda_1(h)}\right\} \leq \frac{1}{\left(\ln \frac{1}{h^2\lambda_1(h)}\right)^s} \int_{\Omega} \left(\ln \frac{1}{b(x)}\right)^s dx,$$

and for 2)

$$\text{meas}\left\{x : \frac{1}{b(x)} \geq \frac{1}{h^p \lambda_1(h)}\right\} \leq (h^p\lambda_1(h))^s \int_{\Omega} \left(\frac{1}{b(x)}\right)^s dx.$$

The proof ends with estimates such as (2.12) and some technical arguments. □

Remark 2.1 In the case where $p = 2$ and $N \leq 2$, estimate (2.11) is not enough sharp so we use the formula of Lieb and Thirring. See [3] for details.

Now we apply the previous theorem to the radial functions.

Corollary 2.4 *Suppose that $0 \in \Omega$. 1) If $b(x) = \exp(-\frac{1}{\|x\|^\beta})$ with $\beta < 2$ then any solution of (1.1) satisfies the* ***TCS*** *property.*
2) If $b(x) = \|x\|^\beta$ with $p \leq N$ and $\beta < p(1-q)/(p-2)$ then any solution of (1.2) satisfies the ***TCS*** *property.*
One has the same conclusion if $p > N$ and $\beta < N(1-q)/(p-2)$.
3) If $b(x) = \|x\|^\beta$ with $N \geq 2$ and $\beta < 2(1-q)/(m-1)$ then any solution of (1.3) satisfies the ***TCS*** *property.*
One has the same conclusion if $N = 1$ and $\beta < (1-q)/(m-1)$.

3 A lower bound for the first eigenvalue

This section is dedicated to estimating the first eigenvalue, in $W^{1,p}(\Omega)$, of the operator $u \mapsto -\Delta_p u + h^{-p}V(.)u^{p-1}$. We have seen that a lower bound is fundamental for applications. First,we introduce a sequence of definitions. We consider a non-empty connected open subset $\Omega \subset \mathbb{R}^N$ and a mesurable function V defined in Ω. We set

$$W^{1,p,V}(\Omega) = \{\psi \in W^{1,p}(\Omega) \ : \ V(x)|\psi^p| \in L^1(\Omega)\}.$$

If $W^{1,p,V}(\Omega) \neq \{0\}$ and $\psi \in W^{1,p,V}(\Omega)$, we set

$$F_V(\psi) = \int_\Omega |\nabla\psi|^p + V(x)|\psi|^p \, dx, \tag{3.1}$$

and define

$$\lambda_1 = \inf\left\{F_V(\psi) : \psi \in W^{1,p,V}(\Omega), \int_\Omega |\psi|^p \, dx = 1\right\}, \tag{3.2}$$

and for $h > 0$,

$$\lambda_1(h) = \inf\left\{F_{h^{-p}V}(\psi) : \psi \in W^{1,p,V}(\Omega), \int_\Omega |\psi|^p \, dx = 1\right\}, \tag{3.3}$$

Thus $\lambda_1(h)$ is the first eigenvalue of the operator

$$u \mapsto -\Delta_p u + h^{-p}V(.)|u|^{p-2}u. \tag{3.4}$$

in $W^{1,p,V}(\Omega)$ with Neumann boundary condition if the infimum is achieved by a regular enough element of $W^{1,p,V}(\Omega)$ and $\partial\Omega$ $\mathcal{C}^1$.
We start with a simple result which enlights our arguments. On the contrary to the linear case ($p = 2$), our proof is not based on the theory of pseudodifferential operators but on the continuous injections of $W^{1,p}(\Omega)$ into the L^s spaces for suitable s.

Theorem 3.1 *Suppose $N > p > 1$. Then either $\lambda_1 = -\infty$ or*

$$\Big(\int_{V(x)\leq\lambda_1} (\lambda_1 - V(x))^{N/p} \, dx\Big)^{p/N} \geq C(p, N), \tag{3.5}$$

where $C = C(p, N) > 0$ is the positive constant of the Sobolev inequality. In addition, if there exists a minimizer in $W^{1,p,V}(\mathbb{R}^N)$,

$$\Big(\int_{V(x)<\lambda_1} (\lambda_1 - V(x))^{N/p} \, dx\Big)^{p/N} \geq C(p, N). \tag{3.6}$$

Proof. Let ψ be in $W^{1,p,V}(\mathbb{R}^N)$ with $\|\psi\|_{L^p(\mathbb{R}^N)} = 1$ then

$$\int_{\mathbb{R}^N} |\nabla\psi|^p \, dx + \int_{\mathbb{R}^N} V(x)|\psi|^p \, dx = F_V(\psi) = F_V(\psi) \int_{\mathbb{R}^N} |\psi|^p \, dx.$$

The integral with V is split in two parts, that is,
$\mathbb{R}^N = \{x : V(x) < F_V(\psi)\} \cup \{x : V(x) \geq F_V(\psi)\}$. Therefore,

$$\int_{\mathbb{R}^N} |\nabla\psi|^p \, dx \leq \int_{V(x)<F_V(\psi)} (F_V(\psi) - V(x))|\psi|^p \, dx. \tag{3.7}$$

Hölder's inequality leads to

$$\int_{\mathbb{R}^N} |\nabla\psi|^p \, dx \leq \Big(\int_{V(x)<F_V(\psi)} (F_V(\psi) - V(x))^{N/p} \, dx \Big)^{p/N} \Big(\int_{\mathbb{R}^N} |\psi|^{p^*} \, dx \Big)^{1-\frac{p}{N}}. \tag{3.8}$$

since $\{x : V(x) < F_V(\psi)\} \subset \mathbb{R}^N$. Non zero constants do not belong to $W^{1,p,V}(\mathbb{R}^N)$ and so all functions ψ satisfy $\int_{\mathbb{R}^N} |\nabla\psi|^p \, dx > 0$. We can apply Sobolev inequality. The Beppo-Levi theorem completes the proof. □

Remark 3.1 If Ω is any open domain of $\mathbb{R}^N$, we define

$$W_0^{1,p,V}(\Omega) = \{\psi \in W_0^{1,p}(\Omega) : V(x)|\psi^p| \in L^1(\Omega)\},$$

and if $W_0^{1,p,V}(\Omega) \neq \{0\}$,

$$\tilde{\lambda_1} = \inf \big\{ F_V(\psi) : \psi \in W_0^{1,p,V}(\Omega), \int_\Omega |\psi|^p \, dx = 1 \big\},$$

then the estimates in Theorem 3.1 hold for $\tilde{\lambda_1}$.

When Ω is a $\mathcal{C}^1$ bounded domain of $\mathbb{R}^N$ and V is a measurable function such that

$$V \in L^\infty(\Omega), \quad \operatorname*{ess\,inf}_{\Omega} V = 0 \quad \text{and} \quad \int_\Omega V(x) \, dx > 0, \tag{3.9}$$

we set u_h the first eigenfunction related to the first eigenvalue $\lambda_1(h)$.

Recall that γ is a positive number which satisfies

$$\gamma \begin{cases} = \frac{N}{p} & \text{for } 1 < p < N, \\ \in (1, +\infty) & \text{for } p = N, \\ = 1 & \text{for } p > N, \end{cases} \tag{3.10}$$

with $\frac{\gamma}{\gamma-1} = +\infty$ if $\gamma = 1$. This γ is such that $W^{1,p}$ imbeds $L^q(\Omega)$ continuously with $q = p\frac{\gamma}{\gamma-1}$.

Theorem 3.2 *Assume that (3.9) holds. Then for h small enough,*

$$\Big(\int_{V(x)<h^p\lambda_1(h)} \big(\lambda_1(h) - \frac{V(x)}{h^p}\big)^\gamma \, dx \Big)^{1/\gamma} \geq C,$$

where $C = C(p, N, \gamma, \Omega, V)$ is a positive real constant.

Proof. We start with (3.8) because the beginning is similar. Replacing $\mathbb{R}^N$, ψ and V by Ω, u_h and $\frac{V}{h^p}$ the Hölder's inequality gives

$$\int_\Omega |\nabla u_h|^p\,dx \le \Big(\int_{V(x)<h^p\lambda_1(h)} \big(\lambda_1(h)-\frac{V(x)}{h^p}\big)^\gamma\,dx\Big)^{1/\gamma}\Big(\int_\Omega |u_h|^q\,dx\Big)^{p/q},$$

where $q=p\frac{\gamma}{\gamma-1}$. Thus, by the imbeddings,

$$\Big(\int_{V(x)<h^p\lambda_1(h)} \big(\lambda_1(h)-\frac{V(x)}{h^p}\big)^\gamma\,dx\Big)^{1/\gamma} \ge C\frac{\|\nabla u_h\|^p_{L^p(\Omega)}}{1+\|\nabla u_h\|^p_{L^p(\Omega)}},$$

with $C=C(p,N,\Omega,\gamma)$ a positive real number. The main idea is to prove that

$$\liminf_{h\to 0}\|\nabla u_h\|_{L^p(\Omega)}>0.$$

Suppose that there exists a sequence (h_n) of positive real numbers which goes to zero such that

$$\lim_{n\to+\infty}\|\nabla u_{h_n}\|_{L^p(\Omega)}=0.$$

Hence (u_{h_n}) is bounded in $W^{1,p}(\Omega)$, so there exists a function u_0 in $W^{1,p}(\Omega)$ such that, up to a subsequence, $u_{h_n}\rightharpoonup u_0$ weakly in $W^{1,p}(\Omega)$. Obviously, $\|\nabla u_0\|_{L^p(\Omega)}=0$. Therefore, $u_0=C$ where C is a real. Thanks to the Rellich-Kondrachov theorem, up to a subsequence, $u_{h_n}\to C$ strongly in $L^p(\Omega)$ so $C=(\frac{1}{\mathrm{meas}(\Omega)})^{\frac{1}{p}}$. We deduce that $\lim_{n\to+\infty}h_n^p\lambda_1(h_n)=\frac{\int_\Omega V(x)\,dx}{\mathrm{meas}(\Omega)}$. But from lemma 3.2 in [3], $\lim_{h\to 0}h^p\lambda_1(h)=0$ which leads to a contradiction. □

A simpler form is provided in the following corollary.

Corollary 3.1 *If (3.9) holds then for h small enough,*

$$\lambda_1(h)(meas\{x:V(x)<h^p\lambda_1(h)\})^\gamma\ge C,$$

where $C=C(p,N,\gamma,\Omega,V)$.

We end this section by quoting a theorem. For Ω a domain of $\mathbb{R}^N$ bounded or not, regular or not and V a mesurable function defined on Ω such that $W^{1,p,V}(\Omega)\ne\{0\}$, we define a well for a mesurable function V [1].

Definition. We say that V has a well in U if U is a $\mathcal{C}^1$ bounded, connected, non-empty open set of Ω and if there exists $\psi_0\in W^{1,p,V}(\Omega)$ with $\|\psi_0\|_{L^p(\Omega)}=1$ such that $\displaystyle\int_\Omega V(x)|\psi_0|^p\,dx<a=\operatorname*{essinf}_{\Omega\backslash U}V$ with $\mathrm{meas}(\Omega\backslash U)>0$.

The term of well generalizes the definition in [8].

Theorem 3.3 ([3]) *If V has a well in U, for h small enough,*

$$\Big(\int_{V(x)\le h^p\lambda_1(h)}(\lambda_1(h)-h^{-p}V(x))^\gamma\,dx\Big)^{1/\gamma}\ge C,$$

where C is a positive constant which does not depend on h.

In addition, if there exists a minimizer in $W^{1,p,V}(\Omega)$,

$$\Big(\int_{V(x)<h^p\lambda_1(h)} (\lambda_1(h) - h^{-p}V(x))^\gamma \, dx\Big)^{1/\gamma} \geq C.$$

The proof is technical but some arguments have already been used for Theorem 3.2.

4 Summary and open questions

For the sake of completeness, we quote another theorem of.

Theorem 4.1 ([3]) *Suppose that b is a continuous and nonnegative function defined in $\overline{\Omega}$ which satisfies for some $x_0 \in \Omega$*

$$\lim_{r\to 0} r^2 \ln(1/\|b\|_{L^\infty(B_r(x_0))}) = \infty.$$

*If u is a weak solution of (1.1) then u does not satisfies the **TCS** property.*

Up to now, we have the following:

	$p = 2$	$p > 2$	$m > 1$
Integral criterion	$\int_0^1 \frac{\ln \mu(t)}{t\mu(t)} dt < \infty$	$\int_0^1 \frac{(\ln \mu(t,p))^{p-1}}{t^{p-1}\mu(t,p)} dt < \infty$	$\int_0^1 \frac{(\ln \mu'(t,m))^m}{t^m \mu'(t,m)} dt < \infty$
$1/b$ criterion with	$\ln(1/b) \in L^s$ $s > \frac{N}{2}$	$1/b \in L^s$ $s > \frac{p-2}{1-q}\frac{N}{p}$, $N \geq p$ $s > \frac{p-2}{1-p}$, $N < p$	$1/b \in L^s$ $s > \frac{m-1}{1-q}\frac{N}{2}$, $N \geq 2$ $s > \frac{m-1}{1-q}$, $N = 1$
Radial case for $\beta \geq 0$ and	$\exp(-1/\|x\|^\beta)$ $\beta < 2$	$\|x\|^\beta$ $\frac{p(1-q)}{p-2}$, $N \geq p$ $\beta < \frac{N(1-q)}{p-2}$, $N < p$	$\|x\|^\beta$ $\beta < \frac{2(1-q)}{m-1}$, $N \geq 2$ $\beta < \frac{(1-q)}{m-1}$, $N = 1$
Converse	yes	no	no
Non **TCS** property for	$\exp(-1/\|x\|^\beta)$ $\beta > 2$	$\vdots$	$\vdots$

Open questions

1. What happens for $p = 2$ and $\beta = 2$? It does not seem within sight.

2. We have no genuine converse for $p > 2$ and $m > 1$. A converse has been found for $p = 2$ because $L^2(\Omega)$ has an inner product. More precisely, for $p > 2$, $\int_\Omega u^{p-1} v\, dx \neq \int_\Omega v^{p-1} u\, dx$ in general. We search for another test-functions (see [3] for details).

3. When $p > 2$, we have a good generalization of the Cwikel, Lieb and Rosenblyum formula, that is, for large dimension ($N > p$). The estimate for $N \leq p$ is far from the optimum. When $p = 2$, the Lieb and Thirring formula works well. We hope that we will find an equivalent.

4. In [7], they also deal with second order elliptic equations with a strong absortion, i.e., $u_{tt} + \Delta u - a(x)u^q = 0$. Heuristically speaking, changing $\mu(\alpha)$ into $\sqrt{\mu(\alpha)}$ gives a sufficient condition for the **TCS** property. We are working on this type of equation when a depends also on t.

5. More generally, the following problem $\Delta_p u - a(x)u^{p-1} = 0$ in an outside domain is difficult to handle. On $\mathbb{R}^N$ minus a ball, a similar technique may be possible.

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